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NEFNESS OF ADJOINT BUNDLES FOR AMPLE VECTOR BUNDLES

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Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a smooth complex projective variety X of dimension n. This paper gives a classification of pairs $(X, \mathcal{E})$ whose adjoint bundles $K_X + \det \mathcal{E}$ are not nef in the case when $r = n - 2$.

1. Introduction.

Let $\mathcal{E}$ be an ample vector bundle of rank r on a smooth complex projective variety X of dimension n. Ye and Zhang investigated the nefness of the adjoint bundle $K_X + \det \mathcal{E}$ and proved the following

Theorem 0.1. ([9], Theorem 1). *If $r \geq n + 1$, then $K_X + \det \mathcal{E}$ is always nef.*

Theorem 0.2. ([9], Theorem 2). *If $r = n$, then $K_X + \det \mathcal{E}$ is nef unless $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.*

Theorem 0.3. ([9], Theorem 3). *If $r = n - 1$, then $K_X + \det \mathcal{E}$ is nef except the following cases.*

(1) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)})$.

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(2) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$.
(3) $X \cong \mathbb{Q}^n$, *a smooth hyperquadric in* $\mathbb{P}^{n+1}$, *and* $\mathcal{E} \cong \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-1)}$.
(4) *There is a vector bundle* $\mathcal{F}$ *on a smooth curve* C *such that* $X \cong \mathbb{P}_C(\mathcal{F})$ *and* $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-1)}$ *for any fiber* $F(\cong \mathbb{P}^{n-1})$ *of* $X \to C$.

Moreover, when $r = n - 2$, Zhang proved the following

Theorem 0.4. ([10], Theorem 1.1). *If* $r = n - 2$, *then* $K_X + \det \mathcal{E}$ *is nef except the following cases.*

(1) *There exists a birational morphism* $f : X \to Z$ *and the exceptional locus of* f *is a divisor* F *such that* $(F, \mathcal{E}_F) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)})$, f *is the contraction morphism of* F *(blowing-down of* F*) to some smooth point on* Z.
(2) *There exists a surjective morphism* $f : X \to W$ *such that* $\dim W = 1$ *or* 2 *and the generic fiber* F *is a smooth projective variety.* F *is either* $\mathbb{P}^{n-1}$, $\mathbb{P}^{n-2}$, *or a hyperquadric.*
(3) $X \cong \mathbb{P}^n$, $\mathbb{Q}^n$, *or a Del Pezzo manifold.*

If we compare Theorems 0.1, 0.2 and 0.3 with Theorem 0.4, Theorem 0.4 does not seem a satisfactory result due to the lack of descriptions of $\mathcal{E}$ in some cases. The purpose of this paper is to give a complete description of pairs $(X, \mathcal{E})$ whose adjoint bundles $K_X + \det \mathcal{E}$ are not nef in the case when $r = n-2$, and to improve on the Theorem 0.4. The precise statement of our result is as follows:

Theorem. *Let* $\mathcal{E}$ *be an ample vector bundle of rank* $r \geq 2$ *on a smooth complex projective variety* X *of dimension* n. *If* $r = n-2$, *then* $K_X + \det \mathcal{E}$ *is nef except the following cases.*

(1) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$.
(2) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)})$.
(3) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)})$.
(4) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-4)})$.
(5) $(X, \mathcal{E}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)})$.
(6) $(X, \mathcal{E}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(2) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-3)})$.
(7) $(X, \mathcal{E}) \cong (\mathbb{Q}^4, \mathcal{S} \otimes \mathcal{O}_{\mathbb{Q}^4}(2))$, *where* $\mathcal{S}$ *is a spinor bundle on* $\mathbb{Q}^4$.
(8) *There exists an effective divisor* E *on* X *such that*

$$(E, \mathcal{E}_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)})$$

and that $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.
(9) X *is a Fano manifold of index* $n - 1$ *with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *and* $\mathcal{E} \cong L^{\oplus(n-2)}$, *where* L *is the ample generator of* $\mathrm{Pic}(X)$.

