MAXIMUM NUMBER OF GENERATORS OF AN IDEAL OF POINTS ON AN IRREDUCIBLE SURFACE OF LOW DEGREE

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Let \( X \) be a finite set of distinct points lying on an irreducible surface with Hilbert Function \( H(X, i) \). In this paper we will prove that the maximum number of generators, in each degree, of the ideal defining \( X \) is obtained by the second difference of the Hilbert Function.

Introduction.

Let \( Y \) be a finite set of distinct points in the projective space \( \mathbb{P}^2_k \), \( k \) algebraically closed field, with Hilbert Function \( H(Y, i) \) and let \( \mathcal{I} \) be the defining ideal of \( Y \).

In [3] Dubreil's Theorem says that the number \( \nu(\mathcal{I}) \) of generators of \( \mathcal{I} \) is bounded by:
\[
\nu(\mathcal{I}) \leq a(\mathcal{I}) + 1
\]
where \( a(\mathcal{I}) \) is the smallest degree of a generator of \( \mathcal{I} \).

The next step was done in [2] and in [8] (Theorem 1.2); if \( \mathcal{a}_i \) is the number of generators of degree \( i \) of the ideal \( \mathcal{I} \), then
\[
\min\{-\Delta^3 H(Y, i), 0\} \leq \mathcal{a}_i \leq -\Delta^2 H(Y, i).
\]

Let \( V \) be a non degenerate arithmetically Cohen-Macaulay projective variety in \( \mathbb{P}^r \) and let \( \mathcal{I} \) be the defining ideal of \( V \). In [12], Valla finds upper bounds for

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the Betti numbers of $I$ when: i) $V$ ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension and degree
and when: ii) $V$ ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension, degree and initial degree.
In particular, when $r = 3$ and $V$ is a set $X$ of points in $\mathbb{P}^3$ one obtains bounds depending on the number of points and on the degree of the first surface containing $X$.

In [4] and [1] it is proved that among all homogeneous ideals with a given Hilbert Function the lex-segment ideal has the maximum Betti numbers. Hence, one can compute the maximum number of generators of an ideal with a giver Hilbert Function (see, for instance, [11]). Using these results, one can obtain bounds for the number of graded generators of the defining ideal $I$ of $X$ in terms of the Hilbert Function and we can note that the maximum number of generators is obtained when the minimal surface containing $X$ is reducible.

One of the main questions on this line is to find bounds for the Betti numbers of a set of points in Uniform Position in $\mathbb{P}^3$. A beginning step of this direction is to obtain bounds for a set $X$ lying on an irreducible surface of degree $s$. Here we treat the cases $s = 2, 3, 4$ and we improve the bounds found in [4].

For $s = 2$, denoting $\Sigma$ the irreducible quadric containing $X$, in [9] it is proved that $\Delta H(X, i)$ is not increasing for $n \geq b$, where $b \geq 2$ is the least degree of a surface containing $X$ but not $\Sigma$; so the behaviour of the Hilbert Function of a set of points in $\mathbb{P}^2_k$ on an irreducible quadric is analogous to the behaviour of the Hilbert Function of a set of points in $\mathbb{P}^2_k$. The same thing happens for the maximum number of generators of the ideal defining a set of points lying on an irreducible quadric. Particularly, if $X$ lies on an irreducible quadric, we prove that

$$\alpha_i \leq -\Delta^2 H(X, i)$$

where $\alpha_i$ is the number of generators of degree $i$ of the ideal defining $X$ and $i$ runs all over the indices in which the number of generators is not determined by the Hilbert Function.

For $s = 3$, if $\Sigma$ is the irreducible cubic surface containing $X$, we obtain

$$\alpha_i \leq \begin{cases} \max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) & i = b + 1 \\ -\Delta^2 H(X, i) & i > b + 1, \end{cases}$$

where $b$ is the least degree of a surface passing through $X$ but not through $\Sigma$.

Similar results are obtained for points on an irreducible surface of degree 4; particularly, in these two cases we improve some result given in [10].

The paper consists of five sections; in the first, the binomial expansion of natural numbers, some related functions and the theory of lex-segment ideal are
introduced; moreover, the definition of graded Betti numbers is given to use the results of [4] and [11].

Section 2, 3, 4 treat, respectively, sets of points on an irreducible surface of degree 2, 3, 4.

In the last section we give some applications of our results.

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1. Preliminaries and notation.

Let $X$ be a finite set of distinct points in $\mathbb{P}^n_k$, $k$ algebraically closed field of characteristic zero. Let $R = k[X_1, \ldots, X_{n+1}]$ and let $I(X)$ be the homogeneous ideal of $R$ that defines the variety $X$.

**Definition 1.1.** For every homogeneous ideal $I$ of $R$, the Hilbert Function of the standard graded Algebra $A = R/I = \oplus_{i \geq 0} A_i$ is the numerical function defined by:

$$H_A(t) = \dim_k (A_t).$$

Particularly, we denote by $H(X, i)$ the Hilbert Function of $X$, that is:

$$H(X, i) = H_{R/I(X)}(i) = \dim_k [R/I(X)]_i.$$

We will use the following terminology:

$$\Delta H(X, i) = H(X, i) - H(X, i - 1)$$

$$\Delta^j H(X, i) = \Delta^{j-1} H(X, i) - \Delta^{j-1} H(X, i - 1) \text{ for all } i \in \mathbb{N} \text{ and } j \geq 1.$$

Let

(1) \hspace{1cm} 0 \rightarrow \oplus R(-j)^{q_0} \rightarrow \cdots \rightarrow \oplus R(-j)^{q_j} \rightarrow \\
\hspace{1cm} \rightarrow \oplus R(-j)^{q_j} \rightarrow R \rightarrow R/I \rightarrow 0

be a minimal graded free resolution of $R/I$.

If $V$ is a variety, we say that $V$ is arithmetically Cohen-Macaulay (ACM for short), if its homogeneous coordinate ring $R/I(V)$ is Cohen-Macaulay.

If $W$ is a $k$-vector space and $u_1, u_2, \ldots, u_t$ are elements of $W$ we will indicate with $\mathcal{L}(u_1, u_2, \ldots, u_t)$ the subspace of $W$ generated by these elements.

Some known result

Now we collect some known result about the maximum Betti numbers of homogeneous ideals and ideals defining a variety of points in $\mathbb{P}^3$ with a given Hilbert
Function, these maximum numbers are given in Theorem 1.9 and Remark 1.12 respectively.

If \( m, k \) are non-negative integers we will use the notation: \( \binom{m}{0} = 1 \) for any \( m \geq 0 \) and \( \binom{m}{k} = 0 \) if \( m < k \).

**Lemma 1.2.** Let \( m, i \) be positive integers. Then \( m \) can be uniquely written as

\[
m = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j}
\]

where \( m(i) > m(i-1) > \cdots > m(j) \geq j \geq 1 \).

**Proof.** See [11] Lemma 4.1. \( \square \)

**Definition 1.3.** The unique expression \( m = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j} \) where \( m(i) > m(i-1) > \cdots > m(j) \geq j \geq 1 \) is called the binomial expansion of \( m \) in base \( i \) and it is denoted by \( m_i \).

