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A CHARACTERIZATION OF A CERTAIN CLASS OF ARITHMETICAL MULTIPLICATIVE FUNCTIONS

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The object of this paper is the set of the "arithmetical multiplicative functions", i.e. the functions $\mathbb{N} \to \mathbb{C}$ for which $f(mn) = f(m)f(n)$, under the condition that m and n have no common factors. This set is a group with respect to the Dirichlet's convolution. We define for such functions the concept of type (briefly, a number that expresses the fact that $f(p^n)$ is zero when n is large enough); moreover, we prove that the set of the completely multiplicative functions which do not assume the value zero coincides with the set of the functions whose inverses are of type 1.

1. Arithmetical functions and multiplicative functions.

Let us define an *arithmetical function* as a function whose domain is the set $\mathbb{N}$ of natural (non zero) numbers and whose values are complex numbers. We may introduce on the set I of all arithmetical functions a ring structure ([1], [11], [6] ex. 4.8), by defining, $\forall f, g \in I$, the operations $+$ and $\times$ as follows:

$$(f+g)(n) = f(n) + g(n);$$
$$(f \times g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

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In other terms, "+" is the ordinary sum of functions, while "$\times$" is the so-called *convolution*, or integral product, or Dirichlet's product (of course the sum is extended over all the divisors of n, also $d = 1$ and $d = n$). It is easy to verify that the additive neutral element is the function $\underline{0}$, which assumes the value 0 for every n, and that the multiplicative neutral element is the function $\alpha(n)$ (sometimes called $\delta(n)$) defined by

$$\alpha(n) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathscr{I}$ denote the ring $(I, +, \times)$ of the arithmetical functions; it is well known that $\mathscr{I}$ is a commutative and unitary ring, that it is an *integrity domain* (i.e., without proper divisors of zero), and that it is a *unique factorization ring* (shortly an UF-ring), because each decomposable element of $\mathscr{I}$ can be written in an "essentially" unique way as a product of indecomposable elements (the word "essentially" means that we can insert in the factorization invertible elements and their inverses as we like) ([1], [3],[11]).

For every $f \in \mathscr{I} - \{\underline{0}\}$ the *order* $\chi(f)$ is defined as the smallest n for which $f(n) \neq 0$ (we may also define $\chi(\underline{0}) = \infty$). The *theorem of the order* asserts that for any $f, g \in \mathscr{I}$ it results $\chi(f \times g) = \chi(f)\chi(g)$. A consequence of this theorem is that a function f is invertible if and only if $\chi(f) = 1$, i.e. if and only if $f(1) \neq 0$. Besides, if $\chi(f)$ is a prime, then f is necessarily indecomposable, but if $\chi(f)$ is composite, it is not true in general that f is decomposable. We shall indicate by $\mathscr{U}$ the set of all invertible functions of $\mathscr{I}$, which is of course a group with respect to the convolution.

A very remarkable subgroup of $\mathscr{U}$ can be defined with the set $\mathscr{M}$ of so-called multiplicative arithmetical functions.

Definition. A function $f \in \mathscr{I}$, $f \neq \underline{0}$, is said to be *multiplicative* if for any $m, n \in \mathbb{N}$ with $(m, n) = 1$, we have $f(mn) = f(m)f(n)$.

Obviously, (m, n) denotes the greatest common divisor of m and n. So $(m, n) = 1$ means that *m and n* are *coprime*, i.e. without common factors other than 1. Many of the most interesting arithmetical functions are multiplicative: for example, the function $d(n)$ (number of the divisors of n), the function $\sigma(n)$ (sum of the divisors of n) and its generalization $\sigma_k(n)$ (sum of the k-th powers of the divisors of n), the Möbius function $\mu(n)$, the Euler's totient $\phi(n)$, and so on. It is to be observed that the definition of multiplicative function implies $f(1 \cdot 1) = f(1)f(1)$, so $f(1)$ may be either 1 or 0; but, if $f(1) = 0$, we see at once that f is identically zero: this is impossible by definition, and therefore we obtain that for every multiplicative function f is must be $f(1) = 1$.

