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# REMARKS ON THE EXISTENCE OF COPIES OF $c_0$ AND $l_\infty$ IN THE SPACE $cabv(\lambda, E)$

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We present some results essentially showing that $cabv(\lambda, E)$ lives inside $cabv(\lambda, E)$ if and only if $l_\infty$ lives inside $cabv(\lambda, E)$. Some applications of this result to other questions (existence of complemented copies of $c_0$ and lifting of the Gelfand-Phillips property ) are given.

Let $S$ be an arbitrary set, $\Sigma$ be a $\sigma$-algebra of subsets of $S$ and $\lambda$ a finite or infinite measure, such that the space $(S, \Sigma, \lambda)$ is atomless. Let also $E$ denote a Banach space. Kwapien ([14]) and Mendoza ([15]) have, respectively, proved that $c_0$ and $l_\infty$ embed into the Bochner function space $L_1(\lambda, E)$ if and only if they embed into $E$ (for Köthe function spaces similar results were got in [3] and [11]).

A natural question rising from the quoted results is the following: *are similar results true for the space $cabv(\lambda, E)$, i.e. the Banach space of countably additive $\lambda$-continuous measures $\mu$ with bounded variation $\|\mu\|$ taking their values into $E$?*

In this short note we shall study the question considered above of the existence of copies of $c_0$ and $l_\infty$ in the larger space $cabv(\lambda, E)$. Some partial results to the question put are already known; for instance, a result due to E. Saab and P. Saab [17] can be used to show the following fact

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**Proposition 1.** *If $E = Y^*$ is a dual Banach space, thus $c_0$ (resp. $l_\infty$) embeds into $cabv(\lambda, E)$ if and only if it does into $E$.*

*Proof.* It is clear that just the "only if" part needs a proof. The Stone-Kakutani Theorem (see [2]) allows a reduction of our study to the case of $(S, \Sigma, \lambda)$ a regular Borel measure space on a compact space. It is well known that the dual space of the space $C(S, Y)$ of continuous $Y$-valued functions is the space of all regular Borel measures with bounded variation and values inside $E$; hence $cabv(\lambda, E)$ is a closed subspace of such a space (actually complemented thanks to the Lebesgue decomposition Theorem, [6]). Hence $c_0$ (resp. $l_\infty$) lives inside $(C(S, Y))^*$. It is now enough to apply the main result from [17] to get our thesis.

So that we need to consider just the case of a not dual space $E$. In such a case, an example due to Talagrand ([21]) shows that $cabv(\lambda, E)$ is so big that $c_0$ can embed into it even if it does not into $E$ (in passing we remark that Talagrand constructed a copy of $c_0$ inside the larger space of all countably additive vector measures with bounded variation, but, since the unit vector basis of $c_0$ is a weakly compact set, there exists a control measure $\lambda$ for it, [6]; hence that copy actually lives in some $cabv(\lambda, E)$). Precisely in that case we shall prove (Theorem 2) that even $l_\infty$ lives inside $cabv(\lambda, E)$. The proof of this result will also allow to perform our construction of a copy of $l_\infty$ in $cabv(\lambda, E)$ even when $c_0$ embeds into $E$ (Theorem 3), so that we have that *$c_0$ embeds into $cabv(\lambda, E)$ if and only if $l_\infty$ embeds into $cabv(\lambda, E)$*; we can thus affirm that no analogue of the Kwapien and Mendoza theorems holds true in general.

We also recall that in [10] it is shown that if $c_0$ embeds into $E$, thus it embeds complementably into $L_1(\lambda, E)$. What about the existence of complemented copies of $c_0$ inside $cabv(\lambda, E)$? The recalled results from [14] and [10] imply that $c_0$ embeds complementably into $L_1(\mu, E)$ if and only if it embeds into $E$, but because of the Talagrand example one could think that $c_0$ embeds also complementably into $cabv(\lambda, E)$, without necessarily embedding into $E$. An application of Theorem 2 will show that this is not the case; indeed, we get that the presence of $c_0$ inside $E$ is a necessary condition for the existence of complemented copies of $c_0$ in $cabv(\lambda, E)$; such a result cannot be reversed in general, as the proof of Proposition 1 proves. These facts show that the situation, in the complementability case, is different from the one in the case of the mere embeddability and not completely clear up to now, because we do not know the complete answer to the following question: *does $c_0$ embeds complementably into $cabv(\lambda, E)$ if and only if it embeds complementably into $E$?*

From our results it will also follow that the so called Gelfand-Phillips property (see below for the definition and [9], [12], [18] for examples, properties and results on it) does not necessarily lift from $E$ to $cabv(\lambda, E)$, contrarily to the case of the space $L_1(\mu, E)$ (see [4]).

## Results.

We start by giving one of the main theorems of the note about the construction of copies of $l_\infty$ inside $cabv(\lambda, E)$ starting from the existence of copies of $c_0$ inside $cabv(\lambda, E)$.

