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ON THE EXISTENCE OF MILD SOLUTIONS OF SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS.

A. G. IBRAHIM - A. M. SOLIMAN

In this paper the existence of local and global mild solution of a semilinear functional differential inclusion in the case when the kernel is not necessarily compact is proved. Also, some topological properties of the solution set are obtained.

1. Introduction

The existence of mild solution for a similinear differential inclusion and functional differential inclusion in a Banach space has been studied by many authors, see for example [1], [3], [5-13], [16], [17] and [19-23]. Let E be a separable Banach space, $r > 0$, $I = [0,T]$, $C(I,E)$ be the Banach space of all continuous functions from I to E with the norm of uniform convergence and $C_0 = C([-r,0],E)$. Let $\{A(t) : t \in I\}$ be a family of densely defined, linear operator (not necessarily bounded or closed) on E, which generates an evolution operator $K : \Delta = \{(t,s) : I \times I : 0 \leq s \leq t \leq T\} \to \mathscr{L}(E)$ (the space of bounded linear operators from E into itself). Let F be a multifunction defined from $I \times C_0$ with nonempty compact and convex values in E and for any $t \in I$, $\tau(t)$ be the mapping from $C([-r,T],E)$ to $C_0 = C([-r,0],E)$ defined by $\tau(t)u(s) = u(s+t)$, $\forall s \in [-r,0]$ and $u \in C([-r,T],E)$.

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Consider the following functional differential inclusion

$$(P)\begin{cases} \dot{u}(t) \in A(t)u(t) + F(t,\tau(t)u) & a.e. \text{ on } I. \\ u(t) = \psi(t) \quad t \in [-r,0], \end{cases}$$

where $\psi \in C_0$.

In [12] Ibrahim proved that, when $\{A(t) : t \in I\}$ is a family of densely defined, closed linear operator on E, the set of mild solutions $S(\psi)$ of (P) is nonempty. In this paper, we consider the case when $\{A(t) : t \in I\}$ is a family of densely defined, linear operator (not necessarily bounded or closed) on E. We do not suppose that the evolution operator K is compact and instead of it we assume that the oriented field F satisfies a compactness type condition. First, we prove a local existence theorem. Then, under a stronger condition on F we prove a global existence theorem. Finally, we give some topological properties of $S(\psi)$. The results obtained in this paper generalizes many results in the literature, see for example [4], [6], [12], [19] and [24].

2. Notations and some auxiliary facts

We will use the following definitions and notations, which can be found in [15] and [24-27]:

- E is a separable Banach space, E^* is the topological dual of E.
- $\mathscr{P}(E)$ is the set of all nonempty subsets of E.
- $\mathscr{P}_C(E)$ is the set of all nonempty and closed subsets of E.
- $\mathscr{P}_K(E)$ is the set of all nonempty and compact subsets of E.
- $\mathscr{P}_{CK}(E)$ is the set of all nonempty, compact and convex subsets of E.
- $r > 0, T > 0$ and $I = [0,T]$.
- $L^1(I,E)$ is the Banach space of Lebesgue-Bochner integrable functions $f : I \to E$ endowed with the usual norm and $\mathscr{L}(E)$ is the Banach space of bounded linear operators from E into itself.
- $C(I,E)$ is the Banach space of all continuous functions from I to E with the norm of uniform convergence, $C_0 = C([-r,0],E)$ and $\psi \in C_0$.
- For any $t \in I$, we denote $\tau(t)$ the mapping $C([-r,t],E) \to C([-r,0],E) = C_0$ defined by $\tau(t)u(s) = u(s+t)$, $\forall s \in [-r,0]$ and $u \in C([-r,T],E)$.

- A multifunction $G: E \to \mathscr{P}(E)$ with closed values is upper semicontinuous (u.s.c.) if and only if $G^{-}(Z) = \{x \in E : G(x) \cap Z \neq \phi\}$ is closed whenever $Z \subseteq E$ is closed.

