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# ON POSITIVE DEFINITE FUNCTIONS AND REPRESENTATIONS OF CLIFFORD $\omega$-SEMIGROUPS

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It is known that a complex valued function $f$ on a Clifford $\omega$-semigroup, $T = \cup_n G_n$ is positive definite if and only if its restriction $f^n$ to $G_n$, is positive definite for any positive integer $n$. Then, by the usual Gelfand-Naimark-Segal construction, $f$ and $f^n (n \in \mathbb{N})$ give rise to the representations $\pi_f$ of $T$, respectively $\pi_{f^n}$ of $G_n$. In this note we study the relationship between the restriction of $\pi_f$ to $G_n$ and the representation of $\pi_{f^n} (n \in \mathbb{N})$.

## Preliminaries.

A Clifford semigroup is an inverse semigroup $T$ which is a disjoint union of groups, or, equivalently, an inverse semigroup in which each idempotent is central. A Clifford $\omega$-semigroup is a Clifford semigroup $T$ in which the semilattice $E_T$ of idempotents is isomorphic to the semilattice $\omega = (\mathbb{N}, \vee)$, $m \vee n = \max(m, n)$. If $T$ is a Clifford $\omega$-semigroup, $T$ is a (disjoint) countable union of groups, $T = \bigcup_n G_n$ and $E_T = \{e_n | n \in \mathbb{N}\}$, where $e_n$ is the identity element of the group $G_n$, $n \in \mathbb{N}$. The product in $T$ is described by a family of group homomorphisms. For $m \geq n$ there is a homomorphism $\Omega_{mn} : G_n \longrightarrow G_m$, and $\{\Omega_{mn}\}_{m \geq n}$ is a coherent family, that is:

(i) $\Omega_{nn}$ is the identity mapping on $G_n$;

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(ii) $\Omega_{mn}\Omega_{nk} = \Omega_{mk}$, for $m \geq n \geq k$.

For $g_m \in G_m$, $g_n \in G_n$ we have $g_m g_n = (\Omega_{jm} g_m)(\Omega_{jn} g_n)$, where $j = m \vee n$ and the product on the right takes place in $G_j$. The homomorphism $\Omega_{mn}$ has an internal description in $T$ by $\Omega_{mn} g_n = e_m g_n = g_n e_m$. We note that $e_0$ is the identity element of $T$.
As usual, we shall denote by $\mathbb{C}(T)$ the *-algebra of all complex valued function with finite support on $T$. For $t \in T$, $\delta_t$ is the function which assumes the value 1 at $t$ and is 0 everywhere else on $T$. The involution and the product on $\mathbb{C}(T)$ are defined as follows:

$$a^* = \sum_{t \in T} \overline{a(t)} \delta_{t^*}, \quad ab = \sum_{t,s \in T} a(t)b(s)\delta_{ts}$$

(where $a = \sum_{t \in T} a(t)\delta_t$, $b = \sum_{s \in S} b(s)\delta_s$).

**Definition 1.** *A complex-valued function $f$ on $T$ is called hermitian if $f(s^*) = \overline{f(s)}$ holds, for all $s \in T$.*

**Definition 2.** *A hermitian function on $T$ is said to be positive definite if*

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} f(s_i^* s_j) \geq 0$$

*whenever $s_1, \ldots, s_m \in T$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ (see [5]).*

**Proposition 1.** *Any positive definite function on an $\omega$-Clifford semigroup (or only on a semigroup which is a disjoint of groups) is bounded.*

*Proof.* Let $g$ be in $T = \bigcup_n G_n$. Then, $\exists n$ such that $g \in G_n$. With the definition of positive definite functions we have that $|f(g)|^2 \leq f(e_0) \cdot f(e_n)$. In particular, for $g = e_n$ it results that $|f(e_n) \leq |f(e_0)|$, so

$$|f(g)|^2 \leq |f(e_0)| \cdot |f(e_n)| \leq |f(e_0)|^2.$$

**Proposition 2.** *Let $f : T \longrightarrow \mathbb{C}$ be a positive definite function on the Clifford $\omega$-semigroup $T = \bigcup_n G_n$, the product of $T$ being described by the coherent family $\{\Omega_{mn}\}_{m \geq n}$. For $m \in \mathbb{N}$ let $f^m : G_m \longrightarrow \mathbb{C}$ be the restriction of $f$ to $G_m$. Then $\forall n \in \mathbb{N}$ and $\forall m \geq n$, the function $g_{mn} : G_n \longrightarrow \mathbb{C}$, $g_{mn} = f^m \circ \Omega_{mn}$ is positive definite on the group $G_n$.*

*Proof.* It is straightforward.