**Theorem 2.** *Let $E$ be a Banach space not containing copies of $c_0$ such that there is an isomorphic copy $X$ of $c_0$ inside $cabv(\lambda, E)$. Then there are a subspace $H$ of $cabv(\lambda, E)$ isomorphic to $c_0$ and a subspace $K$ of $cabv(\lambda, E)$ isomorphic to $l_\infty$ such that $X$ is a subspace of $H$ and $H$ is a subspace of $K$.*

*Proof.* Let $(\mu_n)$ be a basis for $X$ equivalent to the unit vector basis of $c_0$. For $h = (h_n) \in l_\infty$ and $A \in \Sigma$ we consider the series $\sum h_n \mu_n(A)$ that is unconditionally converging ; indeed, the series $\sum \mu_n(A)$ must be weakly unconditionally converging because image of the series $\sum \mu_n$ under the linear and bounded mapping $\mu \to \mu(A)$; since $c_0$ does not embed into $E$, we get that $\sum \mu_n(A)$ is unconditionally converging, a fact that it is well known to give our claim. We may define $\mu_h : \Sigma \to E$ by $\mu_h(A) = \sum h_n \mu_n(A)$ for all $A \in \Sigma$. Clearly $\mu_h$ is finitely additive. Furthermore, if $\sigma = (A_1, \dots A_p)$ is a finite partition of $S$ we get

$$\sum_\sigma \| \sum h_n \mu_n(A_i) \| \leq \sum_\sigma (\| \sum_{i=1}^{q} h_n \mu_n(A_i) \| + \| \sum_{n=q+1}^{\infty} h_n \mu_n(A_i) \|)$$

for all $q \in N$. Since $\sigma$ is finite and $\sum h_n \mu_n(A)$ is unconditionally converging for all $A \in \Sigma$, for each $\epsilon > 0$ one can find a $q_{\epsilon,\sigma} \in N$ such that

$$\sum_\sigma \| \sum_{n=q_{\epsilon,\sigma}+1}^{\infty} h_n \mu_n(A_i) \| < \epsilon$$

so that for all $\epsilon > 0$ we have

$$\sum_\sigma \| \sum h_n \mu_n(A_i) \| \leq \sum_\sigma \| \sum_{i=1}^{q_{\epsilon,\sigma}} h_n \mu_n(A_i) \| + \epsilon \leq$$

$$\leq \| \sum_{i=1}^{q_{\epsilon,\sigma}} h_n \mu_n \| + \epsilon \leq M \|h\|_\infty + \epsilon$$

where the constant $M$ exists and depends only upon $(\mu_n)$ since $\sum \mu_n$ is weakly unconditionally converging in $cabv(\lambda, E)$. Hence $\mu_h$ has bounded variation, for all $h = (h_n) \in l_\infty$. We have so defined a linear and bounded map $\psi : l_\infty \to abv(\Sigma, E)$ (by $abv(\Sigma, E)$ we shall denote the Banach space of finitely additive measures with bounded variation). Now we recall that there is a projection $P$ from $abv(\Sigma, E)$ onto $cabv(\lambda, E)$ (use a result from [22] and again the Lebesgue Decomposition Theorem, [6]), so that we have a linear and bounded map $P \circ \psi : l_\infty \to cabv(\lambda, E)$ such that $P \circ \psi(e_n) = \mu_n$ where $(e_n)$ denotes the unit vector basis of $c_0$. To get $H$ and $K$ as required we have now to apply a famous result due to Rosenthal ([16]). We are done.

So in the case of $E$ the Talagrand space, not only $c_0$, but also $l_\infty$ lives inside $cabv(\lambda, E)$; in passing we observe that if $Z$ is a reflexive subspace of the Talagrand space $X$ ([21]), thus $cabv(\lambda, X/Z)$ contains a copy of $c_0$ even if $X/Z$ does not; that such a $X/Z$ does not contain copies of $c_0$ has been proved in the paper [13]. That $cabv(\lambda, X/Z)$ has a copy of $c_0$ may be proved by contradiction as it follows: suppose that $cabv(\lambda, X/Z)$ does not have copy of $c_0$ inside; since $cabv(\lambda, X/Z)$ is isometrically isomorphic to $cabv(\lambda, X)/cabv(\lambda, Z)$ as proved in [13] and since $c_0$ does not live inside $cabv(\lambda, Z)$ since $Z$ is reflexive, we obtain that $c_0$ is not allowed to embed into $cabv(\lambda, X)$ (see [1]), a contradiction that finishes our proof.

**Remark 1.** It is clear from the proof of Theorem 2 that in order to define $\mu_h$ for all $h = (h_n) \in l_\infty$ we need only to know that $\sum h_n \mu_n(A)$ is unconditionally converging for all $A \in \Sigma$ and all $h = (h_n) \in l_\infty$.