- A multifunction $G: E \to \mathscr{P}(E)$ with closed values is lower semicontinuous (l.s.c.) if and only if $G^{-}(Z) = \{x \in E : G(x) \cap Z \neq \phi\}$ is open whenever $Z \subseteq E$ is open.

- $(S, \mathscr{A}, \mu)$ is a σ-finite measure space. A multifunction $F: S \to \mathscr{P}_C(E)$ is said to be *measurable* if, $F^{-}U = \{s \in S : F(s) \cap U \neq \phi\} \in \mathscr{A}$ for any open subset U of E.

- A multifunction $F: S \to \mathscr{P}_C(E)$ is called *integrably bounded* if there exists an integrable non negative function $g: S \to [0, \infty[$ such that for $a.e.$ $s \in S, \|F(s)\| \leq g(s)$, where
$$\|F(s)\| = \sup\{\|x\| : x \in F(s)\}.$$

- Given a multifunction $F: S \to \mathscr{P}(E)$, we denote $S_F^p := \{f \in L^p(E) : f(s) \in F(s)\ a.e.\}, 1 \leq p \leq \infty$. This set may be empty. It is nonempty if F is measurable and integrably bounded.

- $(\mathscr{B}, \geq)$ is a partially ordered set. A function $\chi : \mathscr{P}(E) \to \mathscr{B}$ is called a *measure of noncompactness* (MNC) in E if
$$\chi(\overline{\text{co}}\Omega) = \chi(\Omega)$$
for every $\Omega \in \mathscr{P}(E)$.

A MNC is called *monotone* if $\Omega_0, \Omega_1 \in \mathscr{P}(E), \Omega_0 \subset \Omega_1$ implies $\chi(\Omega_0) \leq \chi(\Omega_1)$, *nonsingular* if $\chi(\{a\} \cup \Omega) = \chi(\Omega)$ for every $a \in E, \Omega \in \mathscr{P}(E)$ and is called *real* if $\mathscr{B} = [0, \infty]$ with the natural ordering and $\chi(\Omega) < +\infty$ for every bounded Ω.

If $\mathscr{B}$ is a cone in a Banach space, MNC is called *regular* if $\chi(\Omega) = 0$ is equivalent to relative compactness of Ω.

- Let α be the *Hausdorff MNC* on E, which defined by
$$\alpha(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}, \Omega \in \mathscr{P}(E).$$
It is known that the Hausdorff MNC satisfies all the properties above.

- Let W be a closed subset of a Banach space $E, \chi : \mathscr{P}(E) \to \mathscr{B}$ be a MNC on E. A multifunction $F: W \to \mathscr{P}_K(E)$ is said to be χ*-condensing* if for every $\Omega \subset W$, we have
$$\chi(F(\Omega)) \geq \chi(\Omega) \Rightarrow \Omega \text{ is relatively compact}.$$

- A countable set $\{f_n : n \geq 1\} \subset L^1(I,E)$ is said to be *semicompact* if:

(i) the set $\{f_n : n \geq 1\}$ is integrably bounded i.e. there exists $g \in L^1(I,E)$ such that for every $n \geq 1$

$$\|f_n(t)\| \leq g(t) \; a.e. \; t \in I;$$

(ii) the set $\{f_n(t) : n \geq 1\}$ is relatively compact for $a.e. \; t \in I$.

- A function $u \in C([-r,T],E)$ is called a *mild solution* of (P) if

$$\begin{aligned} u &= \psi \text{ on } [-r,0], \text{ and} \\ u(t) &= K(t,0)\psi(0) + \int_0^t K(t,s)f(s)ds \text{ for every } t \in I, \end{aligned}$$

where $f \in L^1(I,E)$ and $f(s) \in F(s,\tau(s)u)$ $a.e.$

We also use the following Definitions and theorems:

Theorem 2.1 ([14], Corollary 3.3.1)**.** *If $\mathscr{M}$ is a closed convex subset of a Banach space E and $\mathscr{F} : \mathscr{M} \to \mathscr{P}_{CK}(\mathscr{M})$ is a closed χ-condensing multifunction, where χ is a nonsingular MNC defined on a subsets of $\mathscr{M}$, then $\mathscr{F}$ has a fixed point.*