**Definition 1.4.** Let \( m, i \) be positive integers and let \( m_i = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j} \). Then we define:

\[
(m_i)_+ = \binom{m(i)+1}{i+1} + \binom{m(i-1)+1}{i} + \cdots + \binom{m(j)+1}{j+1}
\]

We have \( (m_i)_+ \geq m \), in fact:

\[
(m_i)_+ = \sum_{t=j}^{i} \binom{m(t)+1}{t+1} =
\]

\[
= \sum_{t=j}^{i} \left[ \binom{m(t)}{t+1} + \binom{m(t)}{t} \right] =
\]

\[
= \sum_{t=j}^{i} \binom{m(t)}{t+1} + m \geq m.
\]

**Definition 1.5.** In \( R = k[X_1, \ldots, X_{n+1}] \), we say that

\[
X_1^{a_1} X_2^{a_2} \cdots X_n^{a_{n+1}} \preceq \text{lex} \ X_1^{b_1} X_2^{b_2} \cdots X_{n+1}^{b_{n+1}}
\]

in the lexicographic ordering if either \( a_1 < b_1 \) or there exists an index \( i \) such that \( a_1 = b_1, \ldots, a_{i-1} = b_{i-1} \) and \( a_i < b_i \).
Definition 1.6. Let $T_d \subseteq R$ be the set of monomials of degree $d$, a subset $M$ of $T_d$ is called a lexicographic segment (for short lex-segment), if there exist $t_1, t_2 \in T_d$ such that $t \in M$ iff $t_1 \leq t \leq t_2$.

Definition 1.7. $J$ is called a lexicographic segment ideal (for short lex-segment ideal) if one can pick a set of generators such that the generators of minimum degree $d_0$ form a lexicographic segment beginning with $x_1^{d_0}, x_1^{d_0-1}x_2, \ldots$ and the generators of $J$ in degree $d \geq d_0$ form a lexicographic segment starting with the largest monomial of degree $d$ not divisible by a generator of lower degree.

Definition 1.8. The rank of the $i$-th module in the resolution (1) is called the $i$-th Betti number of $I$ and is denoted by $q_i(R/I)$; the graded Betti number of $I$, $q_{ij}(R/I)$, is defined to be the rank of the degree $j$ graded $R$-module $\bigoplus R(-j)^{q_{ij}}$ for each $i, j$. Notice 

\[ q_i(R/I) = \sum_j q_{ij}(R/I). \]

Theorem 1.9. Let $J \subseteq R$ be a lex-segment ideal with a given Hilbert Function $H$ and let $I \subseteq R$ be any other homogeneous ideal with $H_{(R/J)} = H_{(R/I)} = H$. Then 

\[ q_{ij}(R/J) \geq q_{ij}(R/I) \quad \forall i, j. \]


Theorem 1.10. Let $H$ be a Hilbert Function and $J \subseteq R$ the unique lex-segment ideal such that $H_{(R/J)} = H$; then 

\[ q_{1j}(R/J) = (H(j - 1)_{j-1})^+ - H(j). \]


From [5] we know that monomial ideals in $\overline{R} = k[X_1, \ldots, X_n]$ can be lifted to ideals of distinct points in $\mathbb{P}^n$ with the same Betti numbers, then we obtain the following result:

Theorem 1.11. Let $X$ be a set of distinct points in $\mathbb{P}^n$ with Hilbert Function $H(X, i)$ and $J \subseteq \overline{R}$ be the lex-segment ideal such that $H_{(R/J)}(i) = \Delta H(X, i)$, then 

\[ q_{ij}(R/I(X)) \leq q_{ij}(\overline{R}/J) \quad \forall i, j. \]
Proof. Let (1) be the minimal graded free resolution of $A = R/I(X)$, we can suppose that $X_{n+1}$ be a regular element in $A$ and we can consider the ring

$$A/(X_{n+1}) = R/(I(X), X_{n+1}) = \frac{\overline{R}}{(I(X), X_{n+1})/(X_{n+1})}.$$ 

We put $\overline{I} = (I(X), X_{n+1})/(X_{n+1})$, it is easy to see that a minimal graded free resolution of $\overline{R}/\overline{I}$ in $\overline{R}$ is

$$0 \to \oplus \overline{R}(-j)^{q_{ij}} \to \cdots \to \oplus \overline{R}(-j)^{q_{1j}} \to \oplus \overline{R}(-j)^{q_{ij}} \to \overline{R} \to \overline{R}/\overline{I} \to 0,$$

so $H_{(\overline{R}/\overline{I})}(i) = \Delta H(X, i)$.

Using the Theorem 1.9, we get

$$q_{ij}(\overline{R}/J) \geq q_{ij}(\overline{R}/\overline{I}) \quad \forall i, j,$$

but

$$q_{ij}(R/I(X)) = q_{ij}(\overline{R}/\overline{I}),$$

this gives the conclusion. \qed

**Remark 1.12.** When $n = 3$ we know that $I(X)$ has a minimal graded free resolution:

$$0 \to \oplus R(-i)^{\gamma_i} \to \oplus R(-i)^{\beta_i} \to \oplus R(-i)^{\alpha_i} \to I(X) \to 0$$

where $\alpha_i, \beta_i$ and $\gamma_i$ are the graded Betti numbers and it is easy to prove that they are linked with the relation

$$-\alpha_i + \beta_i - \gamma_i = \Delta^4 H(X, i);$$

particularly:

$$\alpha_i = \dim_k ([I(X)]_i/R_1[I(X)]_{i-1}) =$$

$$= \dim_k [I(X)]_i - \dim_k R_1[I(X)]_{i-1}.$$ 

By the Theorem 1.11 we have:

$$\alpha_{i+1} \leq (\Delta H(X, i)_i)^\rightarrow i + 1 \quad \forall i \in \mathbb{N}.$$ 

We note that $-\Delta^2 H(X, i + 1) \leq (\Delta H(X, i)_i)^\rightarrow i + 1$, in fact:

$$-\Delta^2 H(X, i + 1) = -\Delta H(X, i + 1) + \Delta H(X, i) \leq$$

$$\leq -\Delta H(X, i + 1) + (\Delta H(X, i)_i)^\rightarrow.$$
In the following we denote $I = I(X)$.

An algebraic result

Afterwards we will use the following:

**Lemma 1.13.** Let $S = 0$, $S \in R = k[X_1, \ldots, X_{n+1}]$, be the equation of a hypersurface $\Sigma \subset \mathbb{P}^n$ of degree $s$ and $L_1, L_2$ two linear forms that define the hyperplanes $\pi_1, \pi_2$ in $\mathbb{P}^n$, such that:

$$\begin{cases}
\pi_1 \cap \pi_2 = r \\
r \not\subset \Sigma,
\end{cases}$$

then $(L_1, L_2, S)$ is a regular sequence.

