ON POSITIVE DEFINITE FUNCTIONS AND REPRESENTATIONS OF CLIFFORD $\omega$-SEMIGROUPS

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It is known that a complex valued function $f$ on a Clifford $\omega$-semigroup, $T = \bigcup_n G_n$ is positive definite if and only if its restriction $f^n$ to $G_n$, is positive definite for any positive integer $n$. Then, by the usual Gelfand-Naimark-Segal construction, $f$ and $f^n (n \in \mathbb{N})$ give rise to the representations $\pi_f$ of $T$, respectively $\pi_{f^n}$ of $G_n$. In this note we study the relationship between the restriction of $\pi_f$ to $G_n$ and the representation of $\pi_{f^n} (n \in \mathbb{N})$.

Preliminaries.

A Clifford semigroup is an inverse semigroup $T$ which is a disjoint union of groups, or, equivalently, an inverse semigroup in which each idempotent is central. A Clifford $\omega$-semigroup is a Clifford semigroup $T$ in which the semilattice $E_T$ of idempotents is isomorphic to the semilattice $\omega = (\mathbb{N}, \vee)$, $m \vee n = \max(m, n)$. If $T$ is a Clifford $\omega$-semigroup, $T$ is a (disjoint) countable union of groups, $T = \bigcup_n G_n$ and $E_T = \{ e_n | n \in \mathbb{N} \}$, where $e_n$ is the identity element of the group $G_n$, $n \in \mathbb{N}$. The product in $T$ is described by a family of group homomorphisms. For $m \geq n$ there is a homomorphism $\Omega_{mn} : G_n \longrightarrow G_m$, and $\{ \Omega_{mn} \}_{m \geq n}$ is a coherent family, that is:

(i) $\Omega_{nn}$ is the identity mapping on $G_n$.

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(ii) $\Omega_{mn}\Omega_{nk} = \Omega_{mk}$, for $m \geq n \geq k$.

For $g_m \in G_m$, $g_n \in G_n$ we have $g_m g_n = (\Omega_{jm} g_m)(\Omega_{jn} g_n)$, where $j = m \lor n$ and the product on the right takes place in $G_j$. The homomorphism $\Omega_{mn}$ has an internal description in $T$ by $\Omega_{mn} g_n = e_m g_n = g_n e_m$. We note that $e_0$ is the identity element of $T$.

As usual, we shall denote by $\mathbb{C}(T)$ the $\ast$-algebra of all complex valued function with finite support on $T$. For $t \in T$, $\delta_t$ is the function which assumes the value $1$ at $t$ and is $0$ everywhere else on $T$. The involution and the product on $\mathbb{C}(T)$ are defined as follows:

$$a^* = \sum_{t \in T} \overline{a(t)} \delta_t, \quad ab = \sum_{t, s \in T} a(t) b(s) \delta_{ts}$$

(Where $a = \sum_{t \in T} a(t) \delta_t$, $b = \sum_{s \in S} b(s) \delta_s$).

**Definition 1.** A complex-valued function $f$ on $T$ is called hermitian if $f(s^\ast) = \overline{f(s)}$ holds, for all $s \in T$.

**Definition 2.** A hermitian function on $T$ is said to be positive definite if

$$\sum_{i, j=1}^{m} \alpha_i \overline{\alpha_j} f(s_i^\ast s_j) \geq 0$$

whenever $s_1, \ldots, s_m \in T$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ (see [5]).

**Proposition 1.** Any positive definite function on an $\omega$-Clifford semigroup (or only on a semigroup which is a disjoint of groups) is bounded.

**Proof.** Let $g$ be in $T = \bigcup_n G_n$. Then, $\exists n$ such that $g \in G_n$. With the definition of positive definite functions we have that $|f(g)|^2 \leq f(e_0) \cdot f(e_n)$. In particular, for $g = e_n$ it results that $|f(e_n)| \leq |f(e_0)|$, so

$$|f(g)|^2 \leq |f(e_0)| \cdot |f(e_n)| \leq |f(e_0)|^2.$$  

**Proposition 2.** Let $f : T \rightarrow \mathbb{C}$ be a positive definite function on the Clifford $\omega$-semigroup $T = \bigcup_n G_n$, the product of $T$ being described by the coherent family $\{\Omega_{mn}\}_{m \geq n}$. For $m \in \mathbb{N}$ let $f^m : G_m \rightarrow \mathbb{C}$ be the restriction of $f$ to $G_m$. Then $\forall n \in \mathbb{N}$ and $\forall m \geq n$, the function $g_{mn} : G_n \rightarrow \mathbb{C}, g_{mn} = f^m \circ \Omega_{mn}$ is positive definite on the group $G_n$.  