Proposition 2.2 ([14], Proposition 3.5.1)**.** *Let W be a closed subset of a Banach space E and $\mathscr{F} : W \to \mathscr{P}_K(E)$ be a closed multifunction which is χ-condensing on every bounded subset of W, where χ is a monotone MNC in E. If the fixed points set Fix$\mathscr{F}$ is bounded, then it is compact.*

Lemma 2.3 ([14], Proposition 4.2.1)**.** *Every semicompact set is weakly compact in the space $L^1(I,E)$.*

Definition 2.4. Let $K : \Delta \to \mathscr{L}(E)$ be an evolution operator. The operator $G : L^1(I,E) \to C(I,E)$ defined by

$$Gf(t) = \int_0^t K(t,s)f(s)ds, \; t \in I,$$

is called the *generalized Cauchy operator.*

Theorem 2.5 ([4], Theorem 2)**.** *The generalized Cauchy operator G satisfies the properties:*

$(G1)$ there exists $\zeta \geq 0$ such that

$$\|Gf(t) - Gg(t)\| \leq \zeta \int_0^t \|f(s) - g(s)\| ds, \forall t \in I \text{ and } f, g \in L^1(I, E)$$

$(G2)$ for any compact $K \subset E$ and sequence $(f_n)_n,\ f_n \in L^1(I,E)$, such that $\{f_n(t) : n \geq 1\} \subset K$ for *a.e.* $t \in I$, the weak convergence $f_n \xrightarrow{w} f_0$ implies the convergence $Gf_n \to Gf_0$.

Lemma 2.6 ([14], theorem 5.1.1)**.** *Let* $\Pi : L^1(I,E) \to C(I,E)$ *be an operator satisfying condition* $(G2)$ *and the Lipschitz condition (weaker than* $(G1)$*)*

$(G1^{/})$ $\|\Pi f - \Pi g\|_C \leq \zeta \|f - g\|_{L^1(I,E)}$, where $\|.\|$ is the usual sup-norm.

Then for every semicompact set $\{f_n : n \geq 1\} \subset L^1(I,E)$ the set $\{\Pi f_n : n \geq 1\}$ is relatively compact in $C(I,E)$ and, if $f_n \xrightarrow{w} f_0$, then $\Pi f_n \to \Pi f_0$.

Lemma 2.7 ([14], Theorem 4.2.2)**.** *Let the operator* Π *satisfy conditions* $(G1)$ *and* $(G2)$ *and let the set* $\{f_n : n \geq 1\}$ *be integrably bounded with property* $\alpha(\{f_n(t) : n \geq 1\}) \leq \eta(t)$ *for a.e.* $t \in I$*, where* $\eta \in L^1(I, R^+)$ *and* α *is the Hausdorff MNC. Then*

$$\alpha(\{\Pi f_n(t) : n \geq 1\}) \leq 2\zeta \int_0^t \eta(s) ds \text{ for every } t \in I,$$

where $\zeta \geq 0$ *is the constant in condition* $(G1)$.

3. Local and global existence results for (P)

We will use the following assumptions on the data of (P) :

$H(A)$: $\{A(t) : t \in I\}$ is a family of densely defined linear operators (not necessarily bounded or closed), $A(t) : D(A) \subset E \to E$ not depending on t, which generates an evolution operator $K : \Delta = \{(t,s) : I \times I : 0 \leq s \leq t \leq T\} \to \mathscr{L}(E)$, i.e. there exists an evolution system $\{K(t,s) : (t,s) \in \Delta\}$ such that on the region $D(A)$, each operator $K(t,s)$ is strongly differentiable relative to t and s and

$$\frac{\partial K(t,s)}{\partial t} = A(t)K(t,s) \text{ and } \frac{\partial K(t,s)}{\partial s} = -K(t,s)A(t)$$

$H(F)$: $F : I \times C_0 \to \mathscr{P}_{CK}(E)$ is a multifunction such that

(1) F is scalary u.s.c. on $I \times C_0$.

