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A STUDY PARAMETRIC ON MULTIOBJECTIVE NONLINEAR PROGRAMMING PROBLEMS

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This paper deals with multiobjective nonlinear programming problems, with parameters in the objective functions, without differentiability assumptions on the considered functions. The stability notions like the solvability set and stability sets of the first, second, third and fourth kind have been defined and analysed, without differentiability, for this problem.

1. Introduction.

The qualitative analysis of some basic notions like the set of feasible parameters, the solvability set and stability sets of the first and second kind were introduced in [4],[5]. Also, in [6],[7] the stability sets of the third and fourth kind for a general class of convex parametric programming problems were defined and analysed. The relations between multiobjective programming problems and parametric programs have been studied in [2],[3]. In [1] the parametric multiobjective programming problems without differentiability was studied.

In this paper, the solvability set and the stability sets of the first kind, second kind, third kind and fourth kind for multiobjective nonlinear programming problems without differentiability assumptions and with parameters in the objective functions are introduced.

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2. Problem Formulation.

Consider the following multiobjective nonlinear programming (MONLP) problem:

$$P(\lambda): \quad \min F_i(x, \lambda_i) = \left[f_i(x) + \lambda_i h_i(x)\right], \quad i = 1, 2, \ldots, m,$$
$$\text{subject to } M = \left\{x \in R^n \mid g_j(x) \leq 0, \quad j = 1, 2, \ldots, k\right\},$$

where $f_i(x)$, $h_i(x)$, $i = 1, 2, \ldots, m$ and $g_j(x)$, $j = 1, 2, \ldots, k$ are real valued functions convex on M and R^k, respectively, and $\lambda \in R^m$ is any vector parameter.

The efficient solution of $P(\lambda)$ can be characterized in terms of the optimal solutions of the following scalarization of problem $P(\lambda)$:

$$P(w, \lambda): \quad \min \sum_{i=1}^{m} w_i\left[f_i(x) + \lambda_i h_i(x)\right]$$
$$\text{subject to } x \in M,$$

where $w_i \geq 0$, $i = 1, 2, \ldots, m$ and $w = (w_1, w_2, \ldots, w_m) \neq 0$, $\sum_{i=1}^{m} w_i = 1$.

3. The solvability set.

Definition 1. The solvability set of problem $P(\lambda)$, which is denoted by B, is defined by

$$B = \left\{(w, \lambda) \in R^{2m} \mid \text{problem } P(\lambda) \text{ has efficient solutions}\right\}.$$

Let

$$E(w, \lambda) = \left\{x^* \in R^n \mid \sum_{i=1}^{m} w_i\left[f_i(x^*) + \lambda_i h_i(x^*)\right] = \right.$$
$$\left. = \min_{x \in M} \sum_{i=1}^{m} w_i\left[f_i(x) + \lambda_i h_i(x)\right]\right\}.$$

Theorem 1. *The set $E(w, \lambda)$ is convex and closed.*

Proof. Convexity. Let $x^1, x^2 \in E(w, \lambda)$, then the convexity of M and the convexity of the functions $f_i(x)$ and $h_i(x)$, $i = 1, 2, \ldots, m$, yields

$$\sum_{i=1}^{m} w_i\big[f_i(\gamma x^1 + (1-\gamma)x^2) + \lambda_i h_i(\gamma x^1 + (1-\gamma)x^2)\big] \leq$$
$$\leq \gamma \sum_{i=1}^{m} w_i(f_i(x^1) + \lambda_i h_i(x^1)) + (1-\gamma)\sum_{i=1}^{m} w_i(f_i(x^2) + \lambda_i h_i(x^2)) =$$
$$= \min_{x\in M} \sum_{i=1}^{m} w_i(f_i(x) + \lambda_i h_i(x)), \quad 0 < \gamma < 1.$$

Thus $\gamma x^1 + (1-\gamma)x^2 \in E(w, \lambda)$.
Closedness. Let $\{x^k\} \subseteq E(w, \lambda)$ be a sequence of points which converges to $\bar{x}$. Then

$$\sum_{i=1}^{m} w_i\big[f_i(x^k) + \lambda_i h_i(x^k)\big] = \min_{x\in M} \sum_{i=1}^{m} w_i\big[f_i(x) + \lambda_i h_i(x)\big],$$
$$\lim_{k\to\infty} \sum_{i=1}^{m} w_i\big[f_i(x^k) + \lambda_i h_i(x^k)\big] = \min_{x\in M} \sum_{i=1}^{m} w_i\big[f_i(x) + \lambda_i h_i(x)\big].$$