Proof. It is straightforward.

The next proposition can be obtained as a consequence of a result of A.L.T. Paterson ([5], Corollary 3.1). We shall give here an independent proof. We preserve the notations as above.

Proposition 3. The function \( f : T \rightarrow \mathbb{C} \) is a positive definite function on \( T \) if and only if for any \( n \in \mathbb{N} \), \( f^n : G_n \rightarrow \mathbb{C} \) is a positive definite function on \( G_n \).

Proof. It is obvious that if \( f \) is a positive definite function, then \( \forall n \in \mathbb{N}, f^n \) is a positive definite function.

For the converse, we shall start proceed by induction in order to prove that for any \( s_1, \ldots, s_p \in T \), there exists \( n(p) \in \mathbb{N} \) such that \( s_1, \ldots, s_p \in G_{n(p)}, (p \geq 1) \).

For \( p = 1 \), this is clear \( (s_1 \in T = \bigcup_n G_n \implies \exists n \in \mathbb{N} \text{ such that } s_1 \in G_n \text{ and we take } n(1) \text{ equal to one of these } n) \). Assume that \( p > 1 \) and that the assertion has already been proved for smaller values of \( p \). Let \( s_1, \ldots, s_{p+1} \) be in \( T \). By the induction hypothesis, \( \exists n(p) \in \mathbb{N} \) such that \( s_1, \ldots, s_p \in G_{n(p)} \) and \( s_{p+1} \in G_1 \).

Let us denote by \( s_{n(p)} \) the product \( s_1 \cdots s_p \). We have:

\[
s_{n(p)} \cdot s_{p+1} = (\Omega_{(n(p)\vee l)n(p)}(s_{n(p)})) \cdot (\Omega_{(n(p)\vee l)}(s_{p+1})) \in G_{n(p)\vee l}.
\]

If \( n(p) \geq l \), by taking into account that \( G_{n(p)} \) is group and \( s_{n(p)} \cdot s_{p+1} \in G_{n(p)} \), \( s_{n(p)} \in G_{n(p)} \) we can take \( n(p+1) = n(p) \).

If \( n(p) < l \), arguing as above, we conclude that \( s_{n(p)} \in G_1 \). Considering the inverse of \( s_2 \cdots s_p \), respectively \( s_3 \cdots s_p \) in \( G_{n(p)} \) \( (s_2, \ldots, s_p \in G_{n(p)} \text{ by the inductive hypothesis}) \) we have:

\[
s_{n(p)} \cdot (s_2 \cdots s_p)^{-1} = s_1 \cdot e_{n(p)} \in G_{n(p)\vee l} = G_l
\]

and respectively,

\[
s_{n(p)}(s_3 \cdots s_p)^{-1} = s_1 \cdot s_2 \cdot e_{n(p)} = (s_1 \cdot e_{n(p)}) \cdot s_2 \in G_1.
\]

It follows that \( s_2 \in G_l \). Similarly, we obtain \( s_3, \ldots, s_p \in G_l \) and after all \( s_1 \in G_1 \).

Therefore, we can take \( n(p+1) = l \).

Now, let \( s_1, \ldots, s_m \in T \) be and \( \alpha_1, \ldots, \alpha_m \in \mathbb{C} \). Then, as above, \( \exists n(m) \in \mathbb{N} \) such that \( s_1, \ldots, s_m \in G_{n(m)} \). Since \( f^{n(m)} : G_{n(m)} \rightarrow \mathbb{C} \) is positive definite on \( G_{n(m)} \) it follows that

\[
\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} f(s_i^* s_j) = \sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} f^{n(m)}(s_i^* s_j) \geq 0
\]
and the proposition is proved.

Finally, we mention that for any positive definite function \( f \) on \( T \), we shall denote also by \( f \) its prolongement by linearity to the \(*\)-algebra \( \mathbb{C}(T) \). It is clear that this function is a positive functional on \( \mathbb{C}(T)(f(a^*a) \geq 0, \forall a \in \mathbb{C}(T)) \).

Main results.