**Proof.** Obviously $L_1$ is regular in $R$ and $L_2$ in $R/(L_1)$; the ideal $(L_1, L_2)$ is prime, so $S$ is regular in $R/(L_1, L_2)$ iff $S \not\in (L_1, L_2)$ and it is true because $r \subset \pi_1, \pi_2$ but $r \not\subset \Sigma$. \qed

2. Points on an irreducible quadric.

Let $X$ be a set of distinct points in $\mathbb{P}^3$ on an irreducible quadric surface $\Sigma$ defined by $Q = 0$, from [9] Theorem 2.2, we have

$$\Delta H(X, i) = \begin{cases}
2i + 1 & 0 \leq i \leq b - 1 \\
s_i & b \leq i \leq n \\
0 & i \geq n + 1
\end{cases}$$

such that $2b \geq s_b \geq s_{b+1} \geq \ldots \geq s_n \geq 1, b \geq 2$.

We have

$$\Delta^2 H(X, i) = \begin{cases}
1 & i = 0 \\
2 & 1 \leq i \leq b - 1 \\
s_b - 2b + 1 & i = b \\
s_i - s_{i-1} & b + 1 \leq i \leq n \\
-s_n & i = n + 1 \\
0 & i \geq n + 2
\end{cases}$$
\[ \Delta^4 H(X, i) = \begin{cases} 
1 & i = 0 \\
0 & i = 1 \\
-1 & i = 2 \\
0 & 3 \leq i \leq b - 1 \\
s_b - 2b - 1 & i = b \\
s_{b+1} - 3s_b + 4b & i = b + 1 \\
s_{b+2} - 3s_{b+1} + s_b - 2b + 1 & i = b + 2 \\
s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b + 3 \leq i \leq n \\
-3s_n + 3s_{n-1} - s_{n-2} & i = n + 1 \\
3s_n - s_{n-1} & i = n + 2 \\
-s_n & i = n + 3 \\
0 & i \geq n + 4.
\]

Let \( \alpha_i \) be the minimal number of generators of degree \( i \) of the ideal \( I \), then:

\[ \alpha_2 = 1 \]
\[ \alpha_b = 2b + 1 - s_b \] (by the hypothesis on \( s_b \), it is \( \alpha_b \geq 1 \)),
\[ \alpha_i = 0 \quad i \neq 2, i < b \text{ or } i \geq n + 2, \]
\[ \alpha_{b+1} \geq \max(0, -s_{b+1} + 3s_b - 4b). \]

In the case \( b = 2 \) by abuse of notation we will denote again \( \alpha_2 = 1 \) and by \( \alpha_b \) we mean the dimension of the \( k \)-vector space \( I_2 \) minus one.

Furthermore it is easy to compute that

\[ H(X, i) = \begin{cases} 
\binom{i + 3}{3} & i = 0, 1 \\
(i + 1)^2 & 2 \leq i \leq b - 1 \\
b^2 + \sum_{j=b}^{i} s_j & b \leq i \leq n \\
b^2 + \sum_{j=b}^{n} s_j & i \geq n.
\end{cases} \]

**Theorem 2.1.** Let \( X \) be a finite set of distinct points on an irreducible quadric \( \Sigma \). Let now \( h \) be an integer such that \( b + 1 \leq h \leq n + 1 \). With the above notations and \( s_{n+1} = 0 \), then

\[ \alpha_h \leq s_{h-1} - s_h = -\Delta^2 H(X, h). \]
Proof. For all $b - 1 \leq i \leq h$

$$\dim I_i = \binom{i + 3}{3} - H(X, i) = \binom{i + 3}{3} - b^2 - \sum_{j=b}^{i} s_j.$$ 

and

$$\alpha_i = \dim I_i - \dim R_1 I_{i-1}.$$ 

We wish to find a lower bound for $\dim R_1 I_{h-1}$. For this aim, let $\pi_1, \pi_2$ be two planes defined by two linear forms $L_1$ and $L_2$ such that $\pi_i \cap X = \emptyset$ for $i = 1, 2$ and the line $\pi_1 \cap \pi_2$ is not contained in $\Sigma$, so $Q, L_1, L_2$ form a regular sequence in $k[X_1, \ldots, X_4]$.

Let $A_i \subset I_i$ be the $k$-vector space generated by the forms of $I_i$ that do not belong to $R_{i-2} \mathcal{Q}$ and let $B_i \subset A_i$ be the greatest subspace that does not contain forms that are divisible by $L_1$, then it is easy to check that

$$\dim A_i = \binom{i + 3}{3} - b^2 - \sum_{j=b}^{i} s_j - \binom{i + 1}{3} =$$

$$= (i + 1)^2 - b^2 - \sum_{j=b}^{i} s_j$$

and

$$\dim B_i = \dim A_i - \dim A_{i-1} \quad i \geq b.$$ 

Moreover, setting $W \subset R_1 I_{h-1}$ the vector space generated by $R_{h-2} \mathcal{Q}$, $L_1 A_{h-1}$ and $L_2 B_{h-1}$, we have:

$$\dim W = \binom{h + 1}{3} + \dim A_{h-1} + \dim B_{h-1} =$$

$$= \binom{h + 1}{3} + \dim A_{h-1} + \dim A_{h-1} - \dim A_{h-2} =$$

$$= \binom{h + 1}{3} + 2h^2 - 2b^2 - 2\sum_{j=b}^{h-1} s_j - (h - 1)^2 + b^2 + \sum_{j=b}^{h-2} s_j =$$

$$= \binom{h + 1}{3} + 2h^2 - b^2 - \sum_{j=b}^{h-1} s_j - s_{h-1} - h^2 + 2h - 1 =$$

$$= \binom{h + 1}{3} + h^2 + 2h - 1 - b^2 - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$
In fact, let $F \in A_{h-1}, G \in B_{h-1}, A \in R_{h-2}$ be such that in $k[X_1, \ldots, X_4]$

$$AQ + L_1 F + L_2 G = 0. \quad (5)$$

In $k[X_1, \ldots, X_4]/(Q, L_1)$, it results $L_2 G = 0$.

Since $Q, L_1, L_2$ are a regular sequence in $k[X_1, \ldots, X_4]$, then $G = 0$ in $k[X_1, \ldots, X_4]/(Q, L_1)$; it means that in $k[X_1, \ldots, X_4]$

$$G = HQ + KL_1 \quad \text{for some } H \text{ and } K.$$

Since $X$ is contained in the surfaces defined by the forms $G$ and $Q$ it follows that $K$ vanishes in all points of $X$, then $K \in I_{h-2}$ and $G \in (L_1 I_{h-2} \oplus R_{h-3} Q)$, which implies $G = 0$ because $G \in B_{h-1}$. So (5) becomes:

$$AQ + L_1 F = 0;$$

since $Q$ does not divide $F$, then $A = 0$ and $F = 0$. This proves (4).