From the finiteness of the sum and continuity of the functions $f_i(x)$ and $h_i(x)$, $i = 1, 2, \ldots, m$, we have

$$\sum_{i=1}^{m} w_i\big[f_i(\lim_{k\to\infty} x^k) + \lambda_i h_i(\lim_{k\to\infty} x^k)\big] = \sum_{i=1}^{m} w_i\big[f_i(\bar{x}) + \lambda_i h_i(\bar{x})\big] =$$
$$= \min_{x\in M} \sum_{i=1}^{m} w_i\big[f_i(x) + \lambda_i h_i(x)\big].$$

Thus $\bar{x} \in E(w, \lambda)$.

Theorem 2. *If the set $E(w, \lambda)$ is bounded, then the set B is convex.*

Proof. From the boundedness and closedness of $E(w, \lambda)$ (Theorem 1) and the continuity of the functions $f_i(x)$ and $h_i(x)$, $i = 1, 2, \ldots, m$, exist $\bar{x} \in E(w, \lambda)$ with $f_i(\bar{x}) \leq f_i(x^0)$ and $h_i(\bar{x}) \leq h_i(x^0)$ $\forall x^0 \in E(w, \lambda)$. Assume that $(w^1, \lambda^1), (w^2, \lambda^2) \in B$, then there are $x^1, x^2 \in E(w, \lambda)$ such that

$$\sum_{i=1}^{m} w_i^1\big[f_i(x^1) + \lambda_i^1 h_i(x^1)\big] = \min_{x\in M} \sum_{i=1}^{m} w_i^1\big[f_i(x) + \lambda_i^1 h_i(x)\big]$$

and

$$\sum_{i=1}^{m} w_i^2\left[f_i(x^2) + \lambda_i^2 h_i(x^2)\right] = \min_{x \in M} \sum_{i=1}^{m} w_i^2\left[f_i(x) + \lambda_i^2 h_i(x)\right].$$

Then for each $\gamma \in [0, 1]$ we get

$$\sum_{i=1}^{m}(\gamma w_i^1 + (1-\gamma)w_i^2)\left[f_i(\bar{x}) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(\bar{x})\right] \leq$$

$$\leq \gamma \sum_{i=1}^{m} w_i^1\left[f_i(\bar{x}) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(\bar{x})\right] +$$

$$+ (1-\gamma) \sum_{i=1}^{m} w_i^2\left[f_i(\bar{x}) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(\bar{x})\right] =$$

$$= \min_{x \in M} \sum_{i=1}^{m}(\gamma w_i^1 + (1-\gamma)w_i^2)\left[f_i(x) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(x)\right].$$

Hence

$$\left[(\gamma w^1 + (1-\gamma)w^2), (\gamma\lambda^1 + (1-\gamma)\lambda^2)\right] \in B.$$

Theorem 3. *The set B is closed.*

Proof. Consider the sequence $\left\{(w^k, \lambda^k)\right\} \subseteq B$ converging to (w^0, λ^0). If $(w^0, \lambda^0) \notin B$, then we can find

$$\sum_{i=1}^{m} w_i^0\left[f_i(x^0) + \lambda_i^0 h_i(x^0)\right] > \sum_{i=1}^{m} w_i^0\left[f_i(x) + \lambda_i^0 h_i(x)\right],$$

for this inequality we consider the neighbourhood

$$N_{(w^0,\lambda^0)}(\delta) = \Big\{(w, \lambda) \mid \|w - w^0\| < \delta \quad \text{and} \quad \|\lambda - \lambda^0\| < \delta;$$
$$\sum_{i=1}^{m} w_i\left[f_i(x^0) + \lambda_i h_i(x^0)\right] > \sum_{i=1}^{m} w_i\left[f_i(x) + \lambda_i h_i(x)\right]\Big\}$$

of (w^0, λ^0), $\delta > 0$. As we have $\left\{(w^k, \lambda^k)\right\} \to (w^0, \lambda^0)$ there exists j such that $(w^j, \lambda^j) \in N_{(w^0,\lambda^0)}(\delta)$ and

$$\sum_{i=1}^{m} w_i^j\left[f_i(x^0) + \lambda_i^j h_i(x^0)\right] > \sum_{i=1}^{m} w_i^j\left[f_i(x) + \lambda_i^j h_i(x)\right],$$

contradicting that $(w^j, \lambda^j) \in B$.