(10) *There is a vector bundle $\mathcal{F}$ on a smooth curve C such that $X \cong \mathbb{P}_C(\mathcal{F})$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$ for any fiber $F(\cong \mathbb{P}^{n-1})$ of $X \to C$.*

(11) *There is a vector bundle $\mathcal{F}$ on a smooth curve C such that $X \cong \mathbb{P}_C(\mathcal{F})$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-3)}$ for any fiber $F(\cong \mathbb{P}^{n-1})$ of $X \to C$.*

(12) *There is a surjective morphism $f : X \to C$ onto a smooth curve C such that any general fiber F of f is a smooth hyperquadric $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus(n-2)}$.*

(13) *There is a vector bundle $\mathcal{F}$ on a smooth surface S such that $X \cong \mathbb{P}_S(\mathcal{F})$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus(n-2)}$ for any fiber $F(\cong \mathbb{P}^{n-2})$ of $X \to S$.*

The idea of my proof comes from the investigation of ample vector bundles with special zero loci (see [4], Section 3).

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After the first version of this paper was written, Professor M. Andreatta sent me his joint preprint [2] with M. Mella, in which they have proved a very similar result. Their method is different from mine and is based on the Theorem of [1].

I would like to express my gratitude to the referee for improving the original theorem as well as for the other suggestions.

2. Preliminaries.

In this paper varieties are always assumed to be defined over the complex number field $\mathbb{C}$. We use the standard notation from algebraic geometry. The words "vector bundles" and "locally free sheaves" are used interchangeably. The group of line bundles on X is denoted by $\mathrm{Pic}(X)$. The tensor products of line bundles are denoted additively. The pull-back $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. The canonical bundle of a smooth variety X is denoted by K_X. A smooth projective variety X is called a *Fano manifold* if its anticanonical bundle $-K_X$ is ample. For a Fano manifold X, the largest integer which divides $-K_X$ in $\mathrm{Pic}(X)$ is called the *index* of X.

A *polarized manifold* is a pair (X, L) consisting of a smooth projective variety X and an ample line bundle L on X. A polarized manifold (X, L) is said to be a *scroll* over a smooth variety W if $(X, L) \cong (\mathbb{P}_W(\mathcal{F}), H(\mathcal{F}))$ for some ample vector bundle $\mathcal{F}$ on W, where $H(\mathcal{F})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_W(\mathcal{F})$ associated to $\mathcal{F}$. A polarized manifold (X, L) is called a *Del Pezzo manifold* if $K_X = (1-n)L$, where $n = \dim X$.

Let X be a smooth projective variety of dimension n and $Z_1(X)$ the free abelian group generated by integral curves on X. The intersection pairing gives a bilinear map $\mathrm{Pic}(X) \times Z_1(X) \to \mathbb{Z}$ and the *numerical equivalence* $\equiv$ is defined so that the pairing $((\mathrm{Pic}(X)/\equiv) \otimes \mathbb{Q}) \times ((Z_1(X)/\equiv) \otimes \mathbb{Q}) \to \mathbb{Q}$ is non-degenerate. The *closed cone of curves* $\overline{NE}(X)$ is the closed convex cone generated by effective 1-cycles in the $\mathbb{R}$-vector space $(Z_1(X)/\equiv) \otimes \mathbb{R}$. $L \in \mathrm{Pic}(X)$ is called *nef* if the numerical class of L in $(\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$ gives a non-negative function on $\overline{NE}(X) - \{0\}$. Let Z be a 1-cycle on X. We denote by $[Z]$ the numerical class of Z in $(Z_1(X)/\equiv) \otimes \mathbb{R}$. A half line $R = \mathbb{R}_+[Z] (\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\})$ in $\overline{NE}(X)$ is called an *extremal ray* if

(1) $K_X Z < 0$, and
(2) if $z_1, z_2 \in \overline{NE}(X)$ satisfy $z_1 + z_2 \in R$, then $z_1, z_2 \in R$.

A rational curve l on X is called an *extremal rational curve* if $(-K_X)l \leq n+1$ and $\mathbb{R}_+[l]$ is an extremal ray. Let

$$\overline{NE}(X)^+ = \{z \in \overline{NE}(X) \mid K_X z \geq 0\}.$$

Then we have the following basic theorem.