Now let $P_3$ and $P_4$ be the points in which $\Sigma$ intersects the line $\pi_1 \cap \pi_2$; for the generic choice of this line and for the irreducibility of $\Sigma$, we can suppose that there exists a form $H \in A_{h-1}$ such that $P_3$ and $P_4$ do not lie on the surface $\Psi$ defined by $H$. Let us choose $L_3$ and $L_4$ two linear forms defining, respectively, two planes $\pi_3, \pi_4$ such that

$$\begin{cases} P_3 \in \pi_3 \\ P_4 \notin \pi_3 \end{cases} \quad \text{and} \quad \begin{cases} P_3 \notin \pi_4 \\ P_4 \in \pi_4 \end{cases}.$$

We are now going to prove that $L_3 H, L_4 H$ and the elements of $W$ are linearly independent; in fact, let $A \in R_{h-2}, F \in A_{h-1}, G \in B_{h-1}, a, b \in k$ be such that in $k[X_1, \ldots, X_4]$

$$AQ + L_1 F + L_2 G + aL_3 H + bL_4 H = 0. \quad (6)$$

Let us compute (6) in $P_3$; we have

$$bL_4(P_3)H(P_3) = 0;$$

but $L_4(P_3) \neq 0, H(P_3) \neq 0$, thus $b = 0$. Similarly, computing (6) in $P_4$ we have $a = 0$. So (6) becomes

$$AQ + L_1 F + L_2 G = 0$$

that is (5); then $A = 0, F = 0, G = 0.$
Hence
\[ \alpha_h \leq \dim I_h - \dim W - 2 = \]
\[ = \binom{h+3}{3} - b^2 - \sum_{j=b}^{h} s_j - \binom{h+1}{3} - h^2 - 2h + 1 + b^2 + \]
\[ + \sum_{j=b}^{h-1} s_j + s_{h-1} - 2 = \]
\[ = (h + 1)^2 - s_h - h^2 - 2h - 1 + s_{h-1} = \]
\[ = h^2 + 2h + 1 - s_h - h^2 - 2h - 1 + s_{h-1} = \]
\[ = s_{h-1} - s_h, \]
that is the conclusion. □

Example 2.2. These bounds are sharp; in fact, by [6] Proposition 2.3 and [7] Remark 2.12 we have that for any set \( X \) of points on an irreducible quadric, ACM as subscheme of the quadric, is \( \alpha_i = -\Delta^2 H(X, i) \) if \( i = b+1, \ldots, n \).

3. Points on an irreducible cubic surface.

Let \( X \) be a set of points in \( \mathbb{P}^3 \) on an irreducible surface \( \Sigma \) of degree 3, from [10] Theorem 2.1, we have

\[ \Delta H(X, i) = \begin{cases} 
\binom{i+2}{2} & 0 \leq i \leq 2 \\
3i & 3 \leq i \leq b - 1 \\
s_i & b \leq i \leq n \\
0 & i \geq n + 1 
\end{cases} \]

where
\[ s_b \leq 3b - 1, \]
\[ s_b + 1 \geq s_{b+1} \geq s_{b+2} \geq \ldots \geq s_n \geq 1 \text{ and } b \geq 3. \]

Hence

\[ \Delta^2 H(X, i) = \begin{cases} 
i + 1 & i = 0, 1, 2 \\
3 & 3 \leq i \leq b - 1 \\
s_b - 3b + 3 & i = b \\
s_i - s_{i-1} & b + 1 \leq i \leq n \\
-s_n & i = n + 1 \\
0 & i \geq n + 2 \end{cases} \]
\[ \Delta^4 H(X, i) = \begin{cases} 
1 & i = 0 \\
0 & i = 1, 2 \\
-1 & i = 3 \\
0 & 4 \leq i \leq b - 1 \\
s_b - 3b & i = b \\
s_{b+1} - 3s_b + 6b - 3 & i = b + 1 \\
s_{b+2} - 3s_{b+1} + 3s_b - 3b + 3 & i = b + 2 \\
s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b + 3 \leq i \leq n \\
-3s_n + 3s_{n-1} - s_{n-2} & i = n + 1 \\
3s_n - s_{n-1} & i = n + 2 \\
-s_n & i = n + 3 \\
0 & i \geq n + 4. 
\]

Let \( \alpha_i \) be the minimal number of generators of degree \( i \) of the ideal \( I \), then:

\( \alpha_3 = 1 \) and we will call \( S \) the form defining \( \Sigma \),
\( \alpha_b = 3b - s_b \) (\( \alpha_b \geq 1 \) by the hypothesis on \( s_b \)) and we will call these forms \( F_1, \ldots, F_{a_b} \),
\( \alpha_i = 0 \) \quad \text{if} \quad i \neq 3, i < b \) or \( i \geq n + 2 \),
\( \alpha_{b+1} \geq \max(0, -s_{b+1} + 3s_b - 6b + 3) \).

If \( b = 3 \) we will use the same notation as the previous section.

Furthermore it is easy to compute

\[ H(X, i) = \begin{cases} 
\binom{i + 3}{3} + 1 & i = 0, 1, 2 \\
\frac{3i^2 + 3i}{2} + 1 & 3 \leq i \leq b - 1 \\
\frac{3b^2 - 3b}{2} + 1 + \sum_{j=b}^{i} s_j & b \leq i \leq n \\
\frac{3b^2 - 3b}{2} + 1 + \sum_{j=b}^{n} s_j & i \geq n.
\end{cases} \]

**Theorem 3.1.** Let \( X \) be a finite set of distinct points on an irreducible cubic surface \( \Sigma \).

With the previous notations, we have:

\( \alpha_{b+1} \leq \max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) \).
Proof. We have:

\[
\dim I_{b+1} = \dim R_{b+1} - H(X, b + 1) =
\]

\[
= \binom{b + 4}{3} - \frac{3b^2 - 3b}{2} - 1 - s_b - s_{b+1}
\]

and

\[
\alpha_{b+1} = \dim I_{b+1} - \dim R_1 I_b.
\]

We note that

\[
\max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) = -\Delta^4 H(X, b + 1)
\]

iff

\[
s_b = 3b - 1,
\]

so we can distinguish two cases: i) \(s_b < 3b - 1\), ii) \(s_b = 3b - 1\).

If \(s_b < 3b - 1\), then

\[
\alpha_b = 3b - s_b > 3b - 3b + 1
\]

and we can consider two forms \(F_1, F_2\) of degree \(b\) which define, respectively, two surfaces \(\Phi_1\) and \(\Phi_2\).