The next proposition can be obtained as a consequence of a result of A.L.T. Paterson ([5], Corollary 3.1). We shall give here an independent proof. We preserve the notations as above.

**Proposition 3.** *The function $f : T \longrightarrow \mathbb{C}$ is a positive definite function on $T$ if and only if for any $n \in \mathbb{N}$, $f^n : G_n \longrightarrow \mathbb{C}$ is a positive definite function on $G_n$.*

*Proof.* It is obvious that if $f$ is a positive definite function, then $\forall n \in \mathbb{N}$, $f^n$ is a positive definite function.
For the converse, we shall first proceed by induction in order to prove that for any $s_1, \ldots, s_p \in T$, there exists $n(p) \in \mathbb{N}$ such that $s_1, \ldots, s_p \in G_{n(p)}$, $(p \geq 1)$.
For $p = 1$, this is clear ($s_1 \in T = \bigcup_n G_n \Longrightarrow \exists n \in \mathbb{N}$ such that $s_1 \in G_n$ and we take $n(1)$ equal to one of these $n$). Assume that $p > 1$ and that the assertion has already been proved for smaller values of $p$. Let $s_1, \ldots, s_{p+1}$ be in $T$. By the induction hypothesis, $\exists n(p) \in \mathbb{N}$ such that $s_1, \ldots, s_p \in G_{n(p)}$ and $s_{p+1} \in G_l$.
Let us denote by $s_{n(p)}$ the product $s_1 \cdots s_p$. We have:

$$s_{n(p)} \cdot s_{p+1} = (\Omega_{(n(p)\vee l)n(p)}(s_{n(p)})) \cdot (\Omega_{(n(p)\vee l)l}(s_{p+1})) \in G_{n(p)\vee l}\,.$$

If $n(p) \geq l$, by taking into account that $G_{n(p)}$ is group and $s_{n(p)} \cdot s_{p+1} \in G_{n(p)}$, $s_{n(p)} \in G_{n(p)}$ we can take $n(p+1) = n(p)$.
If $n(p) \leq l$, arguing as above, we conclude that $s_{n(p)} \in G_l$. Considering the inverse of $s_2 \cdots s_p$, respectively $s_3 \cdots s_p$ in $G_{n(p)}$ ($s_2, \ldots, s_p \in G_{n(p)}$ by the inductive hypothesis) we have:

$$s_{n(p)} \cdot (s_2 \cdots s_p)^{-1} = s_1 \cdot e_{n(p)} \in G_{n(p)\vee l} = G_l$$

and respectively,

$$s_{n(p)}(s_3 \cdots s_p)^{-1} = s_1 \cdot s_2 \cdot e_{n(p)} = (s_1 \cdot e_{n(p)}) \cdot s_2 \in G_l.$$

It follows that $s_2 \in G_l$. Similarly, we obtain $s_3, \ldots, s_p \in G_l$ and after all $s_1 \in G_l$. Therefore, we can take $n(p+1) = l$.

Now, let $s_1, \ldots, s_m \in T$ be and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Then, as above, $\exists n(m) \in \mathbb{N}$ such that $s_1, \ldots, s_m \in G_{n(m)}$. Since $f^{n(m)} : G_{n(m)} \longrightarrow \mathbb{C}$ is positive definite on $G_{n(m)}$ it follows that

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} f(s_i^* s_j) = \sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} f^{n(m)}(s_i^* s_j) \geq 0$$

and the proposition is proved.

Finally, we mention that for any positive definite function $f$ on $T$, we shall denote also by $f$ its prolongement by linearity to the $^*$-algebra $\mathbb{C}(T)$. It is clear that this function is a positive functional on $\mathbb{C}(T)(f(a^*a) \geq 0, \forall a \in \mathbb{C}(T))$.