Let \( T \) be an inverse semigroup (in particular a Clifford \( \omega \)-semigroup). A representation of \( T \) on a Hilbert space \( \mathcal{H} \) is a star homomorphism of \( T \) into \( \mathcal{L}(\mathcal{H}) \), the algebra of all linear operators on the Hilbert space \( \mathcal{H} \). In [9] one proved that if \( \pi \) is a representation of \( T \), then \( \pi(T) \subset \mathcal{B}(\mathcal{H}) \), the subalgebra of \( \mathcal{L}(\mathcal{H}) \) of bounded linear operators on \( \mathcal{H} \). (In fact, for any \( t \in T \), \( \pi(t) \) is a partial isometry on \( \mathcal{H} \)). For an arbitrary representation \( \pi \) of \( T \) one may consider its natural extension (by linearity) to the algebra \( \mathbb{C}(T) \) (denoted also by \( \pi \)) which will be a representation of the involutive algebra \( \mathbb{C}(T) \) on \( \mathcal{H} \).

Let \( f : T \rightarrow \mathbb{C} \) be a positive definite function on \( T \). By the usual Gelfand-Naimark-Segal construction, \( f \) gives rise to a representation \( \pi_f \) on a Hilbert space \( \mathcal{H}_f \). More precisely, if \( f \) is non-degenerate (i.e. \( f(a^*a) > 0, \forall a \in \mathbb{C}(T), a \neq 0 \)) we can define on \( \mathbb{C}(T) \) the inner product:

\[
(a, b)_f = \sum_{t,s \in T} f(t^*s)a(t)b(s) \quad (a, b \in \mathbb{C}(T)).
\]

We remark that if \( f \) is degenerate, one may consider the quotient space \( \mathbb{C}(T)/\mathcal{N}_f \), \( \mathcal{N}_f = \{ a \in \mathbb{C}(T) | f(a^*a) = 0 \} \) and the inner product defined similarly on the equivalence classes. Everywhere here, for simplicity (of language) \( f \) will be non-degenerate. It follows that \( \mathbb{C}(T) \) equipped with the inner product \( \langle \cdot, \cdot \rangle_f := \langle \cdot, \cdot \rangle \) is a pre-Hilbert space. Its completion will be denoted \( \mathcal{H}_f \). The representation \( \pi_f \) of \( T \) on \( \mathcal{H}_f \) is defined by:

\[
\pi_f(t)(a) = \delta_t \cdot a, \quad \forall t \in T, a \in \mathbb{C}(T).
\]

Now, if \( T = \bigcup_n G_n \) is a Clifford \( \omega \)-semigroup, and \( \pi \) an arbitrary representation of \( T \) on a Hilbert space \( \mathcal{H} \), for any \( n \in \mathbb{N} \), let \( \pi_n \) be the map obtained as follows:

\[
\pi_n : G_n \rightarrow \mathcal{B}(\mathcal{H}_n), \quad \pi_n(g_n) := \pi(g_n),
\]

where \( \mathcal{H}_n \) is the closed subspace \([\pi(G_n)\mathcal{H}] \) spanned by \( \pi(g_n)a, g_n \in G_n, a \in \mathcal{H} \).

In the particular case when \( \pi = \pi_f \) we obtain for any \( n \in \mathbb{N} \) a representation \( \pi_{f,n} \) of \( G_n \) on the Hilbert space \( \mathcal{H}_{f,n} \) (the closed subspace \([\pi(G_n)\mathcal{H}_f] \) spanned
by \( \pi(g_n) a, g_n \in G_n, a \in \mathcal{H}_f \). On the other hand, by taking into account that for any \( n \in \mathbb{N} \), \( f^n \) is a non-degenerate positive definite function on the group \( G_n \) (Proposition 3), \( f^n \) gives rise to a representation \( \pi_{f^n} \) of \( G_n \) on the Hilbert space \( \mathcal{H}_{f^n} \).

In the next theorem we shall analyse the relation between \( \pi_{f^n} \) and \( \pi_{f^n}, (n \in \mathbb{N}) \).

**Theorem.** Let \( T = \bigcup_n G_n \) be a Clifford \( \omega \)-semigroup and \( f \) a positive (non-degenerate) function on \( T \). Then, for any \( n \in \mathbb{N} \), the following statements hold:

1. There exists a surjective continuous intertwining operator of the representations \( \pi_{f,n} \) and \( \pi_{f^n} \) \( (\exists \tilde{\beta} : \mathcal{H}_{f,n} \rightarrow \mathcal{H}_{f^n}, \) a linear continuous map onto \( \mathcal{H}_{f^n} \), such that \( \pi_{f^n}(g) \circ \tilde{\beta} = \tilde{\beta} \circ \pi_{f,n}(g), \forall g \in G_n \).)

2. The representation \( \pi_{f,n} \) is a direct sum of an essential representation equivalent to \( \pi_{f^n} \) and a zero representation.