4. Stability set of the first kind.

Definition 2. Suppose that $(\bar{w}, \bar{\lambda}) \in B$ with a corresponding efficient solution $\bar{x} \in E(w, \lambda)$; then the stability set of the first kind of $P(\lambda)$ corresponding to $\bar{x}$, which is denoted by $K_1(\bar{x})$, is defined by $K_1(\bar{x}) = \{(w, \lambda) \in B \mid \bar{x} \in E(w, \lambda)$ is an efficient solution of $P(\lambda)\}$.

Theorem 4. *The set $K_1(\bar{x})$ is convex.*

Proof. Let $(w^1, \lambda^1), (w^2, \lambda^2) \in K_1(\bar{x})$, then

$$\sum_{i=1}^{m} w_i^1[f_i(\bar{x}) + \lambda_i^1 h_i(\bar{x})] \leq \sum_{i=1}^{m} w_i^1[f_i(x) + \lambda_i^1 h_i(x)] \quad \forall x \in M,$$

and

$$\sum_{i=1}^{m} w_i^2[f_i(\bar{x}) + \lambda_i^2 h_i(\bar{x})] \leq \sum_{i=1}^{m} w_i^2[f_i(x) + \lambda_i^2 h_i(x)] \quad \forall x \in M.$$

Therefore,

$$\sum_{i=1}^{m}(\gamma w_i^1 + (1-\gamma)w_i^2)[f_i(\bar{x}) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(\bar{x})] \leq$$

$$\leq \sum_{i=1}^{m}(\gamma w_i^1 + (1-\gamma)w_i^2)[f_i(x) + (\gamma\lambda_i^1 + (1-\gamma)\lambda_i^2)h_i(x)]$$

$\forall x \in M$ and $0 \leq \gamma \leq 1$, i.e.,

$$[(\gamma w^1 + (1-\gamma)w^2), (\gamma\lambda^1 + (1-\gamma)\lambda^2)] \in K_1(\bar{x}),$$

$0 \leq \gamma \leq 1$, and hence the result follows.

Theorem 5. *The set $K_1(\bar{x})$ is a cone with vertex at $(w, \lambda) = (0, 0)$.*

Proof. It is clear that $(w, \lambda) = (0, 0) \in K_1(\bar{x})$. Suppose that at $(\tilde{w}, \tilde{\lambda}) \in K_1(\bar{x})$; then

$$\sum_{i=1}^{m} \tilde{w}_i[f_i(\bar{x}) + \tilde{\lambda}_i h_i(\bar{x})] \leq \sum_{i=1}^{m} \tilde{w}_i[f_i(x) + \tilde{\lambda}_i h_i(x)] \quad \forall x \in M,$$

and then

$$\sum_{i=1}^{m} \bar{w}_i[f_i(\bar{x}) + \bar{\lambda}_i h_i(\bar{x})] \leq \sum_{i=1}^{m} \bar{w}_i[f_i(x) + \bar{\lambda}_i h_i(x)] \quad \forall x \in M,$$

where $\bar{w}_i = \gamma\tilde{w}_i, \bar{\lambda}_i = \gamma\tilde{\lambda}_i, \gamma \geq 0$, i.e.,

$$(\bar{w}, \bar{\lambda}) = (\gamma\tilde{w}, \gamma\tilde{\lambda}) \in K_1(\bar{x})\ \forall \gamma \geq 0,$$

and hence the result follows.