We work as in Theorem 2.1. Let \(r\) be a line in \(\mathbb{P}^3\) such that:

\[
r \cap \Sigma = \{P_1, P_2, P_3\}
\]

(7)

\[
r \cap \Sigma \cap \Phi_1 = \emptyset
\]

\[
r \cap \Sigma \cap \Phi_2 = \{P_2\}
\]

and let \(L_1, L_2, L_3\) and \(L_4\) be linear forms defining, respectively, the planes \(\pi_1, \pi_2, \pi_3\) and \(\pi_4\) such that \(\pi_1\) and \(\pi_2\) contain \(r\) but not the points of \(X\) and \(\pi_3\) and \(\pi_4\) such that

(8)

\[
\begin{align*}
\{ & P_1 \in \pi_3 \quad \text{and} \quad P_1 \notin \pi_4 \\
& P_2 \notin \pi_3 \quad \text{and} \quad P_2 \in \pi_4, \}
\end{align*}
\]

hence \(P_3 \notin \pi_3\) and \(P_3 \notin \pi_4\). Notice that \(S, L_1, L_2\) form a regular sequence in \(k[X_1, \ldots, X_4]\). The elements given by the \(\binom{b+1}{3}\) generators of the vector subspace \(R_{b-2}S\) of the forms of degree \(b + 1\) containing \(S\) and by the \(2\alpha_b\) forms
of type $L_1 F_i, L_2 F_i$, are linearly independent. In fact, let $A \in R_{b-2}, \overline{F}, \overline{F} \in \mathcal{L}(F_1, \ldots, F_{\alpha_b})$ be such that:

\begin{equation}
AS + L_1 \overline{F} + L_2 \overline{F} = 0,
\end{equation}

we have $L_2 \overline{F} = 0$ in $k[X_1, \ldots, X_4]/(S, L_1)$. Since $S, L_1, L_2$ are a regular sequence in $k[X_1, \ldots, X_4]$, it follows that $\overline{F} = 0$ in $k[X_1, \ldots, X_4]/(S, L_1)$; so

\[ \overline{F} = SH + L_1 K \]

for some $H$ and $K$.

Since $X$ is contained in the surfaces defined by the forms $\overline{F}$ and $S$, it follows that $X$ must be contained in the surface defined by $KL_1$. Since $X \cap \pi_1 = \emptyset$, then the form $K$ vanishes in $X$; but $\deg K = b - 1$, thus $K = 0$. In fact, if $K \neq 0$ we get $K \in (S)$ and so $\overline{F} \in (S)$ and this is false; thus $\overline{F} = 0$ and (9) becomes

\[ AS + L_1 \overline{F} = 0 \]

then $A = 0$ and $\overline{F} = 0$. This proves that the previous $\left(\frac{b+1}{3}\right) + 2\alpha_b$ elements of $R_1 I_b$ are linearly independent.

We claim that if we add to these elements the forms $L_3 F_1, L_3 F_2$ and $L_4 F_1$, we still get independent elements; in fact, let $A \in R_{b-2}, \overline{F}, \overline{F} \in \mathcal{L}(F_1, \ldots, F_{\alpha_b}), a, b, c \in k$ be such that in $k[X_1, \ldots, X_4]$:

\begin{equation}
AS + L_1 \overline{F} + L_2 \overline{F} + aL_3 F_1 + bL_3 F_2 + cL_4 F_1 = 0.
\end{equation}

Let us compute (10), respectively, in $P_1, P_2, P_3$; we have:

\[ cL_4(P_1)F_1(P_1) = 0 \]
\[ aL_3(P_2)F_1(P_2) = 0 \]
\[ aL_3(P_3)F_1(P_3) + bL_3(P_3)F_2(P_3) + cL_4(P_3)F_1(P_3) = 0 \]

and, by (7) and (8), $c = 0, a = 0$ and $b = 0$.

Hence (10) becomes

\[ AS + L_1 \overline{F} + L_2 \overline{F} = 0, \]

then $A = 0, \overline{F} = 0$ and $\overline{F} = 0$.

Therefore

\[ \dim R_1 I_b \geq \left(\frac{b+1}{3}\right) + 2\alpha_b + 3 = \]

\[ = \left(\frac{b+1}{3}\right) + 6b - 2s_b + 3 \]
and, consequently,

\[
\alpha_{b+1} \leq \left(\frac{b + 4}{3}\right) - \frac{3b^2 - 3b}{2} - 1 - s_b - s_{b+1} - \left(\frac{b + 1}{3}\right) - 6b + 2s_b - 3 = \\
= \frac{3b^2 + 9b + 6}{2} + 1 \cdot \frac{3b^2 - 3b}{2} - 4 + s_b - s_{b+1} - 6b = \\
= 6b + 3 - 3 + s_b - s_{b+1} - 6b = \\
= s_b - s_{b+1}
\]

that concludes the first case.

If \( s_b = 3b - 1 \) then \( \alpha_b = 1 \) and, since \( S \) is irreducible, \( \beta_{b+1} = 0 \); so, by (2),

\[
\alpha_{b+1} = -\Delta^4 H(X, b + 1)
\]

(obviously \( \gamma_{b+1} = 0 \)). \( \square \)

**Corollary 3.2.** Let \( X \) be a finite set of distinct points on an irreducible cubic surface \( \Sigma \).

With the previous notations, if \( s_b < 3b - 1 \) then

\[
s_{b+1} \leq s_b.
\]

**Proof.** Under these hypotheses

\[
\max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) = -\Delta^2 H(X, b + 1)
\]

so, by the previous Theorem,

\[
0 \leq \alpha_{b+1} \leq -s_{b+1} + s_b. \quad \square
\]

This Corollary improves the result given in [10] Theorem 2.1.

**Theorem 3.3.** Let \( X \) be a finite set of distinct points on an irreducible cubic surface \( \Sigma \). Let now \( h \) be an integer such that \( b + 1 \leq h \leq n + 1 \). With the above notations and \( s_{n+1} = 0 \), then

\[
\alpha_h \leq \begin{cases} 
\max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) & h = b + 1 \\
-\Delta^2 H(X, h) & h > b + 1.
\end{cases}
\]
Proof. For $h = b + 1$ see Theorem 3.1, so we assume $h > b + 1$. Let $H_1, H_2 \in (I_{h-1} - R_{h-4}S)$ be two forms defining surfaces $\Psi_1$ and $\Psi_2$.

We can choose a line $r$ in $\mathbb{P}^3$ such that:

$$r \cap \Sigma = \{P_1, P_2, P_3\}$$
$$r \cap \Sigma \cap \Psi_1 = \emptyset$$
$$r \cap \Sigma \cap \Psi_2 = \{P_2\};$$

let us introduce the linear forms $L_1, L_2, L_3$ and $L_4$ as before. Notice that $S, L_1, L_2$ form a regular sequence in $k[X_1, \ldots, X_4]$.