It is well known that if f and g are two multiplicative functions, then $f \times g$ is also multiplicative, and if f is multiplicative, so is its inverse function, which we shall indicate by $f^{(-1)}$. Since α is also multiplicative, we have that $(\mathscr{M}, \times)$ is a subgroup of $(\mathscr{U}, \times)$.

Another important subset of $\mathscr{M}$, not closed with respect to $\times$, is the class $\mathscr{C}$ of the *completely multiplicative* functions: they are the arithmetical functions for which $f(mn) = f(m)f(n)$ for *any* m and n. We may also write:

$$f \in \mathscr{C} \iff f \in \mathscr{M}, \quad f(p^i) = [f(p)]^i \quad \forall p \in P, \ \forall i \in \mathbb{N}.$$

For example, given a $k \in \mathbb{Z}$, the function N_k, which is equal to n^k for every n, is completely multiplicative, and so is the function $u(n) = 1$ for every n. But $u \times u = d$, and d is not completely multiplicative. We may say that $f \in \mathscr{M}$ is known if it is assigned for all numbers p^j, i.e. for the powers of the primes, while $f \in \mathscr{C}$ is known if it is assigned for every prime p. For example, by defining $f(p) = p - 1$, we obtain a completely multiplicative function f defined as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ (p_1 - 1)^{a_1}(p_2 - 1)^{a_2} \cdots (p_r - 1)^{a_r} & \text{if } n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \end{cases}$$

Let us recall two well known theorems which we shall use later.

Theorem 1. *If $f \in \mathscr{C}$, then $(fg) \times (fh) = f(g \times h)$ for any $g, h \in \mathscr{I}$.*

Theorem 2. *If $f \in \mathscr{C}$, then $f^{(-1)} = \mu f$.*

2. The type of a multiplicative function.

Now we introduce the concept of *type*. We say that a multiplicative function f is *of type* $\eta(f) = k$, if:

(i) for every prime p, $f(p^k) \neq 0$;
(ii) for every prime p and for every $j > k$, $f(p^j) = 0$.

In other words, we ask that f is zero for the powers of p large enough, while this must never happen for the k-th powers of the primes (if $\eta(f) \geq 2$, it is allowed that $f(p^j) = 0$ for some $j < k$ and for some prime p). Note that there are multiplicative functions which have no type.

A simple example of a function of type 1 is the *Möbius function*, which is the inverse function of u: it is defined by

$$\begin{cases} \mu(1) = 1 \\ \mu(p) = -1 \\ \mu(p^n) = 0 \quad \text{for } n > 1 \end{cases}$$

The only function of type 0 is α; other functions that have a defined type are, e.g. the inverse function of d and the inverse function of σ: both of them are of type 2 (see below).

Let us prove the following

Theorem 3 (theorem of the types). *Let $f, g \in \mathcal{M}$, and suppose $\eta(f) = a$, $\eta(g) = b$; then $\eta(f \times g) = a + b$.*

Proof. Set $h = f \times g$ (we know that h is also multiplicative); for each prime p and for every natural k it is $h(p^k) = \sum_{s+t=k} f(p^s)g(p^t)$. If $k = a + b$, we have to write $a + b$ in all possible ways as the sum of two non-negative integers s and t; but there is only one way of doing this with s not greater than a and t not greater than b, precisely $s = a$ and $t = b$. Then $h\left(p^{a+b}\right)$ reduces only to the term $f(p^a)g(p^b)$, that is different from zero by hypothesis. If $k > a + b$, once k is written as $s + t$, we have necessarily $s > a$ or $t > b$, then each product $f(p^s)g(p^t)$ is zero by hypothesis. Hence the function h is of type $a + b$.

