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DIFFERENTIAL RECURRENCE FORMULAE FOR ORTHOGONAL POLYNOMIALS

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Part I – By combining a general 2nd-order linear homogeneous ordinary differential equation with the three-term recurrence relation possessed by all orthogonal polynomials, it is shown that sequences of orthogonal polynomials which satisfy a differential equation of the above mentioned type necessarily have a differentiation formula of the type:

$$g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x).$$

Part II – A recurrence formula of the form :

$$r_n(x)Y_n'(x) + s_n(x)Y_{n+1}'(x) + t_n(x)Y_{n-1}'(x) = 0,$$

is derived using the result of part I.

Introduction.

The aim of this paper is to deduce a differential recurrence formula (differentiation formula) of the form

$$(*) \qquad g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x)$$

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for a large class of orthogonal polynomial sequences. It should be noted that the formula $(*)$ is not a special case of the formula derived in section four (pp. 14–19) of [7]. In particular see example 2 on page 18 of [7]. As a matter of fact the above mentioned formula from [7] has been proved only in the hypergeometric case, whereas the differentiation formula derived in this paper is valid for all sequences of orthogonal polynomials which satisfy any 2nd-order linear homogeneous ordinary differential equation e.g., for the generalized Hermite polynomials (p. 157 of [4]) or the Heine polynomials (pp. 203–204 of [4]) neither of which are of the hypergeometric or Strum-Liouville type. Al-Salam and Chihara, [1], have shown that the classical orthogonal polynomials have a differentiation formula of the form $\pi(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x)$, where $\pi(x)$ is a fixed polynomial, which is at most of 2nd-degree. The differentiation formula in this paper includes the Al-Salam formula as a special case. It is well known that the classical orthogonal polynomials satisfy a differential equation of the form $a_2(x)Y''(x) + a_1(x)Y'(x) + a_0(x)Y(x) + \lambda_n Y(x) = 0$ (see S. Bochner's paper [3]). Where $a_2(x), a_1(x), a_0(x)$, are polynomials of degree at most 2,1, and 0, respectively. The only orthogonal polynomial solutions of this differential equation are the classical orthogonal polynomials. The differentiation formula given here is valid for sequences of orthogonal polynomials satisfying the general differential equation:

$$Y_n''(x) + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0,$$

which allows for an n dependency in the coefficient of the first derivative term i.e. $p_n(x)$. Furthermore, the coefficient of the $Y_n(x)$ term i.e. $q_n(x)$ is not restricted to being a constant but may depend on x.

Part I: The differentiation formula.

A well known necessary and sufficient condition for orthogonality of a sequence of polynomials is that they satisfy the three-term recurrence formula:

$$(x - B_n)Y_n(x) = A_n Y_{n+1}(x) + C_n Y_{n-1}(x)\,, \tag{1}$$

see [6]. Differentiating (1) one obtains:

$$(x - B_n)Y_n'(x) + Y_n(x) = A_n Y_{n+1}'(x) + C_n Y_{n-1}'(x). \tag{2}$$

Differentiating (2) results in:

$$(x - B_n)Y_n''(x) + 2Y_n'(x) = A_n Y_{n+1}''(x) + C_n Y_{n-1}''(x). \tag{3}$$

A general 2nd-order linear homogeneous differential equation, where the n subscript indicates that it may depend on the parameter n as well on x, is:

$$\alpha_n(x)Y_n''(x) + \beta_n(x)Y_n'(x) + \gamma_n(x)Y_n(x) = 0, \tag{4}$$

which may be rewritten as:

$$Y_n''(x) + \frac{\beta_n(x)}{\alpha_n(x)}\, Y_n'(x) + \frac{\gamma_n(x)}{\alpha_n(x)}\, Y_n(x) = 0, \tag{5}$$

assuming $\alpha_n(x)$ is not identically zero.