Theorem 1.1. (Cone Theorem). *Let X be a smooth projective variety. Then $\overline{NE}(X)$ is the smallest closed convex cone containing $\overline{NE}(X)^+$ and all the extremal rays:*

$$\overline{NE}(X) = \overline{NE}(X)^+ + \Sigma R_j,$$

where the R_j are extremal rays of $\overline{NE}(X)$ for X. For any open convex cone U containing $\overline{NE}(X)^+ - \{0\}$ there exist only a finite number of extremal rays that do not lie in $U \cup \{0\}$. Furthermore, every extremal ray is spanned by a numerical class of an extremal rational curve.

For the proof, we refer to [5], Theorem 1.5. We need also the well-known

Lemma 1.2. *Let $\mathcal{E}$ be an ample vector bundle of rank r on a rational curve C. Then $c_1(\mathcal{E}) \geq r$.*

Let X be a smooth projective variety. If R is an extremal ray, then its *length* $l(R)$ is defined as follows.

$$l(R) := \min\{(-K_X)C \mid C \text{ is a rational curve such that } [C] \in R\}.$$

We note that $0 < l(R) \leq \dim X + 1$ from Theorem 1.1 and the definition of an extremal rational curve. Let $\mathcal{E}$ be an ample vector bundle of rank r on X and $\Omega(X, \mathcal{E})$ the set of extremal rays R such that $(K_X + \det \mathcal{E})R < 0$. Then it follows from Theorem 1.1 that the set $\Omega(X, \mathcal{E})$ is finite. For any extremal ray R in $\Omega(X, \mathcal{E})$ we define a positive integer

$$\Lambda(X, \mathcal{E}, R) = (-K_X - \det \mathcal{E})C,$$

where C is an extremal rational curve such that $(-K_X)C = l(R)$. If $\Omega(X, \mathcal{E})$ is non-empty, then we define a positive integer

$$\Lambda(X, \mathcal{E}) = \max\{\Lambda(X, \mathcal{E}, R) \mid R \in \Omega(X, \mathcal{E})\}.$$

Suppose that $\Lambda(X, \mathcal{E}) = 1$. Then for any extremal ray R in $\Omega(X, \mathcal{E})$ there exists an extremal rational curve C such that $(K_X + \det \mathcal{E})C = -1$. Let

$$m = \min\{(\det \mathcal{E})C \mid C \text{ extremal rational curve with, } (K_X + \det \mathcal{E})C = -1\}. \tag{1.3}$$

Then $m \geq r$ by Lemma 1.2. Set

$$L = (m-1)K_X + m \det \mathcal{E}. \tag{1.4}$$

Then we have the following

Proposition 1.5. *L is ample and $K_X + mL$ is not nef.*

For the proof, we refer to [8], Proposition 3.5. We note that its proof is valid without assuming the spannedness of $\mathcal{E}$.

3. Proof of the Theorem.

Let $\mathcal{E}$ be an ample vector bundle of rank $r = n - 2 \geq 2$ on a smooth projective variety X of dimension n. Assume that $K_X + \det \mathcal{E}$ is not nef. Then, since $K_X + \det \mathcal{E}$ is not non-negative on $\overline{NE}(X) - \{0\}$, by the cone theorem 1.1 we can find an extremal ray R with $(K_X + \det \mathcal{E})R < 0$, and so $\Omega(X, \mathcal{E}) \neq \emptyset$.

(2.1) If $\Lambda(X, \mathcal{E}) \geq 2$, then it follows from [10], Proposition 1.1' that $(X, \mathcal{E})$ is one of the following

(2.1.1) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)})$.
(2.1.2) $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)})$.
(2.1.3) $(X, \mathcal{E}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)})$.
(2.1.4) There is a vector bundle $\mathcal{F}$ on a smooth curve C such that $X \cong \mathbb{P}_C(\mathcal{F})$ and $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$ for any fiber $F(\cong \mathbb{P}^{n-1})$ of $X \to C$.

(2.2) From now on, we assume that $\Lambda(X, \mathcal{E}) = 1$. In this case we can construct an ample line bundle L on X and a positive integer $m \geq r = n - 2$ such that $K_X + mL$ is not nef by Proposition 1.5. Since $n = r + 2 \geq 4$, we use [3], Theorems 1, 2, 3 and 3' to see that (X, L) is one of the following

(2.2.1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
(2.2.2) $(X, L) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(2.2.3) (X, L) is a scroll over a smooth curve C.
(2.2.4) There exists an effective divisor E on X such that $(E, L_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and that $K_X + (n-1)L$ is trivial in $\mathrm{Pic}(E)$.
(2.2.5) $(X, L) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
(2.2.6) (X, L) is a Del Pezzo manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$.
(2.2.7) There is a surjective morphism $f : X \to C$ onto a smooth curve C such that any general fiber F of f is a smooth hyperquadric $\mathbb{Q}^{n-1}$ in $\mathbb{P}^n$ with $L_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$.
(2.2.8) (X, L) is a scroll over a smooth surface S.