## Main results.

Let $T$ be an inverse semigroup (in particular a Clifford $\omega$-semigroup). A representation of $T$ on a Hilbert space $\mathcal{H}$ is a star homomorphism of $T$ into $\mathcal{L}(\mathcal{H})$, the algebra of all linear operators an the Hilbert space $\mathcal{H}$.. In [9] one proved that if $\pi$ is a representation of $T$, then $\pi(T) \subset \mathcal{B}(\mathcal{H})$, the subalgebra of $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. (In fact, for any $t \in T$, $\pi(t)$ is a partial isometry on $\mathcal{H}$). For an arbitrary representation $\pi$ of $T$ one may consider its natural extension (by linearity) to the algebra $\mathbb{C}(T)$ (denoted also by $\pi$) which will be a representation of the involutive algebra $\mathbb{C}(T)$ on $\mathcal{H}$.
Let $f : T \longrightarrow \mathbb{C}$ be a positive definite function on $T$. By the usual Gelfand-Naimark-Segal construction, $f$ gives rise to a representation $\pi_f$ on a Hilbert space $\mathcal{H}_f$. More precisely, if $f$ is non-degenerate (i.e. $f(a^*a) > 0, \forall a \in \mathbb{C}(T)$, $a \neq 0$) we can define on $\mathbb{C}(T)$ the inner product:

$$\langle a, b\rangle_f = \sum_{t,s\in T} f(t^*s)a(t)\overline{b(s)} \quad (a, b \in \mathbb{C}(T)).$$

We remark that if $f$ is degenerate, one may consider the quotient space $\mathbb{C}(T)/\mathcal{N}_f$, $\mathcal{N}_f = \{a \in \mathbb{C}(T) | f(a^*a) = 0\}$ and the inner product defined similarly on the equivalence classes. Everywhere here, for simplicity (of language) $f$ will be non-degenerate. It follows that $\mathbb{C}(T)$ equipped with the inner product $\langle\cdot,\cdot\rangle_f := \langle\cdot,\cdot\rangle$ is a pre-Hilbert space. Its completion will be denoted $\mathcal{H}_f$. The representation $\pi_f$ of $T$ on $\mathcal{H}_f$ is defined by:

$$\pi_f(t)(a) = \delta_{t^*}a, \quad \forall t \in T, a \in \mathbb{C}(T).$$

Now, if $T = \bigcup_n G_n$ is a Clifford $\omega$-semigroup, and $\pi$ an arbitrary representation of $T$ on a Hilbert space $\mathcal{H}$, for any $n \in \mathbb{N}$, let $\pi_n$ be the map obtained as follows:

$$\pi_n : G_n \longrightarrow \mathcal{B}(\mathcal{H}_n), \quad \pi_n(g_n) := \pi(g_n),$$

where $\mathcal{H}_n$ is the closed subspace $[\pi(G_n)\mathcal{H}]$ spanned by $\pi(g_n)a$, $g_n \in G_n$, $a \in \mathcal{H}$. In the particular case when $\pi = \pi_f$ we obtain for any $n \in \mathbb{N}$ a representation $\pi_{f,n}$ of $G_n$ on the Hilbert space $\mathcal{H}_{f,n}$ (the closed subspace $[\pi(G_n)\mathcal{H}_f]$ spanned

by $\pi(g_n)a$, $g_n \in G_n$, $a \in \mathcal{H}_f$). On the other hand, by taking into account that for any $n \in \mathbb{N}$, $f^n$ is a non-degenerate positive definite function on the group $G_n$ (Proposition 3), $f^n$ gives rise to a representation $\pi_{f^n}$ of $G_n$ on the Hilbert space $\mathcal{H}_{f^n}$.
In the next theorem we shall analyse the relation between $\pi_{f,n}$ and $\pi_{f^n}$, $(n \in \mathbb{N})$.

**Theorem.** *Let $T = \bigcup_n G_n$ be a Clifford $\omega$-semigroup and $f$ a positive (non-degenerate) function on $T$. Then, for any $n \in \mathbb{N}$, the following statements hold:*

(1) *There exists a surjective continuous intertwining operator of the representations $\pi_{f,n}$ and $\pi_{f^n}$ ($\exists \widetilde{\beta} : \mathcal{H}_{f,n} \longrightarrow \mathcal{H}_{f^n}$, a linear continuous map onto $\mathcal{H}_{f^n}$ such that $\pi_{f^n}(g) \circ \widetilde{\beta} = \widetilde{\beta} \circ \pi_{f,n}(g)$, $\forall g \in G_n$).*

(2) *The representation $\pi_{f,n}$ is a direct sum of an essential representation equivalent to $\pi_{f^n}$ and a zero representation.*