5. Stability set of the second kind.

Definition 3. Suppose that $(\bar{w}, \bar{\lambda}) \in B$ and $J \subset \{1, 2, \ldots, k\}$. Let $\sigma(\bar{w}, \bar{\lambda}, J)$ denote the side of constraints defined by

$$\sigma(\bar{w}, \bar{\lambda}, J) = \left\{x \in R^n \mid g_j(x) = 0, \text{ for } j \in J; \text{ and } g_j(x) < 0, \text{ for } j \notin J\right\},$$

then the stability set of the second kind of $P(\lambda)$ corresponding to $\sigma(\bar{w}, \bar{\lambda}, J)$, denoted by $K_2(\sigma(\bar{w}, \bar{\lambda}, J))$, is defined by

$$K_2(\sigma(\bar{w}, \bar{\lambda}, J)) = \left\{(w, \lambda) \in B \mid K_2(\sigma(\bar{w}, \bar{\lambda}, J)) \text{ contains an efficient solution of } P(\lambda)\right\}.$$

Proposition 1. *If the functions $f_i(x)$ and $h_i(x)$, $i = 1, 2, \ldots, m$, are strictly convex on M, and $J_1 \neq J_2$, then $K_2(\sigma(w^1, \lambda^1, J_1)) \cap K_2(\sigma(w^2, \lambda^2, J_2)) = \emptyset$.*

Proof. Suppose that $(\bar{w}, \bar{\lambda}) \in K_2(\sigma(w^1, \lambda^1, J_1)) \cap K_2(\sigma(w^2, \lambda^2, J_2))$, then

$$E(\bar{\omega}, \bar{\lambda}) \cap \sigma(w^1, \lambda^1, J_1) \neq \emptyset \quad \text{and} \quad E(\bar{\omega}, \bar{\lambda}) \cap \sigma(w^2, \lambda^2, J_2) \neq \emptyset,$$

which is a contradiction since $E(\bar{\omega}, \bar{\lambda})$, by the assumption, is only a single point.

Remark 1. From the Definitions 2 and 3 we have

$$K_2(\sigma(w, \lambda, J)) = \bigcup_{i \in D} K_1(x^i),$$

where

$$D = \left\{i \mid x^i \in \sigma(\bar{w}, \bar{\lambda}, J) \text{ is an efficient solution of } P(\lambda)\right\}.$$

Corollary 1. *If D is a finite set, then the set $K_2(\bar{\sigma}(\bar{w}, \lambda, J)) \cup \{0\}$ is a closed cone.*

Proof. Follows from Remark 1 and Theorem 5.

Theorem 6. *The set $K_2(\sigma(\bar{w}, \bar{\lambda}, J)) \cup \{0\}$ is star shaped* [4], *with the point $(w, \lambda) = (0, 0)$ as its common visibility point.*

Proof. Let $(w, \lambda) \in K_2(\sigma(\bar{w}, \bar{\lambda}, J))$, then from Remark 1, $(w, \lambda) \in K_1(x^s)$ for at least one index $s \in D$. Then $(\gamma w, \gamma\lambda) \in K_1(x^s) \cup \{0\}$, $\gamma \geq 0$ from the convexity of $K_1(x^s)$, i.e., $(\gamma w, \gamma\lambda) \in K_2(\sigma(\bar{w}, \bar{\lambda}, J))$, and hence the result follows.

Remark 2. We have, from Definition 3, that

$$K_2(\bar{\sigma}(\bar{w}, \bar{\lambda}, J)) = \bigcup_{i \in \tau} K_2(\sigma(w^i, \lambda^i, J_i)),$$

where

$$\tau = \left\{ i \mid J \leq J_i \text{ and } \bar{\sigma}(\bar{w}, \bar{\lambda}, J) \quad \text{is the closure of} \quad \sigma(\bar{w}, \bar{\lambda}, J) \right\}.$$

Theorem 7. *If $f_i(x)$ and $h_i(x)$, $i = 1, 2, \ldots, m$, are continuous and strictly convex on M, and $K_2(\sigma(\bar{w}, \bar{\lambda}, J)) \subset B \cup \{0\}$, then*

$$\overline{K}_2(\sigma(\bar{w}, \bar{\lambda}, J)) \subset K_2(\bar{\sigma}(\bar{w}, \bar{\lambda}, J)) \cup \{0\},$$

where $K_2(\bar{\sigma}(\bar{w}, \bar{\lambda}, J))$ and $\bar{K}_2(\sigma(\bar{w}, \bar{\lambda}, J))$ are, respectively, the boundary and the closure of $K_2(\sigma(\bar{w}, \bar{\lambda}, J))$.