(2) for every nonempty bounded set $\Omega \subset C_0$ it exists a function $\mu_\Omega \in L(I,R^+)$ such that for every $g \in \Omega$

$$\|F(t,g)\| \leq \mu_\Omega(t) \quad a.e. \text{ on } I.$$

(3) there exists a function $k \in L(I,R^+)$ such that for every bounded set $\Omega \subset C_0$

$$\alpha(F(t,\Omega)) \leq k(t)\alpha(\{g(0) : g \in \Omega\}) \quad a.e. \text{ on } I.$$

To prove our theorem we need the following lemma:

Lemma 3.1. *Under assumptions $H(F)(1)$ and $H(F)(3)$ if we consider the sequences $(x_n), x_n \in C([-r,d],E)$ and $(f_n), f_n \in L^1(I,E)$, where $f_n \in S^1_{F(.,\tau(.)x_n)}$, such that $x_n \to x^0$ and $f_n \xrightarrow{w} f^0$, then $f^0 \in S^1_{F(.,\tau(.)x^0)}$.*

Proof. We first prove that $\tau(.)x_n \to \tau(.)x^0$. So, we have for every $t \in I$

$$\begin{aligned}\|\tau(t)x_n - \tau(t)x\| &= \sup_{s\in[-r,0]} \|x_n(t+s) - x(t+s)\| \\ &\leq \sup_{-r\leq h\leq t} \|x_n(h) - x(h)\|,\end{aligned}$$

since, we have that $x_n \to x^0$, then for every $t \in I$

$$\lim_{n\to\infty} \sup_{-r\leq h\leq t} \|x_n(h) - x(h)\| = 0.$$

Then, $\tau(.)x_n \to \tau(.)x^0$. Now, we can apply the Convergence Theorem (see for instance [2], Theorem 1.4.1), which completes the proof. □

Now, we prove the local existence theorem:

Theorem 3.2. *If hypotheses $H(A)$ and $H(F)$ hold and if $\psi \in C_0$, then there exists $d \in I$ and a mild solution $u_* \in C([-r,d],E)$ of (P).*

Proof. Let $\varepsilon > 0$ be a given number and let us consider the closed unit ball

$$B^* := \overline{B}_\varepsilon(x^*),$$

where $x^* \in C([-r,T],E)$ is the function defined by

$$\begin{aligned}x^* &= \psi \text{ on } [-r,0] \text{ and} \\ x^*(t) &= \psi(0), t \in I.\end{aligned}$$

Let

$$B = \{\tau(t)x : x \in B^*, t \in I\}.$$

Clearly $B \subset C_0$. Indeed B is a bounded subset of C_0, because for every $t \in I$ and $x \in B^*$

$$\begin{aligned} \|\tau(t)x\| &= \sup_{s\in[-r,0]} \|\tau(t)x(s)\| \\ &= \sup_{s\in[-r,0]} \|x(t+s)\| \\ &\leq \sup_{h\in[-r,T]} \|x(h)\| \\ &\leq \|x^*\| + \varepsilon. \end{aligned}$$

Thanks to condition $H(F)(2)$, we have

$$\|F(s,z)\| \leq \mu_B(s), \text{ for every } z \in B \text{ and } a.e. \text{ for } s \in I,$$

where μ_B is the function defined in $H(F)(2)$.Also, since the evolution operator K is strongly continuous on Δ, then there exists a natural number N such that

$$\|K(t,s)\|_{\mathscr{L}(E)} < N \text{ for all } (t,s) \in \Delta. \tag{1}$$

Now, we can choose $d_1 \in (0,T]$ such that

$$N\int_0^{d_1} \mu_B(s)ds \leq \varepsilon/2. \tag{2}$$

Since the evolution operator K is strongly continuous on Δ, then there exists $d_2 \in (0,T]$ such that

$$\|(K(t,0)-K(0,0))\psi(0)\| \leq \varepsilon/2 \;\text{ for all } t \in [0,d_2] \tag{3}$$