Let $\frac{\beta_n(x)}{\alpha_n(x)}$ and $\frac{\gamma_n(x)}{\alpha_n(x)}$ be written as $p_n(x)$ and $q_n(x)$ respectively:

$$Y_n''(x) + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0. \tag{6}$$

By using (1), (2), (3) and (6) one can obtain a necessary condition for orthogonal polynomials satisfying a 2nd-order linear homogeneous differential equation (6). Solve (3) for $(x - B_n)Y_n''(x)$ and substitute that result, as well as (1) and (2) into (6) multiplied by $(x - B_n)$ obtaining:

$$\begin{aligned} &-2Y_n'(x) + A_nY_{n+1}''(x) + C_nY_{n-1}''(x) + \\ &+ p_n(x)\Big[-Y_n(x) + A_nY_{n+1}'(x) + C_nY_{n-1}'(x)\Big] + \\ &+ q_n(x)\Big[A_nY_{n+1}(x) + C_nY_{n-1}(x)\Big] = 0. \end{aligned} \tag{7}$$

Letting $n \to n-1$ in (6) yields:

$$Y_{n-1}''(x) + p_{n-1}(x)Y_{n-1}'(x) + q_{n-1}(x)Y_{n-1}(x) = 0. \tag{8}$$

Letting $n \to n+1$ in (6) gives:

$$Y_{n+1}''(x) + p_{n+1}(x)Y_{n+1}'(x) + q_{n+1}(x)Y_{n+1}(x) = 0. \tag{9}$$

Solve (8) and (9) for $Y_{n-1}''(x)$ and $Y_{n+1}''(x)$ and substitute these into (7):

$$\begin{aligned} &-2Y_n'(x) - A_n\Big[p_{n+1}(x)Y_{n+1}'(x) + q_{n+1}(x)Y_{n+1}(x)\Big] - \\ &- C_n\Big[p_{n-1}(x)Y_{n-1}'(x) + q_{n-1}(x)Y_{n-1}(x)\Big] + \\ &+ p_n(x)\Big[-Y_n(x) + A_nY_{n+1}'(x) + C_nY_{n-1}'(x)\Big] + \\ &+ q_n(x)\Big[A_nY_{n+1}(x) + C_nY_{n-1}(x)\Big] = 0, \end{aligned} \tag{10}$$

rearranging:

$$\begin{aligned}(11)\qquad -2Y_n'(x) &= p_n(x)Y_n(x) + A_n\Big[q_{n+1}(x) - q_n(x)\Big]Y_{n+1}(x) + \\ &+ C_n\Big[q_{n-1}(x) - q_n(x)\Big]Y_{n-1}(x) + A_n\Big[p_{n+1}(x) - p_n(x)\Big]Y_{n+1}'(x) + \\ &+ C_n\Big[p_{n-1}(x) - p_n(x)\Big]Y_{n-1}'(x).\end{aligned}$$

From (1):

$$(12)\qquad A_nY_{n+1}(x) = (x - B_n)Y_n(x) - C_nY_{n-1}(x).$$

From (2):

$$(13)\qquad A_nY_{n+1}'(x) = (x - B_n)Y_n'(x) + Y_n(x) - C_nY_{n-1}'(x).$$

Substituting (12) and (13) into (11):

$$\begin{aligned}(14)\qquad -2Y_n'(x) &= p_n(x)Y_n(x) + \Big[q_{n+1}(x) - q_n(x)\Big]\cdot \\ &\cdot\Big[(x - B_n)Y_n(x) - C_nY_{n-1}(x)\Big] + C_n\Big[q_{n-1}(x) - q_n(x)\Big]Y_{n-1}(x) + \\ &+ \Big[p_{n+1}(x) - p_n(x)\Big]\Big[(x - B_n)Y_n'(x) + Y_n(x) - C_nY_{n-1}'(x)\Big] + \\ &+ C_n\Big[p_{n-1}(x) - p_n(x)\Big]Y_{n-1}'(x),\end{aligned}$$

or grouping:

$$\begin{aligned}(15)\qquad &\Big\{\Big[p_n(x) - p_{n+1}(x)\Big](x - B_n) - 2\Big\}\,Y_n'(x) = \\ &= \Big\{p_{n+1}(x) + \Big[q_{n+1}(x) - q_n(x)\Big](x - B_n)\Big\}\,Y_n(x) + \\ &+ C_n\Big[q_{n-1}(x) - q_{n+1}(x)\Big]Y_{n-1}(x) + C_n\Big[p_{n-1}(x) - p_{n+1}(x)\Big]Y_{n-1}'(x).\end{aligned}$$