Remark. One might think to define an alternative "type" of a multiplicative function as follows: $\eta(f) = k$ if $f(p^j) = 0$ for every prime p and for every $j > k$, but $f(p^k) \neq 0$ for *at least* a prime p. Using such a definition of type, the Theorem 3 is not true in general; precisely, it must be corrected as follows: $\eta(f \times g) \leq \eta(f) + \eta(g)$. Anyway, in what follows we shall use the former definition of type.

From Theorem 3 we have, as announced, that $\eta(e) = 2$, where $e = d^{(-1)}$: in fact $d = u \times u$; therefore $e = \mu \times \mu$, and the thesis follows from $\eta(\mu) = 1$. So we may easily write the explicit expression of $e(n)$: it is sufficient to calculate

$$e(p) = \mu(1)\mu(p) + \mu(p)\mu(1) = -2,$$
$$e(p^2) = \mu(1)\mu(p^2) + \mu(p)\mu(p) + \mu(p^2)\mu(1) = 1,$$

to obtain

$$\begin{cases} e(1) = 1 \\ e(p) = -2 \\ e(p^2) = 1 \\ e(p^n) = 0 & \text{for } n > 2 \end{cases}$$

We also deduce from the theorem of the types that if $\eta(f) > 0$, then $f^{(-1)}$ cannot have a type. One might think to assign a negative type to a function whose inverse has a positive type (for example, $\eta(u) = -1$, $\eta(d) = -2$); this seems to be justified by the theorem itself, because $\eta(\alpha) = 0$. But it is impossible to extend the concept of type in such a way, for *it is not true* in general that

$\eta(f \times g) = a + b$ for any a and b in $\mathbb{Z}$: in fact, if it was so, by calculating $f \times g$ with $\eta(f) = 2$ and $\eta(g) = -2$, we should always obtain $\eta(f \times g) = 0$, which is of course false in general (e.g., $e \times \sigma$ does not coincide with α).

Nevertheless, for every non-negative integer k we may define $\mathcal{M}_k$ as the class of the multiplicative functions of type k, and $\mathcal{M}_{-k}$ as the class of the functions *whose inverses* are of type k; where the class $\mathcal{M}_0$ contains of course only α. It is plain that these classes are all disjoint, but we may also note that the union of all the classes $\mathcal{M}_k$ with $k \in \mathbb{Z}$) *does not* coincide with $\mathcal{M}$, for there exist functions that have no type, and such that their inverses do not have a type either; let us consider in fact the function ϕ, that is the Euler's totient, and let ψ be $\phi^{(-1)}$: we may easily see that $\psi(p^n) = 1 - p$, *independently* of the exponent n (it is an example of *a strongly multiplicative* function, i.e. a multiplicative function for which $f(p^n) = f(p)$ for every p and for $n > 1$); it is plain that neither ϕ nor ψ have a type.

Apart from α, completely multiplicative functions have no type, for, if it were $f(p^n) = 0$ for n large enough and for every prime p, f should be zero for every p. Hence the class $\mathcal{C}$ is disjoint with all the classes $\mathcal{M}_k$ with positive k. One may ask now what is the behaviour of $\mathcal{C}$ with respect to the classes $\mathcal{M}_k$ with negative k. We shall answer this question with the Theorem 4.

Let us define a subclass of $\mathcal{C}$, which we will denote by $\mathcal{C}^*$, constituted by the completely multiplicative functions that are *never* equal to zero, or (which is the same) that are never zero for any prime p. For example, the function N_k, which is equal to n^k for each n, belongs to $\mathcal{C}^*$, while the function that is equal to 1 if and only if n is odd is a completely multiplicative function but does not belong to $\mathcal{C}^*$.