For all $b - 1 \leq i \leq h$, we have

$$\dim I_i = \binom{i + 3}{3} - H(X, i) =$$

$$\binom{i + 3}{3} - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j.$$

Now we proceed as in the quadric case. Let $A_i \subset I_i$ be the $k$-vector space generated by the forms of $I_i$ that do not belong to $R_{i-3}S$ and let $B_i \subset A_i$ be the greatest subspace that does not contain forms that are divisible by $L_1$, then it is easy to check that

$$\dim A_i = \binom{i + 3}{3} - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j - \binom{i}{3} =$$

$$\frac{3i^2 + 3i}{2} + 1 - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j =$$

$$\frac{3i^2 + 3i}{2} - \frac{3b^2 - 3b}{2} - \sum_{j=b}^{i} s_j$$

and

$$\dim B_i = \dim A_i - \dim A_{i-1} = 3i - s_i \quad i \geq b.$$ 

Moreover, denoting $W \subset R_1 I_{h-1}$ the vector space generated by $R_{h-3}S$, $L_1 A_{h-1}$ and $L_2 B_{h-1}$, we get as in the previous Theorem:

$$\dim W = \binom{h}{3} + \dim A_{h-1} + \dim B_{h-1} =$$

$$= \binom{h}{3} + \frac{3(h - 1)^2 + 9(h - 1)}{2} - \frac{3b^2 + 3b}{2} - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$
Again we can add the forms \( L_3 H_1, L_3 H_2, L_4 H_1 \) still obtaining independent elements; so:

\[
\alpha_h \leq \dim I_h - \dim W - 3 = s_{h-1} - s_h.
\]

\[ \square \]

**Example 3.4.** We give now an example of a set of points in which these bounds are sharp. In fact, let \( \Sigma \) be an irreducible cubic surface and let \( r \) be a line contained in \( \Sigma \); let us consider 5 points in \( r \) and other 15 generic points in \( \Sigma \), we call \( X \) the set consisting of these points. Thus the Hilbert Function of \( X \) is given by:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
H & 1 & 4 & 10 & 19 & 20 & 20 & 20 & 20 & \ldots \\
\Delta H & 1 & 3 & 6 & 9 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\Delta^2 H & 1 & 2 & 3 & 3 & -8 & -1 & 0 & 0 & 0 & \ldots \\
\Delta^3 H & 1 & 1 & 1 & 0 & -11 & 7 & 1 & 0 & 0 & \ldots \\
\Delta^4 H & 1 & 0 & 0 & -1 & -11 & 18 & -6 & -1 & 0 & \ldots \\
\end{array}
\]

In this case \( b = n = 4, \alpha_3 = 1, \alpha_4 = 11 \) and \( \alpha_5 = 1 \) because the generators of degree lesser than 5 must vanish in all points of \( r \), hence \( \alpha_5 = -\Delta^2 H(X, 5) \).

4. **Points on an irreducible surface of degree 4.**

Let \( X \) be a set of points in \( \mathbb{P}^3 \) on an irreducible surface \( \Sigma \) of degree 4, from [10] Theorem 2.1, we have

\[
\Delta H(X, i) = \begin{cases}
\binom{i+2}{2} & 0 \leq i \leq 3 \\
4i - 2 & 4 \leq i \leq b - 1 \\
s_i & b \leq i \leq n \\
0 & i \geq n + 1
\end{cases}
\]

with

\[
\begin{align*}
s_b & \leq 4b - 3, \\
s_{b+1} & \leq s_b + 2, \\
s_{b+2} & \leq s_{b+1} + 1, \\
s_{b+3} & \geq s_{b+2} \geq \ldots \geq s_n \geq 1 \text{ and } b \geq 4.
\end{align*}
\]
We have
\[ \Delta^2 H(X, i) = \begin{cases} 
    i + 1 & 0 \leq i \leq 3 \\
    4 & 4 \leq i \leq b - 1 \\
    s_{b} - 4b + 6 & i = b \\
    s_i - s_{i-1} & b + 1 \leq i \leq n \\
    -s_n & i = n + 1 \\
    0 & i \geq n + 2 
\end{cases} \]

\[ \Delta^4 H(X, i) = \begin{cases} 
    1 & i = 0 \\
    0 & 1 \leq i \leq 3 \\
    -1 & i = 4 \\
    0 & 5 \leq i \leq b - 1 \\
    s_{b} - 4b + 2 & i = b \\
    s_{b+1} - 3s_{b} + 8b - 8 & i = b + 1 \\
    s_{b+2} - 3s_{b+1} + 3s_{b} - 4b + 6 & i = b + 2 \\
    s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b + 3 \leq i \leq n \\
    -3s_n + 3s_{n-1} - s_{n-2} & i = n + 1 \\
    3s_n - s_{n-1} & i = n + 2 \\
    -s_n & i = n + 3 \\
    0 & i \geq n + 4 
\end{cases} \]

Let \( \alpha_i \) be the minimal number of generators of degree \( i \) of the ideal \( I \), then:
\( \alpha_4 = 1 \) and we will call \( S \) the form defining \( \Sigma \),
\( \alpha_b = 4b - 2 - s_b \) (\( \alpha_b \geq 1 \) by the hypothesis on \( s_b \)) and we will call these forms \( F_1, \ldots, F_{a_b} \),
\( \alpha_i = 0 \) for \( i \neq 4, i < b \) or \( i \geq n + 2 \),
\( \alpha_{b+1} \geq \max(0, -s_{b+1} + 3s_b - 8b + 8) \).

If \( b = 4 \) we will use the same notation as the previous section.
Furthermore it is easy to compute
\[ H(X, i) = \begin{cases} 
    \binom{i + 3}{3} & i = 0, 1, 2 \\
    2i^2 + 2 & 3 \leq i \leq b - 1 \\
    2b^2 - 4b + 4 + \sum_{j=b}^{i} s_j & b \leq i \leq n \\
    2b^2 - 4b + 4 + \sum_{j=b}^{n} s_j & i \geq n. 
\end{cases} \]
Case $i = b + 1$

**Theorem 4.1.** Let $X$ be a finite set of distinct points on an irreducible surface $\Sigma$ of degree 4.

With the previous notations we have:

$$\alpha_{b+1} \leq \max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)).$$

**Proof.** We have:

$$\dim I_{b+1} = \dim R_{b+1} - H(X, b + 1) =$$

$$= \binom{b+4}{3} - 2b^2 + 4b - 4 - s_b - s_{b+1}$$

and

$$\alpha_{b+1} = \dim I_{b+1} - \dim R_1 I_b.$$  

We note that

$$\max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) = -\Delta^2 H(X, b + 1)$$

iff

$$s_b \leq 4b - 4$$

so we can distinguish two cases: i) $s_b \leq 4b - 4$, ii) $s_b = 4b - 3$.

If $s_b \leq 4b - 4$, then

$$\alpha_b = 4b - 2 - s_b \geq 4b - 2 - 4b + 4 = 2,$$

so we can consider two forms $F_1, F_2$ which define, respectively, two surfaces $\Phi_1$ and $\Phi_2$ of degree $b$.

One can easily check that, if $S$ is irreducible, there exists a line $r$ in $\mathbb{P}^3$ such that:

$$r \cap \Sigma = \{P_1, P_2, P_3, P_4\}$$

$$r \cap \Sigma \cap \Phi_1 = \emptyset$$

(11)

$$r \cap \Sigma \cap \Phi_2 = \{P_1, P_2\}$$

$$P_3 \neq P_4.$$
Let $L_1, L_2, L_3$ and $L_4$ be linear forms defining, respectively, the planes $\pi_1, \pi_2, \pi_3$ and $\pi_4$ such that $\pi_1$ and $\pi_2$ contain $r$ but not the points of $X$ and $\pi_3$ and $\pi_4$ such that
\[
\begin{cases}
P_1 \in \pi_3 \\
P_2 \notin \pi_3
\end{cases}
\quad \text{and} \quad \begin{cases}
P_1 \notin \pi_4 \\
P_2 \in \pi_4,
\end{cases}
\]
so that $P_3, P_4 \notin \pi_3$ and $P_3, P_4 \notin \pi_4$. Again one shows that $\binom{b}{3} + 2\alpha_b$ elements given by the $\binom{b}{3}$ generators of the vector subspace $R_{b-3}S$ of the forms of degree $b + 1$ containing $S$ and by the $2\alpha_b$ forms of type $L_1 F_i, L_2 F_i$, are linearly independent.