Take $d = \min(d_1,d_2)$. Now, consider the multifunction

$$\Gamma : C([-r,d],E) \to \mathscr{P}(C([-r,d],E)) \text{ such that}$$

$$\Gamma(x) = \{y \in C([-r,d],E) : y = \psi \text{ on } [-r,0] \quad \text{and}$$

$$y(t) = K(t,0)\psi(0) + \int_0^t K(t,s)f(s)ds,\ f \in S^1_{F(.,\tau(.)x)}\}$$

From the assumption $H(F)$ it is clear that $S^1_{F(.,\tau(.)x)}$ is nonempty. It is obvious that a function $x \in C([-r,d],E)$ is a mild solution of the set of (P) on $[-r,d]$ iff $x \in \Gamma(x)$. So, we have to show that Γ has a fixed point.

In order to apply Theorem (2.1) we will follow the following steps:

Step 1: Γ is closed.

To prove that Γ is closed, i.e. the graph of Γ is closed, consider the sequence (x_n, y_n) in Graph(Γ) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$, our aim is to prove that

$$\bar{y} \in \Gamma(\bar{x}).$$

Let (f_n) be an arbitrary sequence such that $f_n \in S^1_{F(.,\tau(.)x_n)}$. We want to prove that the set $\{f_n : n \geq 1\}$ is semicompact. So, we first prove that the set $\{f_n : n \geq 1\}$ is integrably bounded. Let $\Omega = \{\tau(t)x_n : t \in I, n \geq 1\}$. Since (x_n) is a convergence sequence, then, as we prove in Lemma (3.1), the sequence $(\tau(t)x_n)$ is convergent. Then, Ω is bounded in C_0 and from $H(F)(2)$ we have

$$\|f_n(t)\| \leq \|F(t, \tau(t)x_n)\| \leq \mu_\Omega(t) \quad a.e. \text{ on } I.$$

Hence, the set $\{f_n : n \geq 1\}$ is integrably bounded. Second, the set $\{f_n(t) : n \geq 1\}$ is relatively compact $a.e.$ on I since from $H(F)(3)$ we have for $a.e.\ t \in I$

$$\begin{aligned} \alpha(\{f_n(t)\}_{n=1}^{\infty}) &\leq \alpha(F(t, \{\tau(t)x_n\}_{n=1}^{\infty})) \\ &\leq k(t)\alpha(\{\tau(t)x_n(0) : n \geq 1\}) \\ &= k(t)\alpha(\{x_n(t) : n \geq 1\}), \end{aligned}$$

since for $a.e.\ t \in I$ the sequence $(x_n(t))$ is a convergent sequence, then $\alpha(\{x_n(t) : n \geq 1\}) = 0$.Then, the set $\{f_n : n \geq 1\}$ is semicompact. From Lemma (2.3), the set $\{f_n : n \geq 1\}$ is weakly compact in $L^1(I, E)$. So, we can assume w.l.o.g. that $f_n \overset{w}{\to} \overline{f}$ in $L^1(I, E)$. By Theorem (2.5), we have $Gf_n \to G\overline{f}$. Then applying Lemma (3.1),

$$\begin{aligned} \bar{y} &= \psi \text{ on } [-r, 0] \text{ and} \\ \bar{y}(t) &= K(t, 0)\psi(0) + G\overline{f},\ \overline{f} \in S^1_{F(.,\tau(.)\bar{x})}, t \in [0, d] \end{aligned}$$

Hence, $\bar{y} \in \Gamma(\bar{x})$.

Step 2: Γ has compact convex values.

Clearly, Γ has convex values. So, we will prove that Γ has compact values. Let $x \in C([-r, d], E)$ and consider the sequences (z_n) in $\Gamma(x)$ and (f_n) in $S^1_{F(.,\tau(.)x)}$ such that

$$\begin{aligned} z_n &= \psi \text{ on } [-r, 0] \text{ and} \\ z_n(t) &= K(t, 0)\psi(0) + Gf_n, t \in I. \end{aligned}$$

As in step 1, we can prove that the set $\{f_n : n \geq 1\}$ is semicompact. So, w.lo.g. it converges weakly in $L^1(I, E)$. Then, (Gf_n) converges in $C(I, E)$. Then, (z_n) converges in $C([-r, d], E)$. Then, $\Gamma(x)$ is relatively compact. Since, $\Gamma(x)$ is closed, then it is compact.