*Proof.* (1) In order to define a linear continuous operator on $\mathcal{H}_{f,n}$ to $\mathcal{H}_{f^n}$, by taking into account the definitions of these spaces, it is enough to define it on the set $\{\delta_{g^*t} \mid g \in G_n, t \in T\}$. Thus, if $g \in G_n$ and $t \in T = \bigcup_n G_n$, it follows that there exists $m \in \mathbb{N}$ such that $t \in G_m$. Denoting $t$ by $t_m$, we put

$$\beta(\delta_{g^*t_m}) = \begin{cases} \delta_{g^*\Omega_{nm}(t_m)} & \text{if} \quad m \leq n \\ 0 & \text{otherwise} \end{cases}$$

We extend this map by linearity to a map defined on the subspace $[\pi(G_n)\mathbb{C}(T)]$, which is clearly a continuous operator (since $\langle a, b\rangle_{f^n} = \langle a, b\rangle_f$, $\forall a, b \in \mathbb{C}(G_n) \subset \mathbb{C}(T)$). It follows that we can construct its extension by continuity to $\mathcal{H}_{f,n}$, denoted by $\widetilde{\beta}$. The algebra $\mathbb{C}(T)$ being unitary, $\widetilde{\beta}$ will be onto $\mathcal{H}_{f^n}$. We have only to verify that $\widetilde{\beta}$ is an intertwining operator of the representations $\pi_{f,n}$ and $\pi_{f^n}$. As above, it is enough to see that for any $g, h \in G_n$ and $t := t_m \in T$, $t \in G_m$ we have:

$$(\widetilde{\beta} \circ \pi_{f,n}(g))(\delta_{h^*t_m}) = (\pi_{f^n}(g) \circ \widetilde{\beta})(\delta_{h^*t_m}).$$

Indeed, by a short computation, we obtain:

$$(\widetilde{\beta} \circ \pi_{f,n}(g))(\delta_{h^*t_m}) = \widetilde{\beta}(\delta_{g^*h^*t_m}) =$$

$$= \begin{cases} \delta_{g^*h^*\Omega_{nm}(t_m)} & \text{if} \quad m \leq n \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\pi_{f^n}(g) \circ \widetilde{\beta})(\delta_{h^*t_m}) = \pi_{f^n}(\widetilde{\beta}(\delta_{h^*t_m})) =$$

$$= \delta_{g^*}\widetilde{\beta}(\delta_{h^*t_m}) = \begin{cases} \delta_{g^*h^*\Omega_{nm}(t_m)} & \text{if} \quad m \leq n \\ 0 & \text{otherwise} \end{cases}$$

(2) It is clear that the closed subspace of $\mathcal{H}_{f,n}$, Ker $\widetilde{\beta}$ (and also its orthogonal complement, $(\operatorname{Ker}\widetilde{\beta})^{\perp}$) is invariant under $\pi_{f,n}(G_n)$. Consequently,

$$\pi_{f,n} = \rho_{f,n} \oplus \alpha_{f,n}$$

where $\rho_{f,n}$ is an essential representation of $\mathcal{K}_{f,n} := (\operatorname{Ker}\widetilde{\beta})^{\perp}$ and $\alpha_{f,n}$ is a representation of $\mathcal{L}_{f,n} := \operatorname{Ker}\widetilde{\beta}$. For $\delta_{h^*t_m}$ in $\mathcal{L}_{f,n}$ and $g \in G_n$, we have:

$$\alpha_{f,n}(g)(\delta_{h^*t_m}) = \pi_{f,n}(g)(\delta_{h^*t_m}) = \delta_{g^*h^*t_m} \, .$$

With the definition of $\widetilde{\beta}$ it follows that $\alpha_{f,n}$ is the zero representation of $\mathcal{L}_{f,n}$. We also have that $\rho_{f,n} \cong \pi_{f,n}$, $\widetilde{\beta} : \mathcal{K}_{f,n} \longrightarrow \mathcal{H}_{f,n}$ being a bijective (continuous) intertwining operator.

**Application.**

It is known that if $T$ is an inverse semigroup with identity element $e$, every (*-) representation of $\mathbb{C}(T)$ is a direct sum of an essential representation and a zero representation, and every essential representation is a direct sum of cyclic representations, each of which being unitarily equivalent to $\pi_f$ for some state $f$. With the previous theorem, we obtain that the restriction of every (*-) representation of $\mathbb{C}(T)$ to $\mathbb{C}(G_n)$ (in particular of the universal representation) is a direct sum of cyclic representations of $\mathbb{C}(G_n)$ and a zero representation.

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