Now we can add to them the four forms $L_3 F_1, L_2 F_2, L_4 F_1$ and $L_4 F_2$ still getting independent elements; in fact, let $A \in R_{b-3}, \overline{F}, \overline{F} \in \mathcal{L}(F_1, \ldots, F_{\alpha_b}), a, b, c, d \in k$ be such that in $k[X_1, \ldots, X_4]$:
\[
(13) \quad AS + L_1 \overline{F} + L_2 \overline{F} + aL_3 F_1 + bL_3 F_2 + cL_4 F_1 + dL_4 F_2 = 0.
\]
Let us compute (13), respectively, in $P_1, P_2$; we get
\[
cL_4(P_1)F_1(P_1) = 0 \\
aL_3(P_2)F_1(P_2) = 0
\]
so, by (11) and (12), $c = 0$ and $a = 0$; computing (13) in $P_3$ and $P_4$ we obtain the following system:
\[
(14) \quad \begin{cases}
bL_3(P_3)F_2(P_3) + dL_4(P_3)F_2(P_3) = 0 \\
bL_3(P_4)F_2(P_4) + dL_4(P_4)F_2(P_4) = 0,
\end{cases}
\]
since $F_2(P_3) \neq 0$ and $F_2(P_4) \neq 0$, then (14) becomes:
\[
\begin{cases}
bL_3(P_3) + dL_4(P_3) = 0 \\
bL_3(P_4) + dL_4(P_4) = 0,
\end{cases}
\]
since $L_3 \cap L_4$ and $r$ are skew, then (14) has only the solution $b = 0, d = 0$. Hence (13) becomes
\[
AS + L_1 \overline{F} + L_2 \overline{F} = 0,
\]
so again $A = 0, \overline{F} = 0$ and $\overline{F} = 0$.

Therefore
\[
\dim R_1 I_b \geq \binom{b}{3} + 2\alpha_b + 4 = \binom{b}{3} + 8b - 2\alpha_b.
\]
and
\[
\alpha_{b+1} \leq \left(\frac{b + 4}{3}\right) - 2b^2 + 4b - 4 - s_b - s_{b+1} - \left(\frac{b}{3}\right) - 8b + 2s_b = \\
= 2b^2 + 4b + 2 + 2 - 2b^2 - 4b - 4 + s_b - s_{b+1} = \\
= s_b - s_{b+1}
\]
that concludes the first case.
If \( s_b = 4b - 3 \) then \( \alpha_b = 1 \) and, being \( S \) irreducible, \( \beta_{b+1} = 0 \); so, by (2),
\[
\alpha_{b+1} = -\Delta^4 H(X, b + 1)
\]
(obviously \( \gamma_{b+1} = 0 \)). □

**Corollary 4.2.** Let \( X \) be a finite set of distinct points on an irreducible surface \( \Sigma \) of degree 4.
With the previous notations, if \( s_b < 4b - 3 \) then
\[
s_{b+1} \leq s_b.
\]

**Proof.** Under these hypotheses
\[
\max(-\Delta^2 H(X, b + 1), -\Delta^4 H(X, b + 1)) = -\Delta^2 H(X, b + 1)
\]
so, by the previous Theorem,
\[
0 \leq \alpha_{b+1} \leq -s_{b+1} + s_b. \quad □
\]

**Case** \( i = b + 2 \)

**Theorem 4.3.** Let \( X \) be a finite set of distinct points on an irreducible surface \( \Sigma \) of degree 4.
With the previous notations it follows:
\[
\alpha_{b+2} \leq \begin{cases} 
-\Delta^4 H(X, b + 2) & \text{if } s_{b+1} = s_b + 2 \\
-\Delta^2 H(X, b + 2) & \text{otherwise.}
\end{cases}
\]
Proof. We know that:

\[
\dim I_{b+2} = \dim R_{b+2} - H(X, b + 2) = \\
= \left(\binom{b+5}{3}\right) - 2b^2 + 4b - 4 - s_b - s_{b+1} - s_{b+2}
\]

and

\[
\alpha_{b+2} = \dim I_{b+2} - \dim R_1 I_{b+1},
\]

If \(s_{b+1} \leq s_b + 1\) we can have two cases: either \(s_b < 4b - 3\) then \(\alpha_b \geq 2\), or \(s_b = 4b - 3\) and \(s_{b+1} \leq s_b + 1\) thus in this case \(\alpha_b = 1\) but \(\alpha_{b+1} \geq 1\); so, in both cases, we can find two surfaces \(\Psi_1\) and \(\Psi_2\) defined by \(H_1\) and \(H_2\) in \((I_{b+1} - R_{b-3}S)\) and a line \(r\) satisfying the conditions:

\[
\begin{align*}
& r \cap \Sigma = \{P_1, P_2, P_3, P_4\} \\
& r \cap \Sigma \cap \Psi_1 = \emptyset \\
& r \cap \Sigma \cap \Psi_2 = \{P_1, P_2\}
\end{align*}
\]

(15)

\(P_3 \neq P_4\).

Let \(L_1, L_2, L_3, L_4\) be the linear forms as in the previous case; if \(t = \dim I_{b+1}\), let \(B = \{G_1, \ldots, G_t\}\) be a basis of \(I_{b+1}\), we can suppose that \(G_i \in R_{b-3}S\) for \(i = 1, \ldots, \left(\binom{b}{3}\right)\) and \(G_i = L_1 F_i - \left(\binom{b+1}{3}\right)\) for \(i = \left(\binom{b}{3}\right) + 1, \ldots, \left(\binom{b}{3}\right) + \alpha_b\).

As in the previous case we can choose independent elements in \(I_{b+2}\) in the following way. Pick \(\left(\binom{b+1}{3}\right)\) generators of \(R_{b-2}S\), elements of type \(L_1 G_i\) with \(i \geq \left(\binom{b}{3}\right)\) (i.e. \(G_i \in B\) and \(S\) does not divide \(G_i\); they are in number of \(\dim I_{b+1} - \left(\binom{b}{3}\right)\)), and elements of type \(L_2 G_i\) with \(i > \left(\binom{b}{3}\right) + \alpha_b\) (they are in number of \(\dim I_{b+1} - \left(\binom{b}{3}\right) - \alpha_b\)).

So we find

\[
N = \left(\binom{b+1}{3}\right) + \dim I_{b+1} - \dim R_{b-3}S + \dim I_{b+1} - \dim R_{b-3}S - \alpha_b = \\
= \left(\binom{b+1}{3}\right) + 12b + 2 - s_b - 2s_{b+1}
\]

surfaces in \(R_1 I_{b+1}\) that are linearly independent.