From (1):

$$(16)\qquad Y_{n-1}(x) = \frac{1}{C_n}\Big[(x - B_n)Y_n(x) - A_nY_{n+1}(x)\Big].$$

From (2):

$$(17)\qquad Y_{n-1}'(x) = \frac{1}{C_n}\Big[(x - B_n)Y_n'(x) + Y_n(x) - A_nY_{n+1}'(x)\Big].$$

Substitute (16) and (17) into (15):

$$\begin{aligned}
&\Big\{\Big[p_n(x) - p_{n+1}(x)\Big](x - B_n) - 2\Big\}\, Y_n'(x) = \\
&= \Big\{p_{n+1}(x) + \Big[q_{n+1}(x) - q_n(x)\Big](x - B_n)\Big\}Y_n(x) + \\
&+ \Big[q_{n-1}(x) - q_{n+1}(x)\Big]\Big[(x - B_n)Y_n(x) - A_n Y_{n+1}(x)\Big] + \\
&+ \Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big[(x - B_n)Y_n'(x) + Y_n(x) - A_n Y_{n+1}'(x)\Big].
\end{aligned} \tag{18}$$

Letting $n \to n + 1$ in (15):

$$\begin{aligned}
&\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}Y_{n+1}'(x) = \\
&= \Big\{p_{n+2}(x) + \Big[q_{n+2}(x) - q_{n+1}(x)\Big](x - B_{n+1})\Big\}Y_{n+1}(x) + \\
&+ C_{n+1}\Big[q_n(x) - q_{n+2}(x)\Big]Y_n(x) + C_{n+1}\Big[p_n(x) - p_{n+2}(x)\Big]Y_n'(x).
\end{aligned} \tag{19}$$

Substituting (19) into (18) multiplied by $\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}$:

$$\begin{aligned}
&\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \\
&\cdot\Big\{\Big[p_n(x) - p_{n+1}(x)\Big](x - B_n) - 2\Big\}Y_n'(x) = \\
&= \Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \\
&\cdot\Big\{p_{n+1}(x) + \Big[q_{n+1}(x) - q_n(x)\Big](x - B_n)\Big\}Y_n(x) + \\
&+ \Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \\
&\cdot\Big[q_{n-1}(x) - q_{n+1}(x)\Big]\Big[(x - B_n)Y_n(x) - A_n Y_{n+1}(x)\Big] + \\
&+ \Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \\
&\cdot\Big[p_{n-1}(x) - p_{n+1}(x)\Big](x - B_n)Y_n'(x) + \\
&+ \Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \\
&\cdot\Big[p_{n-1}(x) - p_{n+1}(x)\Big]Y_n(x) - A_n\Big[p_{n-1}(x) - p_{n+1}(x)\Big]\cdot \\
&\cdot\Big\{\{p_{n+2}(x) + \Big[q_{n+2}(x) - q_{n+1}(x)\Big](x - B_{n+1})\}Y_{n+1}(x) + \\
&+ C_{n+1}\Big[q_n(x) - q_{n+2}(x)\Big]Y_n(x) + C_{n+1}\Big[p_n(x) - p_{n+2}(x)\Big]Y_n'(x)\Big\}.
\end{aligned} \tag{20}$$

Grouping coefficients of $Y_n'(x)$, $Y_n(x)$ and $Y_{n+1}(x)$, cancelling, and factoring gives:

$$\begin{aligned}
&\Big\{\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot\\
&\cdot\Big\{\Big[p_n(x) - p_{n-1}(x)\Big](x - B_n) - 2\Big\} +\\
&+ A_nC_{n+1}\Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big[p_n(x) - p_{n+2}(x)\Big]\Big\}Y_n'(x) =\\
&= \Big\{\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot\\
&\cdot\Big\{p_{n-1}(x) + \Big[q_{n-1}(x) - q_n(x)\Big](x - B_n)\Big\} - \qquad (21)\\
&- A_nC_{n+1}\Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big[q_n(x) - q_{n+2}(x)\Big]\Big\}Y_n(x) -\\
&- A_n\Big\{\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\Big[q_{n-1}(x) - q_{n+1}(x)\Big] +\\
&+ \Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big\{p_{n+2}(x) +\\
&+\Big[q_{n+2}(x) - q_{n+1}(x)\Big](x - B_{n+1})\Big\}\Big\}Y_{n+1}(x).
\end{aligned}$$