Proof. If either $K_2(\sigma(\bar{w}, \bar{\lambda}, J))$ is closed or $K_2(\sigma(\bar{w}, \bar{\lambda}, J)) = k_1(\bar{x})$, the result is clear.

Let (w^*, λ^*) be a boundary point of $K_2(\sigma(\bar{w}, \bar{\lambda}, J))$; if $(w^*, \lambda^*) = (0, 0)$ the result is clear. Otherwise, choose a sequence $(w^{(n)}, \lambda^{(n)}) \geq (0, 0)$ which converge to (w^*, λ^*) such that $(w^{(n)}, \lambda^{(n)}) \in K_2(\sigma(\bar{w}, \bar{\lambda}, J))$ with corresponding efficient solutions $x^{(n)} \in K_2(\sigma(\bar{w}, \bar{\lambda}, J))$. Then

$$\sum_{i=1}^{m} w_i^{(n)} \left[f_i(x^{(n)}) + \lambda_i^{(n)} h_i(x^{(n)}) \right] \leq \sum_{i=1}^{m} w_i^{(n)} \left[f_i(x) + \lambda_i^{(n)} h_i(x) \right]$$

$\forall x \in M$ and n. Therefore,

$$\lim_{n \to \infty} \sum_{i=1}^{m} w_i^{(n)} \left[f_i(x^{(n)}) + \lambda_i^{(n)} h_i(x^{(n)}) \right] \leq \lim_{n \to \infty} \sum_{i=1}^{m} w_i^{(n)} \left[f_i(x) + \lambda_i^{(n)} h_i(x) \right],$$

$\forall x \in M$. From the finiteness of the sum and the continuity of f_i and h_i, $i = 1, 2, \ldots, m$, it follows that:

$$\sum_{i=1}^{m} w_i^* \left[f_i(\lim_{n \to \infty} x^{(n)}) + \lambda_i^* h_i(\lim_{n \to \infty} x^{(n)}) \right] \leq \sum_{i=1}^{m} w_i^* \left[f_i(x) + \lambda_i^* h_i(x) \right] \forall x \in X,$$

i.e.,

$$\sum_{i=1}^{m} w_i^* \left[f_i(x^*) + \lambda_i^* h_i(x^*) \right] \leq \sum_{i=1}^{m} w_i^* \left[f_i(x) + \lambda_i^* h_i(x) \right]$$

$\forall x \in M$ where $\lim_{n \to \infty} x^{(n)} = x^*$ exists since $B \cup \{0\}$ is closed, and it is an efficient by the fact that $E(w^*, \lambda^*) = \{x^*\}$. Therefore, $x^* \in \bar{\sigma}(\bar{w}, \bar{\lambda}, J)$ and hence the result follows.

6. Stability set of the third kind.

Definition 4. Suppose that the problem $P(\lambda)$ is solvable with a corresponding efficient solution $\bar{x}$, x^* is any feasible point, and $\delta > 0$; then the stability set of the third kind of $P(\lambda)$, which is denoted by $K_3(\bar{w}, \bar{\lambda}, x^*, \delta)$, is defined by

$$K_3(\bar{w}, \bar{\lambda}, x^*, \delta) = \left\{(w, \lambda) \in R^{2m} \mid \|\psi(x^*, w, \lambda) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \delta\right\},$$

where

$$\psi(x, w, \lambda) = \sum_{i=1}^{m} w_i\left[f_i(x) + \lambda_i h_i(x)\right].$$

Lemma 1. *The set $K_3(\bar{w}, \bar{\lambda}, x^*, \delta)$ is convex.*

Proof. Let (w^1, λ^1), $(w^2, \lambda^2) \in K_3(\bar{w}, \bar{\lambda}, x^* \delta)$, then

$$\|\psi(x^*, w^1, \lambda^1) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \delta, \quad \|\psi(x^*, w^2, \lambda^2) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \delta.$$

Therefore,

$$(1-\gamma)\|\psi(x^*, w^1, \lambda^1) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < (1-\gamma)\delta$$

and

$$\gamma\|\psi(x^*, w^2, \lambda^2) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \gamma\delta, \quad 0 < \gamma < 1.$$