Step 3: Γ is condensing on bounded sets with respect to the well-defined, monotone, nonsingular and regular MNC ν in the space $C([-r,d],E)$ defined by (see [14])

$$\nu(Z) = \max_{\mathscr{D}\in\Delta(Z)} \beta(\mathscr{D}),$$

where Z is bounded subset of $C([-r,d],E)$ and $\Delta(Z)$ is the collection of all denumerable subsets of Z and β is the real MNC defined as

$$\beta(\mathscr{D}) = \sup_{t\in[-r,d]} e^{-Lt}\alpha(\{\phi(t) : \phi \in \mathscr{D}\}),$$

where $L > 0$ is a constant chosen such that

$$q := 2N \sup_{t\in[0,d]} \int_0^t e^{-L(t-s)} k(s)ds < 1.$$

To prove that Γ is condensing on bounded sets with respect to ν, let $Z \subset C([-r,d],E)$ be a bounded set such that

$$\nu(\Gamma(Z)) \geq \nu(Z),$$

we want to prove that Ω is relatively compact. Since ν is regular, it is enough to prove that $\nu(\Omega) = 0$. From the definition of ν, there is a denumerable set $D = \{y_n : n \geq 1\} \subset \Gamma(\Omega)$ such that

$$\nu(\Gamma(Z)) = \beta(Z).$$

For each $n \geq 1$, let $x_n \in Z$ such that $y_n \in \Gamma(x_n)$. This means that

$$\begin{aligned} y_n &= \psi \text{ on } [-r,0] \text{ and} \\ y_n(t) &= K(t,0)\psi(0) + Gf_n, f_n \in S^1_{F(.,\tau(.)x_n)}, t \in [0,d]. \end{aligned}$$

From $H(F)(3)$ we have for $a.e.$ $s \in [0,d]$

$$\begin{aligned} \alpha(\{f_n(s) &: n \geq 1\}) \leq \alpha(F(s,\{\tau(s)x_n : n \geq 1\})) \\ &\leq k(s)\alpha(\{\tau(s)x_n(0) : n \geq 1\}) \\ &= k(s)\alpha(\{x_n(s) : n \geq 1\}) \\ &\leq e^{Ls}k(s) \sup_{\xi\in[-r,d]} e^{-L\xi}\alpha(\{x_n(\xi) : n \geq 1\}) \\ &= e^{Ls}k(s)\beta(\{x_n : n \geq 1\}) = \eta(s). \end{aligned}$$

Then, invoking to Lemma (2.7) we get

$$\begin{aligned}\alpha(\{Gf_n(t) \quad &: \quad n \geq 1\}) \leq 2N\int_0^t \eta(s)ds \\ &= \quad 2N\int_0^t e^{Ls}k(s)\beta(\{x_n : n \geq 1\})ds \\ &\leq \quad 2N\beta(\{x_n : n \geq 1\})\int_0^t e^{Ls}k(s)ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}\beta(\{x_n \quad &: \quad n \geq 1\}) \leq \nu(Z) \leq \nu(\Gamma(Z)) \\ &= \quad \beta(D) \\ &= \quad \beta(\{y_n : n \geq 1\}) \\ &= \quad \sup_{t\in[-r,d]} e^{-Lt}\alpha(\{y_n(t) : n \geq 1\}) \\ &\leq \quad \sup_{t\in[-r,0]} e^{-Lt}\alpha(\{y_n(t)\}_{n=1}^{\infty}) + \sup_{t\in[0,d]} e^{-Lt}\alpha(\{y_n(t)\}_{n=1}^{\infty}) \\ &= \quad \sup_{t\in[-r,0]} e^{-Lt}\alpha(\{\psi(t)\}) + \sup_{t\in[0,d]} e^{-Lt}\alpha(\{Gf_n(t)\}_{n=1}^{\infty}) \\ &\leq \quad \sup_{t\in[0,d]} e^{-Lt}2N\beta(\{x_n : n \geq 1\})\int_0^t e^{Ls}k(s)ds \\ &= \quad 2N\beta(\{x_n : n \geq 1\}) \sup_{t\in[0,d]}\int_0^t e^{-L(t-s)}k(s)ds \\ &= \quad \beta(\{x_n : n \geq 1\})q.\end{aligned}$$