In case (2.2.1) m must be n, $n-1$, or $n-2$ because $K_X + (n+1)L$ is nef. In case (2.2.2) or (2.2.3) m must be $n-1$ or $n-2$ because $K_X + nL$ is nef. In case (2.2.4), (2.2.5), (2.2.6), (2.2.7), or (2.2.8) m must be $n-2$ because $K_X + (n-1)L$ is nef. We proceed now by cases.

(2.3) Case (2.2.1).

Take an arbitrary extremal rational curve C with $(K_X + \det\mathcal{E})C = -1$. Then $(\det\mathcal{E})C = (-K_X)C - 1 \geq n$ because $-K_X = \mathcal{O}_{\mathbb{P}^n}(n+1)$, and so $m \geq n$ by (1.3). Thus $m = n$. By (1.4),

$$\begin{aligned} n \det\mathcal{E} &= L - (n-1)K_X \\ &= \mathcal{O}_{\mathbb{P}^n}(1) + (n-1)\mathcal{O}_{\mathbb{P}^n}(n+1) \\ &= \mathcal{O}_{\mathbb{P}^n}(n^2), \end{aligned}$$

and hence $\det\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(n)$. Let $\mathcal{E}' = \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^n}(1)$. Then $\mathcal{E}'$ is an ample vector bundle of rank $n-1$, and $K_X + \det\mathcal{E}' = \mathcal{O}_{\mathbb{P}^n}$. Since $n = r + 2 \geq 4$, we use [7], Main theorem 0.3 and Proposition 7.4 to see that $\mathcal{E}'$ is either $\mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-2)}$ or $\mathcal{O}_{\mathbb{P}^n}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}$. Thus $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(3) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-3)}$ or $\mathcal{O}_{\mathbb{P}^n}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-4)}$. So we are in case (3) or (4).

(2.4) Case (2.2.2).

Take an arbitrary extremal rational curve C such that $(K_X + \det\mathcal{E})C = -1$. Then, since $-K_X = \mathcal{O}_{\mathbb{Q}^n}(n)$, we have $(\det\mathcal{E})C = (-K_X)C - 1 \geq n-1$, so that by (1.3) $m = n-1$. By (1.4),

$$\begin{aligned} (n-1)\det\mathcal{E} &= L - (n-2)K_X \\ &= \mathcal{O}_{\mathbb{Q}^n}(1) + (n-2)\mathcal{O}_{\mathbb{Q}^n}(n) \\ &= \mathcal{O}_{\mathbb{Q}^n}((n-1)^2), \end{aligned}$$

and so $\det\mathcal{E} = \mathcal{O}_{\mathbb{Q}^n}(n-1)$. Set $\mathcal{E}' = \mathcal{E} \oplus \mathcal{O}_{\mathbb{Q}^n}(1)$. Then $\mathcal{E}'$ is an ample vector bundle of rank $n-1$ such that $K_X + \det\mathcal{E}' = \mathcal{O}_{\mathbb{Q}^n}$. It follows from [7], Main theorem 0.3 and Proposition 7.4 that $\mathcal{E}'$ is either $\mathcal{O}_{\mathbb{Q}^n}(2) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-2)}$ or $(\mathcal{S} \otimes \mathcal{O}_{\mathbb{Q}^4}(2)) \oplus \mathcal{O}_{\mathbb{Q}^4}(1)$, where $\mathcal{S}$ is a spinor bundle on $\mathbb{Q}^4$; hence $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^n}(2) \oplus \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus(n-3)}$ or $\mathcal{S} \otimes \mathcal{O}_{\mathbb{Q}^4}(2)$. Consequently we are in case (6) or (7).

(2.5) Case (2.2.3).

