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GROTHENDIEK OPERATORS ON THE PROJECTIVE TENSOR PRODUCTS OF SPACES

DUMITRU POPA

In the paper it is proved that if $L(X, Y^*) = K(X, Y^*)$, $U \in L(X\widehat{\otimes}Y, Z)$ is such that $U^\# x \in Gr(Y, Z)$ for each $x \in X$ and $U^\# : X \to Gr(Y, Z)$ is a Grothendieck operator, then $U \in Gr(X\widehat{\otimes}Y, Z)$.

In the sequel by X, Y, Z we denote Banach spaces, $L(X, Y)$ will be the Banach space of all linear and continuous operators from X to Y equipped with the operator norm, $K(X, Y) \subset L(X, Y)$ will be the subspace of all compact operators. By $X\widehat{\otimes}Y$ we denote the projective tensor product of X and Y. As is well known $L(X, Y^*) = (X\widehat{\otimes}Y)^*$ (see [1]).

Let $U \in L(X\widehat{\otimes}Y, Z)$ be; for $x \in X$ we consider the operator

$$U^\# x : Y \to Z, (U^\# x)(y) = U(x \otimes y), y \in Y$$

which is evidently a linear and continuous operator from Y to Z and hence $U^\# : X \to L(Y, Z)$ is also linear and continuous. Similarly

$$U_\# : Y \to L(X, Z), (U_\# y)(x) = U(x \otimes y), x \in X, y \in Y$$

is linear and continuous.

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Recall that $U \in L(X, Y)$ is called a Grothendieck operator if its dual is weak*-weak sequentially continuous i.e.

$$y_n^* \to 0 \text{ weak}^* \Rightarrow U^*(y_n^*) \to 0 \text{ weak}.$$

We denote by $Gr(X, Y)$ the class of all Grothendieck operators from X to Y on which always we consider the operator norm. As is well-known Gr is an ideal of operators, see [3].

For each notations and notions used and not defined we refer the reader to [1].

Our main result is the following.

Theorem 1. *If $L(X, Y^*) = K(X, Y^*)$ and $U \in L(X\widehat{\otimes}Y, Z)$ is such that $U^{\#}x \in Gr(Y, Z)$ for each $x \in X$ and $U^{\#} : X \to Gr(Y, Z)$ is a Grothendieck operator then $U \in Gr(X\widehat{\otimes}Y, Z)$.*

Proof. For $z^* \in Z^*$ and $x \in X$ we denote by $y_1^* = [U^*(z^*)](x)$ and $y_2^* = z^* \circ U^{\#}x$. Evidently: $y_1^*(y) = y_2^*(y) = (z^*(U(x \otimes y)))$, for each $y \in Y$. As Y is weak*-dense in Y^{**} we obtain that: $y^{**}(y_1^*) = y^{**}(y_2^*)$ for each $y^{**} \in Y^{**}$ i.e.

$$[y^{**} \circ U^*(z^*)](x) = y^{**}(z^* \circ U^{\#}x). \tag{1}$$

Also for $z^* \in Z^*$, $y^{**} \in Y^{**}$ we denote by $T_{y^{**},z^*} : L(Y, Z) \to R(C)$ the linear and continuous functional defined by:

$$T_{y^{**},z^*}(V) = y^{**}(V^*(z^*)), \quad V \in L(Y, Z).$$

The relation (1) shows that:

$$T_{y^{**},z^*} \circ U^{\#} = y^{**} \circ U^*(z^*). \tag{2}$$

Let now $z_n^* \to 0$ weak* be. For each $V \in Gr(Y, Z)$ it follows that $V^*(z_n^*) \to 0$ weak, from where $y^{**}(V^*(z_n^*)) \to 0$ for each $y^{**} \in Y^{**}$ or $T_{y^{**},z_n^*}(V) \to 0$ i.e. $T_{y^{**},z_n^*} \to 0$ weak* (in $(Gr(Y, Z))^*$).

As $U^{\#} : X \to Gr(Y, Z)$ is a Grothendieck operator it follows that; $(U^{\#})^*(T_{y^{**},z_n^*}) \to 0$ weak i.e. $T_{y^{**},z_n^*} \circ U^{\#} \to 0$ weak; hence using (2) $y^{**} \circ U^*(z_n^*) \to 0$ weak for each $y^{**} \in Y^{**}$.

As $L(X, Y^*) = K(X, Y^*)$ this last relation shows that $U^*(z_n^*) \to 0$ weak in $L(X, Y^*)$ (see [4]) i.e. U is a Grothendieck operator.

Remark 2. Evidently the conclusion of the Theorem 1 is true if $L(X, Y^*) = K(X, Y^*)$ and $U_\# : Y \to Gr(X, Z)$ is a Grothendieck operator, because in this case $L(Y, X^*) = K(Y, X^*)$ (see [2] the proof of Theorem 6) and if we denote by $V : Y\widehat{\otimes}X \to Z$, $V(y \otimes x) = U(x \otimes y)$, $x \in X$, $y \in Y$, then $V^\# = U_\#$ and we can use Theorem 1.

The following corollary is an extension of Theorem 5 from [2].

Corollary 3. *If* $L(X, Y^*) = K(X, Y^*)$, $U \in Gr(X, X_1)$, $V \in Gr(Y, Y_1)$ *then* $U\widehat{\otimes}V \in Gr(X\widehat{\otimes}Y, X_1\widehat{\otimes}Y_1)$.

Proof. Let $T = U\widehat{\otimes}V$ be. For $x \in X$ let $A_x : Y_1 \to X_1\widehat{\otimes}Y_1$ be the operator $A_x(y_1) = (Ux) \otimes y_1$, $y_1 \in Y_1$. Evidently $T^\# x = A_x \circ V$. For $x_1 \in X_1$ let $B_{x_1} : Y_1 \to X_1\widehat{\otimes}Y_1$ be the operator $B_{x_1}(y_1) = x_1 \otimes y_1$. Also let $B : X_1 \to Gr(Y, X_1\widehat{\otimes}Y_1)$ be the operator $B(x_1)(y) = x_1 \otimes (Vy)$, $x_1 \in X_1$, $y \in Y$. We have $B(x_1) = B_{x_1} \circ V$. Hence $T^\# = B \circ U$. As U and V are Grothendieck operators using the ideal property of the class of all Grothendieck operators, the above relations show that T verifies the hypothesis of Theorem 1; hence T is a Grothendieck operator.

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Department of Mathematics,
University of Constanta,
8700 Constanta (ROMANIA)