Using (1) to eliminate $Y_{n+1}(x)$ in (21) gives:

$$\begin{aligned}
&\Big\{\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot\\
&\cdot\Big\{\Big[p_n(x) - p_{n-1}(x)\Big](x - B_n) - 2\Big\} +\\
&+ A_nC_{n+1}\Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big[p_n(x) - p_{n+2}(x)\Big]\Big\}Y_n'(x) =\\
&= \Big\{\Big\{\Big[p_{n+1}(x) - p_{n+2}(x)\Big](x - B_{n+1}) - 2\Big\}\cdot \qquad (22)\\
&\cdot\Big\{p_{n-1}(x) + \Big[q_{n+1}(x) - q_n(x)\Big](x - B_n)\Big\} -\\
&- \Big[p_{n-1}(x) - p_{n+1}(x)\Big]\Big\{A_nC_{n+1}\Big[q_n(x) - q_{n+2}(x)\Big] +\\
&+ \{p_{n+2}(x) + \Big[q_{n+2}(x) - q_{n+1}(x)\Big](x - B_{n+1})\}(x - B_n)\Big\}\Big\}Y_n(x) +
\end{aligned}$$

$$+ C_n\left\{ \left\{\left[p_{n+1}(x) - p_{n+2}(x)\right](x - B_{n+1}) - 2\right\} \left[q_{n-1}(x) - q_{n+1}(x)\right] + \right.$$
$$+ \left[p_{n-1}(x) - p_{n+1}(x)\right]\left\{p_{n+2}(x) + \right.$$
$$\left. \left. + \left[q_{n+2}(x) - q_{n+1}(x)\right](x - B_{n+1})\right\}\right\} Y_{n-1}(x).$$

Note: For the classical orthogonal polynomials (21) and (22) reduce to, respectively:

(23)
$$\begin{aligned} -2Y_n'(x) = \left\{p(x) + \left[q_{n-1}(x) - q_n(x)\right](x - B_n)\right\} Y_n(x) + \\ + A_n\left[q_{n+1}(x) - q_{n-1}(x)\right] Y_{n+1}(x) \end{aligned}$$

and

(24)
$$\begin{aligned} -2Y_n'(x) = \left\{p(x) + \left[q_{n+1}(x) - q_n(x)\right](x - B_n)\right\} Y_n(x) + \\ + C_n\left[q_{n-1}(x) - q_{n+1}(x)\right] Y_{n-1}(x). \end{aligned}$$

Hence a necessary (but not sufficient) condition for orthogonal polynomials satisfying a 2nd-order linear homogeneous differential equation is that they have a recurrence formula of the form:

(25)
$$G_n(x)Y_n'(x) = F_n(x)Y_n(x) + H_n(x)Y_{n-1}(x)$$

or

(26)
$$J_n(x)Y_n'(x) = I_n(x)Y_n(x) + K_n(x)Y_{n+1}(x),$$

where the $F_n(x)$, $G_n(x)$, $H_n(x)$, $I_n(x)$, $J_n(x)$ and $K_n(x)$ are given above in (21) and (22).
Now consider a special case of (25) i.e., $H_n(x) = 0$ $(F_n(x), G_n(x) \neq 0)$.
Then from (25):

(27)
$$G_n(x)Y_n'(x) = F_n(x)Y_n(x).$$

This (27) is a 1sr-order linear differential equation but it is known that orthogonal polynomials satisfying linear homogeneous ordinary differential equations do not satisfy 1sr-order differential equations.
(See [5]). Therefore no sequences of orthogonal polynomials are possible in this

case, therefore $H_n(x) \neq 0$.
Since $H_n(x) \neq 0$ (25) may be written in a simpler form:

$$g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x), \tag{28}$$

where the $H_n(x)$ has been incorporated into the new $g_n(x)$ and $f_n(x)$. Equation (28) may also be written in the form:

$$h_n(x)Y_n'(x) = i_n(x)Y_n(x) + Y_{n+1}(x). \tag{29}$$

Note: $p_n(x)$ and $q_n(x)$ are not assumed to be polynomials or even rational functions. It remains to be shown what restriction must be placed on them so that sequences of orthogonal polynomial solutions are possible.