Hence,

$$\begin{aligned}\|\psi(x^*, (\gamma w^1 + (1-\gamma)w^2), (\gamma\lambda^1 + (1-\gamma)\lambda^2)) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| \leq \\ \gamma\|\psi(x^*, w^1, \lambda^1) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| + (1-\gamma)\|\psi(x^*, w^2, \lambda^2) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \\ \gamma\delta + (1-\gamma)\delta = \delta.\end{aligned}$$

Then

$$\left[(\gamma w^1 + (1-\gamma)w^2), (\gamma\lambda^1 + (1-\gamma)\lambda^2)\right] \in K_3(\bar{w}, \bar{\lambda}, x^*, \delta),$$

and hence the result follows.

Now, under the assumption $\psi(x, w, \lambda) = \sum_{i=1}^{m} w_i[f_i(x) + \lambda_i h_i(x)]$, the determination of subset from the set $K_3(\bar{w}, \bar{\lambda}, x^*, \delta)$ is given as follows:

$$\|\psi(x^*, w, \lambda) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| =$$

$$\left\| \sum_{i=1}^{m} w_i[f_i(x^*) + \lambda_i h_i(x^*)] - \sum_{i=1}^{m} \bar{w}_i[f_i(\bar{x}) + \bar{\lambda}_i h_i(\bar{x})] \right\| =$$

$$\left\| \sum_{i=1}^{m} \bar{w}_i[f_i(x^*) + \bar{\lambda} h_i(x^*)] - \sum_{i=1}^{m} \bar{w}_i[f_i(x^*) + \bar{\lambda}_i h_i(x^*)] + \right.$$

$$\left. + \sum_{i=1}^{m} w_i[f_i(x^*) + \lambda_i h_i(x^*)] - \sum_{i=1}^{m} \bar{w}_i[f_i(\bar{x}) + \bar{\lambda}_i h_i(\bar{x})] \right\| =$$

$$\left\| \sum_{i=1}^{m} (w_i - \bar{w}_i) f_i(x^*) + \sum_{i=1}^{m} (w_i\lambda_i - \bar{w}_i\bar{\lambda}_i) h_i(x^*) + \right.$$

$$\left. + \sum_{i=1}^{m} \bar{w}_i[f_i(x^*) - f_i(\bar{x})] + \sum_{i=1}^{m} \bar{w}_i\bar{\lambda}_i(h_i(x^*) - h_i(\bar{x})) \right\| \leq$$

$$\sum_{i=1}^{m} \{\|w_i - \bar{w}_i\| \, \|f_i(x^*)\| + \|w_i\lambda_i - \bar{w}_i\bar{\lambda}_i\| \, \|h_i(x^*)\| +$$

$$+ \|\bar{w}_i\| \, \|f_i(x^*) - f_i(\bar{x})\| + \|\bar{w}_i\bar{\lambda}_i\| \, \|h_i(x^*) - h_i(\bar{x})\|\} < \delta,$$

$$\sum_{i=1}^{m} \{\|w_i - \bar{w}_i\| \, \|f_i(x^*)\| + \|w_i\lambda_i - \bar{w}_i\bar{\lambda}_i\| \, \|h_i(x^*)\|\} <$$

$$\delta - \sum_{i=1}^{m} \{\|\bar{w}_i\| \, \|f_i(x^*) - f_i(\bar{x})\| + \|\bar{w}_i\bar{\lambda}_i\| \, \|h_i(x^*) - h_i(\bar{x})\|\}.$$

If $I(x^*)$ denotes the set

$$I(x^*) = \Big\{(w, \lambda) \in R^{2m} \mid \sum_{i=1}^{m} \{\|w_i - \bar{w}_i\| \|f_i(x^*)\| + $$
$$+ \|w_i\lambda_i - \bar{w}_i\bar{\lambda}_i\| \|h_i(x^*)\|\} < \varepsilon\Big\},$$

then $I(x^*) \subset K_3(\bar{w}, \bar{\lambda}, x^*, \delta)$. In order that $I(x^*) \neq \emptyset$, then it is clear that either δ is large or $\sum_{i=1}^{m} \|f_i(x^*) - f_i(\bar{x})\|$ and $\sum_{i=1}^{m} \|h_i(x^*) - h_i(\bar{x})\|$ are sufficiently small.