**Proof.** (1) In order to define a linear continuous operator on \( \mathcal{H}_{f,n} \) to \( \mathcal{H}_{f^n} \), by taking into account the definitions of these spaces, it is enough to define it on the set \( \{ \delta_g^{*t_n} | g \in G_n, t \in T \} \). Thus, if \( g \in G_n \) and \( t \in T = \bigcup_n G_n \), it follows that there exists \( m \in \mathbb{N} \) such that \( t \in G_m \). Denoting \( t \) by \( t_m \), we put

\[
\beta(\delta_g^{*t_m}) = \begin{cases} 
\delta_g^{*\Omega_{nm}(t_m)} & \text{if } m \leq n \\
0 & \text{otherwise}
\end{cases}
\]

We extend this map by linearity to a map defined on the subspace \( [\pi(G_n)\mathbb{C}(T)] \), which is clearly a continuous operator (since \( \langle a, b \rangle_{f^n} = \langle a, b \rangle_f, \forall a, b \in \mathbb{C}(G_n) \subset \mathbb{C}(T) \)). It follows that we can construct its extension by continuity to \( \mathcal{H}_{f,n} \), denoted by \( \beta \). The algebra \( \mathbb{C}(T) \) being unitary, \( \beta \) will be onto \( \mathcal{H}_{f^n} \). We have only to verify that \( \beta \) is an intertwining operator of the representations \( \pi_{f,n} \) and \( \pi_{f^n} \). As above, it is enough to see that for any \( g, h \in G_n \) and \( t := t_m \in T, t \in G_m \) we have:

\[
(\beta \circ \pi_{f,n}(g))(\delta_h^{*t_m}) = (\pi_{f^n}(g) \circ \beta)(\delta_h^{*t_m}).
\]

Indeed, by a short computation, we obtain:

\[
(\beta \circ \pi_{f,n}(g))(\delta_h^{*t_m}) = \beta(\delta_g^{*h^{*t_m}}) = \\
= \begin{cases} 
\delta_g^{*h^{*\Omega_{nm}(t_m)}} & \text{if } m \leq n \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
(\pi_{f^n}(g) \circ \beta)(\delta_h^{*t_m}) = \pi_{f^n}(\beta(\delta_h^{*t_m})).
\]
\[
= \delta_{g^*\tilde{\beta} (\delta_{h^*l_m})} = \begin{cases} 
\delta_{g^*h^*\Omega_{nm}(l_m)} & \text{if } m \leq n \\
0 & \text{otherwise}
\end{cases}
\]

(2) It is clear that the closed subspace of \(\mathcal{H}_{f,n}\), \(\text{Ker} \tilde{\beta}\) (and also its orthogonal complement, \((\text{Ker} \tilde{\beta})^\perp\)) is invariant under \(\pi_{f,n}(G_n)\). Consequently,

\[
\pi_{f,n} = \rho_{f,n} \oplus \alpha_{f,n}
\]

where \(\rho_{f,n}\) is an essential representation of \(\mathcal{K}_{f,n} := (\text{Ker} \tilde{\beta})^\perp\) and \(\alpha_{f,n}\) is a representation of \(\mathcal{L}_{f,n} := \text{Ker} \tilde{\beta}\). For \(\delta_{h^*l_m}\) in \(\mathcal{L}_{f,n}\) and \(g \in G_n\), we have:

\[
\alpha_{f,n}(g)(\delta_{h^*l_m}) = \pi_{f,n}(g)(\delta_{h^*l_m}) = \delta_{g^*h^*l_m}.
\]

With the definition of \(\tilde{\beta}\) it follows that \(\alpha_{f,n}\) is the zero representation of \(\mathcal{L}_{f,n}\). We also have that \(\rho_{f,n} \cong \pi_{f,n}\), \(\tilde{\beta} : \mathcal{K}_{f,n} \longrightarrow \mathcal{H}_{f,n}\) being a bijective (continuous) intertwining operator.

**Application.**

It is known that if \(T\) is an inverse semigroup with identity element \(e\), every \((\ast\)-) representation of \(\mathbb{C}(T)\) is a direct sum of an essential representation and a zero representation, and every essential representation is a direct sum of cyclic representations, each of which being unitarily equivalent to \(\pi_f\) for some state \(f\). With the previous theorem, we obtain that the restriction of every \((\ast\)-) representation of \(\mathbb{C}(T)\) to \(\mathbb{C}(G_n)\) (in particular of the universal representation) is a direct sum of cyclic representations of \(\mathbb{C}(G_n)\) and a zero representation.

**REFERENCES**


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