We can write $(X, L) = (\mathbb{P}_C(\mathcal{F}), H)$ for some ample vector bundle $\mathcal{F}$ on C, where H is the tautological line bundle on $\mathbb{P}_C(\mathcal{F})$. Let π be the projection $X \to C$. Then $K_X = -nH + \pi^*(K_C + \det\mathcal{F})$, so $K_F + nH_F = \mathcal{O}_F$ for every fiber $F(\cong \mathbb{P}^{n-1})$ of π. Now, by (1.4),

$$L = (m-1)K_X + m\det\mathcal{E},$$

hence

$$\begin{aligned}\mathcal{O}_F = K_F + nH_F &= K_F + n(m-1)K_F + nm\det\mathcal{E}_F \\ &= nm(K_F + \det\mathcal{E}_F) + (1-n)K_F.\end{aligned}$$

Thus $m(K_F + \det\mathcal{E}_F) = \mathcal{O}_{\mathbb{P}^{n-1}}(-(n-1))$ and $K_F + \det\mathcal{E}_F$ is not nef. By virtue of Theorem 0.3, $\mathcal{E}_F$ is either $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$ or $\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-3)}$. In the former case, since $K_F + \det\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$, we have $2m = n-1$. Recalling that $m \geq n-2$, we have $2(n-2) \leq n-1$. Therefore $n \leq 3$, contrary to the hypothesis that $n = r+2 \geq 4$. Consequently $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-3)}$ for any fiber F and we are in case (11).

(2.6) Case (2.2.4).

Since $K_X + (n-1)L$ is trivial in $\mathrm{Pic}(E)$, we have

$$(K_X)_E + (n-1)L_E = \mathcal{O}_E,$$

and hence $(K_X)_E = \mathcal{O}_{\mathbb{P}^{n-1}}(1-n)$. Moreover, by the adjunction formula,

$$\mathcal{O}_E(E) = K_E - (K_X)_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-n) - \mathcal{O}_{\mathbb{P}^{n-1}}(1-n) = \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

Since $(n-2)\det\mathcal{E} = L - (n-3)K_X$ by (1.4), we have

$$\begin{aligned}(n-2)\det\mathcal{E}_E &= L_E - (n-3)(K_X)_E \\ &= \mathcal{O}_{\mathbb{P}^{n-1}}(1) + (n-3)\mathcal{O}_{\mathbb{P}^{n-1}}(n-1) \\ &= \mathcal{O}_{\mathbb{P}^{n-1}}((n-2)^2),\end{aligned}$$

so $\det \mathcal{E}_E = \mathcal{O}_{\mathbb{P}^{n-1}}(n-2)$. This implies that $\mathcal{E}_E$ is a uniform bundle of splitting type $(1, \dots, 1)$. Thus $\mathcal{E}_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus(n-2)}$ by virtue of [6], p. 55, Theorem 3.2.3 and we are in case (8).

(2.7) Case (2.2.5).

Take an arbitrary extremal rational curve C with $(K_X + \det \mathcal{E})C = -1$. Then $(\det \mathcal{E})C = (-K_X)C - 1 \geq 4$ since $-K_X = \mathcal{O}_{\mathbb{P}^4}(5)$, and so $m \geq 4$ by (1.3). But this is impossible because in this case $m = n - 2 = 2$.

(2.8) Case (2.2.6).

We have $K_X + (n-1)L = 0$. Let M be the ample generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$. Then there exists a positive integer a such that $L = aM$. Thus $K_X + a(n-1)M = 0$. Since the index of the Fano manifold X is less than or equal to $\dim X + 1$, we have $a(n-1) \leq n+1$. Suppose that $a \geq 2$. Then $2(n-1) \leq n+1$, and so $n \leq 3$, contrary to assumption. Thus $a = 1$ and L is the ample generator of $\mathrm{Pic}(X)$. This implies that X is a Fano manifold of index $n-1$.

Since $m = n - 2$ in this case, we have by (1.4)

$$(n-2) \det \mathcal{E} = L - (n-3)K_X,$$

hence

$$(n-2) \det \mathcal{E} = L + (n-3)(n-1)L = (n-2)^2 L.$$

Thus $\det \mathcal{E} = (n-2)L$. Now let $\mathcal{E}' = \mathcal{E} \oplus L$. Then $\mathcal{E}'$ is an ample vector bundle of rank $n-1$ and we have $K_X + \det \mathcal{E}' = \mathcal{O}_X$, so that by [7], Main theorem 0.3 and Proposition 7.4 we see that $\mathcal{E}' \cong L^{\oplus(n-1)}$, and hence that $\mathcal{E} \cong L^{\oplus(n-2)}$. We are in case (9).