Remark 3. We note that $I(x^*) \subset K_1(\bar{x})$.

7. Stability set of the fourth kind.

Definition 5. Suppose that the problem $P(\lambda)$ is solvable at $(\bar{w}, \bar{\lambda}) \in B$ with a corresponding noninferior solution $\bar{x}$, and $\delta > 0$, then the stability set of the fourth kind of $P(\lambda)$, which is denoted by $K_4(\bar{w}, \bar{\lambda}, \delta)$, is defined by

$$K_4(\bar{w}, \bar{\lambda}, \delta) = \left\{(w, \lambda) \in R^{2m} \mid \exists x \in M,\ \|\psi(x; w, \lambda) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \delta\right\}.$$

Lemma 2. *The set $K_4(\bar{w}, \bar{\lambda}, \delta)$ is convex in λ and closed in x.*

Proof. The first part of the proof is clear from Lemma 1 at any feasible point $x \in M$. To prove the second part, let $\tilde{x}_n \in K_4(\bar{w}, \bar{\lambda}, \delta)$, $n = 1, 2, \ldots$, be a sequence of points which converges to $\tilde{x}$; then $\|\psi(\tilde{x}_n, w^*, \lambda^*) - \psi(\bar{x}; \bar{w}, \bar{\lambda})\| < \delta$, and $\|\psi(\tilde{x}_n, w^*, \lambda^*) - \psi(\tilde{x}, w^*, \lambda^*)\| \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}
&\|\psi(\tilde{x}, w^*, \lambda^*) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| = \\
&\|\psi(\tilde{x}, w^*, \lambda^*) - \psi(\tilde{x}_n, w^*, \lambda^*) + \psi(\tilde{x}_n, w^*, \lambda^*) - \psi(\bar{x}, \bar{w}, \bar{\lambda}\| \leq \\
&\|\psi(\tilde{x}, w^*, \lambda^*) - \psi(\tilde{x}_n, w^*, \lambda^*)\| + \|\psi(\tilde{x}_n, w^*, \lambda^*) - \psi(\bar{x}, \bar{w}, \bar{\lambda})\| < \delta,
\end{aligned}$$

which means that $\tilde{x} \in K_4(\bar{w}, \bar{\lambda}, \delta)$. Hence, the result follows.

Remark 4. It is clear that $K_3(\bar{w}, \bar{\lambda}, x^*, \delta) \subset K_4(\bar{w}, \bar{\lambda}, \delta)$.

Following the same steps as those for determining $I(x^*)$, it is clear that if we define $V = \bigcup_{x \in M} I(x)$, then $V \subset K_4(\bar{w}, \bar{\lambda}, \delta)$.
A simple expression for V can be deduced in the following case. If $f_i(x)$ and $h_i(x)$ are construction mapping on M, i.e., there exist a proper fraction q such that $\|f(x^*) - f(\bar{x})\| \leq q\|x - \bar{x}\|$ and proper fraction p such that $\|h(x^*) - h(\bar{x})\| \leq p\|x - \bar{x}\|$, then using Cauchy's inequality [8], it follows that

$$\sum_{i=1}^{m} \left\{Z_i \|w_i - \bar{w}_i\| + Y_i \|w_i\lambda_i - \bar{w}_i\bar{\lambda}_i\|\right\} < \delta - \sum_{i=1}^{m} \|x - \bar{x}\| \left\{q \|\bar{w}_i\| + p\|\bar{w}_i\bar{\lambda}_i\|\right\} = \beta(x).$$

where $Z_i = \|f_i(x^*)\|$, $Y_i = \|h_i(x^*)\|$.
If $I'(x)$ denotes the set

$$I'(x) = \left\{(w, \lambda) \in R^{2m} \mid \sum_{i=1}^{m} \{Z_i \|w_i - \bar{w}_i\| + Y_i \|w_i\lambda_i - \bar{w}_i\bar{\lambda}_i\|\} < \beta(x)\right\},$$

then

$$V' = \bigcup_{x \in M} I'(x).$$

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