Theorem 4. *With the symbols already defined, we have* $\mathcal{C}^* = \mathcal{M}_{-1}$*; in other words, a function is of type* 1 *if and only if its inverse is completely multiplicative and never equal to zero.*

Proof. If $f \in \mathcal{C}^*$, then $f^{(-1)} = \mu f$ (by Theorem 2); so:

$$f^{(-1)}(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ (-1)^r f(n) & \text{if} \quad n = p_1 \cdots p_r \\ 0 & \text{otherwise} \end{cases}$$

Since f is never equal to zero, $f^{(-1)}$ is of type 1, so we have proved that $f \in \mathcal{M}_{-1}$. On the other hand, if f is a function of type 1, we have to prove that its inverse (which we already known to be multiplicative) is completely multiplicative and never equal to zero. Let g be $f^{(-1)}$; by calculating $(f \times g)(p)$ we have:

$$0 = f(1)g(p) + f(p)g(1),$$

from which $g(p) = -f(p)$; now let us show by induction that $g(p^n) = [-f(p)]^n$. Since this is plainly true for $n = 1$, we must only see the passage from n to $n+1$; let us calculate $(f \times g)(p^{n+1})$:

$$0 = f(1)g(p^{n+1}) + f(p)g(p^n) + f(p^2)g(p^{n-1}) + \ldots + f(p^{n+1})g(1).$$

But $\eta(f) = 1$, hence this equality reduces to $0 = g(p^{n+1}) + f(p)g(p^n)$, i.e., for the inductive hypothesis, $g(p^{n+1}) = [-f(p)][-f(p)]^n = [-f(p)]^{n+1}$. This proves that g is completely multiplicative, and now it is clear that it is never equal to zero, since $f(p) \neq 0$ for every p.

As a corollary of Theorem 4, we may state that $\mathscr{M}_{-1} \subset \mathscr{C}$. A function f of the set $\mathscr{C} - \mathscr{M}_{-1}$, with the exception of α, does not belong to any of the classes $\mathscr{M}_k$: in fact we have already said that this is true for $k > 0$; but it is obviously true also for $k < 0$, because $f^{(-1)}$ is equal to μf, which is zero for $n = p^2, p^3$, etc., but is not different from zero for *all* the primes, and therefore has no type.

As a further application, let us give the explicit expression of the function $\tau = \sigma^{(-1)}$. We know that $\sigma = N \times u$, where N is the "identity function", $N(n) = n$ for every n (in fact for every n we have $(N \times u)(n) = \sum_{d|n} N(d)u\left(\frac{n}{d}\right) = \sum_{d|n} d = \sigma(n)$). But N is completely multiplicative, and so $N^{(-1)}$ is equal to μN. Therefore we have $\tau = \mu N \times \mu$, which is a function of type 2; now it is sufficient to calculate

$$\tau(p) = \mu(1)N(1)\mu(p) + \mu(p)N(p)\mu(1) = -(p+1),$$

$$\tau(p^2) = \mu(1)N(1)\mu(p^2) + \mu(p)N(p)\mu(p) + \mu(p^2)N(p^2)\mu(1) = p,$$

to obtain

$$\begin{cases} \tau(1) = 1 \\ \tau(p) = -(p+1) \\ \tau(p^2) = p \\ \tau(p^n) = 0 \quad \text{for } n > 2 \end{cases}$$

The problem of characterizing in some way the classes $\mathscr{M}_k$ with $k < -1$ remains of course still open.

Notes and references. Pellegrino [11] used the symbol $f^{\times^{-1}}$ for the inverse function of f with respect to $\times$ (and $f^{\times^n}$ for the convolution $f \times f \times \cdots \times f n$ times, see [12]), but many of the authors that studied this subject used the symbol f^{-1} for the inverse function of f. In this paper the symbol $f^{(-1)}$has been used instead of f^{-1}.

The fact that $\mathscr{I}$ is an UF-ring was proved in 1959 by E.D. Cashwell and C.J. Everett [3]. This proof is based on a homomorphism between $\mathscr{I}$ and a ring of polynomials in infinite variables.

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