(2.9) Case (2.2.7).

By (1.4),

$$(n-2) \det \mathcal{E} = L - (n-3)K_X.$$

Let F be any general fiber of f. Then its canonical bundle is the restriction of K_X. Thus

$$\begin{aligned}(n-2) \det \mathcal{E}_F &= L_F - (n-3)K_F \\ &= \mathcal{O}_{\mathbb{Q}^{n-1}}(1) + (n-3)\mathcal{O}_{\mathbb{Q}^{n-1}}(n-1) \\ &= \mathcal{O}_{\mathbb{Q}^{n-1}}((n-2)^2),\end{aligned}$$

so $\det \mathcal{E}_F = \mathcal{O}_{\mathbb{Q}^{n-1}}(n-2)$. Since $\mathcal{E}_F \otimes \mathcal{O}_F(-1)$ is trivial on any line on $\mathbb{Q}^{n-1}$, $\mathcal{E}_F \otimes \mathcal{O}_F(-1)$ is trivial by [8], Lemma 3.6.1. Thus $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{Q}^{n-1}}(1)^{\oplus(n-2)}$ and we are in case (12).

(2.10) Case (2.2.8).

We can write $(X, L) = (\mathbb{P}_S(\mathcal{F}), H)$ for some ample vector bundle $\mathcal{F}$ on S, where H is the tautological line bundle on $\mathbb{P}_S(\mathcal{F})$. Let π be the projection $X \to S$. Then $K_X = -(n-1)H + \pi^*(K_S + \det \mathcal{F})$, and so $K_F + (n-1)H_F = \mathcal{O}_F$ for any fiber $F(\cong \mathbb{P}^{n-2})$ of π. We have by (1.4)

$$L = (n-3)K_X + (n-2)\det \mathcal{E},$$

hence

$$\begin{aligned}\mathcal{O}_F = K_F + (n-1)H_F &= K_F + (n-1)(n-3)K_F + (n-1)(n-2)\det \mathcal{E}_F \\ &= (n-1)(n-2)(K_F + \det \mathcal{E}_F) + (2-n)K_F,\end{aligned}$$

i.e., $(n-1)(n-2)(K_F + \det \mathcal{E}_F) = (n-2)K_F$. Thus $K_F + \det \mathcal{E}_F$ is not nef. Applying Theorem 0.2 to $\mathcal{E}_F$, we have $\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus(n-2)}$ and we are in case (13). This completes the proof.

REFERENCES

[1] M. Andreatta - E. Ballico - J. A. Wiśniewski, *Vector bundles and adjunction,* Internat. J. Math., 3 (1992), pp. 331–340.

[2] M. Andreatta - M. Mella, *Contractions on a manifold polarized by an ample vector bundle,* preprint.

[3] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive,* Algebraic Geometry, Sendai, 1985 (T. Oda,ed), Adv. Stud. Pure Math., vol.10, Kinokuniya, Tokyo, 1987, pp. 167–178.

[4] A. Lanteri - H. Maeda, *Ample vector bundles with sections vanishing on projective spaces or quadrics,* Internat. J. Math. (to appear).

[5] S. Mori, *Threefolds whose canonical bundles are not numerically effective,* Ann. of Math., 116 (1982), pp. 133–176.

[6] C. Okonek - M. Schneider - H. Spindler, *Vector Bundles on Complex Projective Spaces,* Progr. Math., vol.3, Birkhäuser, Boston, 1980.

[7] T. Peternell - M. Szurek - J. A. Wiśniewski, *Fano manifolds and vector bundles,* Math. Ann., 294 (1992), pp. 151–165.

[8] J. A. Wiśniewski, *Length of extremal rays and generalized adjunction,* Math. Z., 200 (1989), pp. 409–427.

[9] Y.-G. Ye - Q. Zhang, *On ample vector bundles whose adjunction bundles are not numerically effective,* Duke Math. J., 60 (1990), pp. 671–687.

[10] Q. Zhang, *A theorem on the adjoint system for vector bundles,* Manuscripta Math., 70 (1991), pp. 189–201.

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