This remark allows us to present the following result for which we do not give a proof, since it can be performed with the same techniques used in the proof of Theorem 2.2 from [5]

**Theorem 3.** *Let $c_0$ embed into $E$. Then there is a subspace $K$ of $cabv(\lambda, E)$ isomorphic to $l_\infty$.*

Theorem 2 and Theorem 3 actually show that $cabv(\lambda, E)$ is so "big" that the following equivalence always holds true

**Corollary 4.** *For any Banach space $E$ the following facts are equivalent:*

1) *$c_0$ embeds into $cabv(\lambda, E)$*
2) *$l_\infty$ embeds into $cabv(\lambda, E)$.*

The above results have some other interesting consequences; the first one is about the question of the existence of complemented copies of $c_0$ in $cabv(\lambda, E)$

**Corollary 5.** *Let $c_0$ embed complementably into $cabv(\lambda, E)$. Then $c_0$ embeds into $E$.*

*Proof.* Suppose that $c_0$ does not live into $E$ or, at least, that $\sum h_n \mu_n(A)$ is unconditionally converging for all $h = (h_n) \in l_\infty$ and all $A \in \Sigma$, according to Remark 1. Let us consider $H$ and $K$ as in Theorem 2. Since $H$ is complemented in $X$ ($X$ denotes the complemented copy of $c_0$ living in $cabv(\lambda, E)$), $H$ is complemented also in $cabv(\lambda, E)$; this gives that $H$ is complemented in $K$, which is well known to be false.

Proposition 1 and Corollary 5 suggest the following question: *does $c_0$ embed complementably into a (not dual Banach space $E$) if and only if it does the same into $cabv(\lambda, E)$?*

Sometimes the answer to the above question is positive; indeed, it enough to consider a Banach space with the so-called (BD) property (see [8]; we recall that among them one can find Banach spaces with the Gelfand-Phillips property and Banach spaces not containing copies of $l_1$), because it is known that $c_0$ embeds complementably into such a space as soon as it lives inside it ([18]), so the claim follows from Corollary 5; but we do not have a complete answer to the above question at the moment.

The next result is about the Gelfand-Phillips property in $cabv(\lambda, E)$ and it also makes use of the previous theorems; first we recall that a Banach space $E$ is said to possess the *Gelfand-Phillips property* (see [7]) if any limited subset of $X$(i.e a set $M$ such that for any weak*-null sequence $(x_n^*)$ in the dual space one has $\lim_n \sup_M |x_n^*(x)| = 0$) in $E$ is relatively compact. Many examples of spaces possessing the Gelfand-Phillips property are known in the literature (we refer to [7], [12] for lists of such spaces).

**Corollary 6.** *The Gelfand-Phillips property does not necessarily lift from the Banach space $E$ to $cabv(\lambda, E)$.*

*Proof.* Thanks to Corollary 4 in order to get our claim it is enough to find a Banach space $E$ possessing the Gelfand-Phillips property such that $c_0$ embeds into $cabv(\lambda, E)$, because in such a case even $l_\infty$ embeds into $cabv(\lambda, E)$. Since it is well known that the Gelfand-Phillips property is inherited by subspaces and $l_\infty$ does not possess it, we shall be done. So any Banach space with the Gelfand-Phillips property having a copy of $c_0$ inside, as well as the Talagrand space ([21]), works well to reach our target.

However, sometimes the Gelfand-Phillips property lifts from $E$ to $cabv$ $(\lambda, E)$; of course we necessarily have that $c_0$ does not embed into $cabv(\lambda, E)$ and hence into $E$. So if, for instance, $E$ has the Gelfand-Phillips property and the Radon-Nikodym property, then the Gelfand-Phillips property lifts from $E$ to $cabv(\lambda, E)$ (use results in [4]). Furthermore, let us suppose that $E$ is a Banach lattice not containing copies of $c_0$; thus $cabv(\lambda, E)$ is a Banach lattice ([19]). So if $E$ is a dual Banach lattice or if $E^{**}$ is weakly sequentially complete, then $cabv(\lambda, E)$ does not contain copies of $c_0$ ([20]) and so it has the Gelfand-Phillips property , as any Banach lattice not containing copies of $c_0$ does. Using similar reasonings,the quoted result from the paper [19], Proposition 1, Corollary 4 and Corollary 5 we can also state the following

**Proposition 7.** *If E is a Banach lattice , then the following facts are equivalent:*

1) $c_0$ *does not embed into* $cabv(\lambda, E)$
2) $l_\infty$ *does not embed into* $cabv(\lambda, E)$
3) $cabv(\lambda, E)$ *has the Gelfand-Phillips property .*

*If E is a dual Banach lattice , then* (1), (2) *and* (3) *above are equivalent to*

4) $c_0$ *does not embed into* $E$
5) $l_\infty$ *does not embed into* $E$
6) $c_0$ *does not embed complementably into* $cabv(\lambda, E)$
7) $E$ *has the Gelfand-Phillips property .*

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