Part II: A recurrence formula for orthogonal polynomials of the form $r_n(x)Y_n'(x) + s_nY_{n+1}'(x) + t_nY_{n-1}'(x) = 0$.

$$(x - B_n)Y_n(x) = A_nY_{n+1}(x) + C_nY_{n-1}(x). \tag{30}$$

$$g_n(x)Y_n'(x) = f_n(x)Y_n(x) + Y_{n-1}(x). \tag{31}$$

$$Y_n''(x) + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0. \tag{32}$$

Differentiate (30) and (31):

$$(x - B_n)Y_n'(x) + Y_n(x) = A_nY_{n+1}'(x) + C_nY_{n-1}'(x), \tag{33}$$

$$g_n(x)Y_n''(x) + g_n'(x)Y_n'(x) = f_n'(x)Y_n(x) + f_n(x)Y_n'(x) + Y_{n-1}'(x). \tag{34}$$

Dividing (34) by $g_n(x)$ gives:

$$Y_n''(x) = -\frac{g_n'(x)Y_n'(x)}{g_n(x)} + \frac{f_n'(x)}{g_n(x)}Y_n(x) + \frac{f_n(x)}{g_n(x)}Y_n'(x) + \frac{1}{g_n(x)}Y_{n-1}'(x). \tag{35}$$

Substitute (35) into (32):

$$\begin{aligned} -\frac{g_n'(x)}{g_n(x)}Y_n'(x) + \frac{f_n'(x)}{g_n(x)}Y_n(x) + \frac{f_n(x)}{g_n(x)}Y_n'(x) + \frac{1}{g_n(x)}Y_{n-1}'(x) + \\ + p_n(x)Y_n'(x) + q_n(x)Y_n(x) = 0. \end{aligned} \tag{36}$$

Grouping gives:

$$
(37)\qquad \begin{aligned} &-\frac{g_n'(x)}{g_n(x)}\,Y_n'(x) + \left[\frac{f_n'(x)}{g_n(x)} + q_n(x)\right] Y_n(x) + \frac{f_n(x)}{g_n(x)}\,Y_n'(x) + \\ &+ \frac{1}{g_n(x)}\,Y_{n-1}'(x) + p_n(x)Y_n'(x) = 0. \end{aligned}
$$

Solving (33) for $Y_n(x)$ and substituting the result into (37):

$$
(38)\qquad \begin{aligned} &-\frac{g_n'(x)}{g_n(x)}\,Y_n'(x) + \left[\frac{f_n'(x)}{g_n(x)} + q_n(x)\right] [A_n Y_{n+1}'(x) + C_n Y_{n-1}'(x) - \\ &(x - B_n)Y_n'(x)] + \frac{f_n(x)}{g_n(x)}\,Y_n'(x) + \frac{1}{g_n(x)}\,Y_{n-1}'(x) + p_n(x)Y_n'(x) = 0. \end{aligned}
$$

Grouping gives:

$$
(39)\qquad \begin{aligned} &\left\{\frac{f_n(x)}{g_n(x)} - \frac{g_n'(x)}{g_n(x)} + p_n(x) - \left[\frac{f_n'(x)}{g_n(x)} + q_n(x)\right](x - B_n)\right\} Y_n'(x) + \\ &+ A_n \left[\frac{f_n'(x)}{g_n(x)} + q_n(x)\right] Y_{n+1}'(x) + \\ &+ \left\{C_n \left[\frac{f_n'(x)}{g_n(x)} + q_n(x)\right] + \frac{1}{g_n(x)}\right\} Y_{n-1}'(x) = 0. \end{aligned}
$$

Formula (39) is of the form:

$$
(40)\qquad r_n(x)Y_n'(x) + s_n(x)Y_{n+1}'(x) + t_n(x)Y_{n-1}'(x) = 0.
$$

Conclusion.

The differentiation formula derived in part I and the recurrence formula derived in part II are both necessary conditions for all orthogonal polynomials satisfying a 2nd-order linear homogeneous ordinary differential equation, whether it is of the hypergeometric or strum-Liouville type or not. Using the differentiation formula and the three-term pure recurrence formula it is possible to derive the general differential equation of [2].

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