Since $q < 1$, we get

$$\beta(\{x_n : n \geq 1\}) = 0$$

Consequently,

$$\nu(Z) = 0$$

Step 4: Let

$$H = \{\tau(t)x_d : x \in B^*, t \in [0,d]\},$$

where x_d is the restriction of x on the interval $[0,d]$. We want to prove that Γ maps H into itself.

Let $D=\Gamma(H)$ and $y\in D$. Then there is $x\in H$ such that $y=\Gamma(x)$. Then,

$$y=\psi \text{ on } [-r,0] \text{ and } y(t)=K(t,0)\psi(0)+Gf, f\in S^1_{F(.,\tau(.)x)}, t\in[0,d].$$

Using (1), (2) and (3), for *a.e.* $t\in[0,d]$ we have

$$\begin{aligned}
\|y(t)-x^*(t)\| &= \|K(t,0)\psi(0)+\int_0^t K(t,s)f(s)ds-\psi(0)\| \\
&\le \|(K(t,0)-K(0,0))\psi(0)\|+\int_0^t \|K(t,s)\|_{\mathscr{L}(E)}\|f(s)\|ds \\
&\le \varepsilon/2+N\int_0^{d_1}\mu_B(s)ds \qquad \text{Since } \tau(s)x\in B \\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

which proves that $y\in H$.

Now, apply Theorem (2.1) to $\Gamma: H\to \mathscr{P}_{CK}(H)$, which completes the proof. □

The next theorem shows that under a stronger condition on F, we obtain a global existence result.

Theorem 3.3. *If hypotheses $H(A)$, $H(F)(1), H(F)(3)$ and instead of $H(F)(2)$ we have*

$H(F)(2)'$ there exists a function $\gamma\in L(I,R^+)$ such that for every $g\in C_0$

$$\|F(t,g)\|\le\gamma(t)(1+\|g(0)\|) \quad a.e. \text{ on } I.$$

hold and if ψ is continuous, then the set of mild solutions $S(\psi)$ of (P) is a nonempty subset of the space $C([-r,T],E)$.

Proof. Let x^* be a function defined by

$$\begin{aligned}
x^* &= \psi \text{ on } [-r,0] \text{ and} \\
x^*(t) &= \psi(0), t\in I.
\end{aligned}$$

Let $L>0$ be a real number such that

$$q^*:=\max_{t\in I} N\int_0^t e^{-L(t-s)}\gamma(s)ds<1,$$

and $\varepsilon > 0$ be a given number such that

$$((N+1)\|\psi(0)\| + N\|\gamma\|_{L(I,R^+)}))(1-q^*)^{-1} \leq \varepsilon,$$

Also, we consider the following norm in the space $C([-r,T],E)$ defined by

$$\|x\|_* = \sup_{t\in[-r,T]} e^{-Lt}\|x(t)\|$$

Clearly, this norm is equivalent to the uniform norm. We denote the closed ball in the space $(C([-r,T],E), \|.\|_*)$ with center at $x^*$and radius ε by B^*.

In order to apply Theorem (2.1), let Γ be a multifunction defined by

$$\Gamma : C([-r,T],E) \to \mathscr{P}(C([-r,T],E)) \text{ such that}$$

$$\Gamma(x) = \{y \in C([-r,T],E) : y = \psi \text{ on } [-r,0]$$

and

$$y(t) = K(t,0)\psi(0) + \int_0^t K(t,s)f(s)ds, f \in S^1_{F(.,\tau(.)x)}\}$$

Similarly, as in the previous theorem, Γ is closed with compact convex values and is condensing on bounded sets with respect to well defined, monotone, non-singular and regular MNC ν in the space $C(I,E)$. So, it is sufficient to prove that Γ maps B^* into itself.