As in the previous case, we can add the four forms \(L_3 H_1, L_4 H_1, L_3 H_2\) and \(L_4 H_2\); so

\[
\alpha_{b+2} \leq \left(\binom{b+5}{3}\right) - 2b^2 + 4b - 4 - s_b - s_{b+1} - s_{b+2} - N - 4 = \\
= s_{b+1} - s_{b+2}
\]
this concludes the proof if \( s_{b+1} \leq s_b + 1 \).

If \( s_{b+1} = s_b + 2 \), for the Corollary 4.2, \( s_b = 4b - 3 \) then \( \alpha_b = 1 \) and \( \alpha_{b+1} = 0 \).

Since \( \Sigma \) is irreducible and of degree 4, then \( \beta_{b+2} = 0 \) and \( \gamma_{b+2} = 0 \); thus

\[
\alpha_{b+2} = -\Delta^4 H(X, b + 2) .
\]

\[
\square
\]

**Corollary 4.4.** Let \( X \) be a finite set of distinct points on an irreducible surface \( \Sigma \) of degree 4.

With the previous notations, if \( s_{b+1} \leq s_b + 1 \), then

\[
s_{b+2} \leq s_{b+1} .
\]

**Proof.** Applying the previous Theorem with \( s_{b+1} \leq s_b + 1 \)

\[
0 \leq \alpha_{b+2} \leq s_{b+1} - s_{b+2} .
\]

\[
\square
\]

**Remark 4.5.** Corollary 4.2 and Corollary 4.4 improve the results given in [10] for the Hilbert Function of a set of points lying on an irreducible surface.

**The general case**

**Theorem 4.6.** Let \( X \) be a finite set of distinct points on an irreducible surface \( \Sigma \) of degree 4. Let now \( h \) be an integer such that \( b + 3 \leq h \leq n + 1 \). With the previous notations and \( s_{n+1} = 0 \), then

\[
\alpha_h \leq -\Delta^2 H(X, h) = s_{h-1} - s_h .
\]

**Proof.** We know

\[
\dim I_h = \binom{h + 3}{3} - H(X, h) = \binom{h + 3}{3} - 2b^2 + 4b - 4 - \sum_{j=b}^{h} s_j .
\]

Let \( H_1, H_2 \in (I_{h-1} - R_{h-4} S) \) be two forms defining surfaces \( \Psi_1 \) and \( \Psi_2 \) such that:

\[
\begin{align*}
 r \cap \Sigma &= \{ P_1, P_2, P_3, P_4 \} \\
 r \cap \Sigma \cap \Psi_1 &= \emptyset \\
 r \cap \Sigma \cap \Psi_2 &= \{ P_1, P_2 \} \\
 P_3 &\neq P_4,
\end{align*}
\]

(16)
let us introduce the linear forms $L_1, L_2, L_3$ and $L_4$ as before.

Let $A_h \subset I_h$ be the $k$-vector space generated by the forms of $I_h$ that do not belong to $R_{h-4}S$ and let $B_h \subset A_h$ be the greatest subspace that does not contain forms that are divisible by $L_1$, then it is easy to check that

$$\dim A_h = \binom{h+3}{3} - 2b^2 + 4b - 4 - \sum_{j=b}^{h} s_j - \binom{h-1}{3} =$$

$$= 2h^2 - 2b^2 + 4b - 2 - \sum_{j=b}^{h} s_j$$

and

$$\dim B_h = \dim A_h - \dim A_{h-1} =$$

$$= 4h - 2 - s_h.$$

Moreover, if $W \subset R_1I_{h-1}$ is the vector space generated by $R_{h-4}S$, $L_1A_{h-1}$ and $L_2B_{h-1}$, again we obtain:

$$\dim W = \binom{h-1}{3} + \dim A_{h-1} + \dim B_{h-1} =$$

$$= \binom{h-1}{3} + 2h^2 - 6 - 2b^2 + 4b - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$

Adding, like before cases, the forms $L_3H_1$, $L_3H_2$, $L_4H_1$ and $L_4H_2$, we obtain:

$$\alpha_h \leq \dim I_h - \dim W - 4 = s_{h-1} - s_h. \quad \square$$

**Example 4.7.** We construct a set of points for which these bounds are sharp; let $\Sigma$ be an irreducible surface of degree 4 containing a line $r$. Let us consider 6 points in $r$ and other 29 generic points in $\Sigma$, we call $X$ the set consisting of these points. Thus the Hilbert Function of $X$ is given by:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>34</td>
<td>35</td>
<td>35</td>
<td>35</td>
<td>35</td>
<td>35</td>
<td>...</td>
</tr>
<tr>
<td>$\Delta H$</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>14</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$\Delta^2 H$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>-13</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$\Delta^3 H$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-17</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$\Delta^4 H$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-17</td>
<td>29</td>
<td>-11</td>
<td>-1</td>
<td>0</td>
<td>...</td>
</tr>
</tbody>
</table>

In this case $b = n = 5$, $\alpha_4 = 1$, $\alpha_5 = 17$ and $\alpha_6 = 1$ because the generators of degree lesser than 6 must vanish in all points of $r$, hence $\alpha_6 = -\Delta^2 H(X, 6)$. 
5. Applications.

Remark 5.1. Let \( b + 1 \leq h \leq n + 1 \), all bounds founded for \( \alpha_h \) are valid just asking that there exists a form \( F \) in \( I_{h-1} \) defining a surface with no common components with a surface \( \Sigma \) of smaller degree containing \( X \).

Example 5.2. Let \( Q \) the form defining \( \Sigma \). The hypothesis that there exists a surface in \( I \) of degree \( h - 1 \) such that \( Q \) and \( F \) have no common components is necessary. Let us consider 10 points in generic position on a plane \( \pi \) and 3 generic points outside \( \pi \); let \( X \) be the set of these 13 points, then the Hilbert Function of \( X \) is given by:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ...
|---|---|---|---|---|---|---|---|---
| \( H \) | 1 | 4 | 9 | 13 | 13 | 13 | 13 | 13 | ...  
| \( \Delta H \) | 1 | 3 | 5 | 4 | 0 | 0 | 0 | 0 | ...
| \( \Delta^2 H \) | 1 | 2 | 2 | -1 | -4 | 0 | 0 | 0 | ...
| \( \Delta^3 H \) | 1 | 1 | 0 | -3 | -3 | 4 | 0 | 0 | ...
| \( \Delta^4 H \) | 1 | 0 | -1 | -3 | 0 | 7 | -4 | 0 | ...

Hence \( \alpha_2 = 1, \alpha_3 = 3, \alpha_4 \geq 3 \) and \( \alpha_4 = \beta_4 \), but \( \beta_4 \) is equal to the number of syzygies of degree 3 for a set consisting of 3 generic points in \( \mathbb{P}^3 \), so \( \alpha_4 = 5 \), while, by Theorem 2.1, \( \alpha_4 \leq 4 \).

This happens because the quadric passing through these 13 points splits in \( \pi \) and in the plane defined by the 3 points outside \( \pi \) and all cubic surfaces passing through \( X \) contain \( \pi \).
REFERENCES


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