Let $x \in B^*$ and $y \in \Gamma(x)$. Then we can write

$$\begin{aligned} y &= \psi \text{ on } [-r,0] \text{ and} \\ y(t) &= K(t,0)\psi(0) + \int_0^t K(t,s)f(s)ds, f \in S^1_{F(.,\tau(.)x)}, t \in I. \end{aligned}$$

Then,

$$e^{-Lt}\|y(t) - x^*(t)\| \leq$$

$$\leq e^{-Lt}\|K(t,0)\|_{\mathscr{L}(E)}\|\psi(0)\| + e^{-Lt}\int_0^t \|K(t,s)\|_{\mathscr{L}(E)}\|f(s)\|ds + e^{-Lt}\|\psi(0)\| \leq$$

$$\leq e^{-Lt}N\|\psi(0)\| + e^{-Lt}\int_0^t N\gamma(s)(1+\|x(s)\|)ds + e^{-Lt}\|\psi(0)\| \leq$$

$$\leq (N+1)\|\psi(0)\| + N\|\gamma\|_{L^1(I,R^+)} + e^{-Lt}N\int_0^t \gamma(s)\|x(s)\|ds =$$

$$= (N+1)\|\psi(0)\| + N\|\gamma\|_{L^1(I,R^+)} + N\int_0^t e^{-L(t-s)}\gamma(s)e^{-Ls}\|x(s)\|ds \leq$$

$$\leq (N+1)\|\psi(0)\| + N\|\gamma\|_{L^1(I,R^+)} + N\|x\|_* \int_0^t e^{-L(t-s)}\gamma(s)ds \leq$$

$$\leq (N+1)\|\psi(0)\| + N\|\gamma\|_{L^1(I,R^+)} + \varepsilon q^* \leq \varepsilon.$$

which proves that $y \in B^*$.

Now, apply Theorem (2.1) to $\Gamma : B^* \to \mathscr{P}_{CK}(B^*)$, which proves that the set of mild solutions $S(\psi)$ of (P) is nonempty. □

The next theorem gives some topological properties of $S(\psi)$:

Theorem 3.4. *Under the conditios in Theorem (3.3), the set of mild solutions $S(\psi)$ of (P) is compact subset of the space $C([-r,T],E)$.*

Proof. To prove that the set of mild solutions $S(\psi)$ of (P) is compact, By Proposition (2.2), it is enough to prove that $S(\psi)$ is bounded.

Since ψ is continuous on $[-r,0]$, then it is bounded on $[-r,0]$. Also, let $x \in S(\psi)$. Then, we have for every $t \in I$

$$\begin{aligned}
\|x(t)\| &\leq \|K(t,0)\|_{\mathscr{L}(E)}\|\psi(0)\| + \int_0^t \|K(t,s)\|_{\mathscr{L}(E)}\|f(s)\|ds, f \in S^1_{F(.,\tau(.)x)} \\
&\leq N\|\psi(0)\| + N\int_0^t \gamma(s)(1+\|\tau(s)x(0)\|)ds \qquad \text{from } H(F)(2)' \\
&\leq N\|\psi(0)\| + N\int_0^t \gamma(s)(1+\|x(s)\|)ds \\
&\leq N(\|\psi(0)\| + \|\gamma\|_{L^1(I,R^+)} + \int_0^t \gamma(s)\|x(s)\|ds).
\end{aligned}$$

Invoking Gronwall's inequality, we get

$$\|x(t)\| \leq N(\|\psi(0)\| + \|\gamma\|_{L^1(I,R^+)})e^{N\|\gamma\|_{L^1(I,R^+)}} = M,$$

which completes the proof. □

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A. G. IBRAHIM
Department of Mathematics
Faculty of Science
Cairo University
Cairo, Egypt
e-mail: agamal2000@yahoo.com

A. M. SOLIMAN
Department of Mathematics
University College for Women
Ain Shams University
Cairo, Egypt
e-mail: asma_812@hotmail.com