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MAXIMUM NUMBER OF GENERATORS OF AN IDEAL OF POINTS ON AN IRREDUCIBLE SURFACE OF LOW DEGREE

ELENA GUARDO - ORNELLA PARISI

Let X be a finite set of distinct points lying on an irreducible surface with Hilbert Function $H(X, i)$. In this paper we will prove that the maximum number of generators, in each degree, of the ideal defining X is obtained by the second difference of the Hilbert Function.

Introduction.

Let Y be a finite set of distinct points in the projective space $\mathbb{P}_k^2$, k algebrically closed field, with Hilbert Function $H(Y, i)$ and let $\overline{I}$ be the defining ideal of Y.

In [3] Dubreil's Theorem says that the number $\nu(\overline{I})$ of generators of $\overline{I}$ is bounded by:

$$\nu(\overline{I}) \leq a(\overline{I}) + 1$$

where $a(\overline{I})$ is the smallest degree of a generator of $\overline{I}$.

The next step was done in [2] and in [8] (Theorem 1.2); if $\overline{\alpha}_i$ is the number of generators of degree i of the ideal $\overline{I}$, then

$$\min\{-\Delta^3 H(Y, i), 0\} \leq \overline{\alpha}_i \leq -\Delta^2 H(Y, i).$$

Let V be a non degenerate arithmetically Cohen-Macaulay projective variety in $\mathbb{P}^r$ and let I be the defining ideal of V. In [12], Valla finds upper bounds for

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the Betti numbers of I when: i) V ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension and degree and when: ii) V ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension, degree and initial degree. In particular, when $r = 3$ and V is a set X of points in $\mathbb{P}^3$ one obtains bounds depending on the number of points and on the degree of the first surface containing X.

In [4] and [1] it is proved that among all homogeneous ideals with a given Hilbert Function the lex-segment ideal has the maximum Betti numbers. Hence, one can compute the maximum number of generators of an ideal with a given Hilbert Function (see, for instance, [11]). Using these results, one can obtain bounds for the number of graded generators of the defining ideal I of X in terms of the Hilbert Function and we can note that the maximum number of generators is obtained when the minimal surface containing X is reducible.

One of the main questions on this line is to find bounds for the Betti numbers of a set of points in Uniform Position in $\mathbb{P}^3$. A beginning step of this direction is to obtain bounds for a set X lying on an irreducible surface of degree s. Here we treat the cases $s = 2, 3, 4$ and we improve the bounds found in [4].

For $s = 2$, denoting Σ the irreducible quadric containing X, in [9] it is proved that $\Delta H(X, i)$ is not increasing for $n \geq b$, where $b \geq 2$ is the least degree of a surface containing X but not Σ; so the behaviour of the Hilbert Function of a set of points in $\mathbb{P}^3_k$ on an irreducible quadric is analogous to the behaviour of the Hilbert Function of a set of points in $\mathbb{P}^2_k$. The same thing happens for the maximum number of generators of the ideal defining a set of points lying on an irreducible quadric. Particularly, if X lies on an irreducible quadric, we prove that

$$\alpha_i \leq -\Delta^2 H(X, i)$$

where α_i is the number of generators of degree i of the ideal defining X and i runs all over the indices in which the number of generators is not determined by the Hilbert Function.

For $s = 3$, if Σ is the irreducible cubic surface containing X, we obtain

$$\alpha_i \leq \begin{cases} \max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) & i = b+1 \\ -\Delta^2 H(X, i) & i > b+1, \end{cases}$$

where b is the least degree of a surface passing through X but not through Σ.

Similar results are obtained for points on an irreducible surface of degree 4; particularly, in these two cases we improve some result given in [10].

The paper consists of five sections; in the first, the binomial expansion of natural numbers, some related functions and the theory of lex–segment ideal are introduced; moreover, the definition of graded Betti numbers is given to use the results of [4] and [11].

Section 2, 3, 4 treat, respectively, sets of points on an irreducible surface of degree 2, 3, 4.

In the last section we give some applications of our results.

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1. Preliminaries and notation.

Let X be a finite set of distinct points in $\mathbb{P}^n_k$, k algebrically closed field of characteristic zero. Let $R = k[X_1, \dots, X_{n+1}]$ and let $I(X)$ be the homogeneous ideal of R that defines the variety X.

Definition 1.1. For every homogeneous ideal I of R, the *Hilbert Function* of the standard graded Algebra $A = R/I = \oplus_{t\geq 0} A_t$ is the numerical function defined by:

$$H_A(t) = \dim_k(A_t).$$

Particularly, we denote by $H(X, i)$ the Hilbert Function of X, that is:

$$H(X, i) = H_{R/I(X)}(i) = \dim_k[R/I(X)]_i\,.$$

We will use the following terminology:

$\Delta H(X, i) = H(X, i) - H(X, i-1)$

$\Delta^j H(X, i) = \Delta^{j-1} H(X, i) - \Delta^{j-1} H(X, i-1)$ for all $i \in \mathbb{N}$ and $j \geq 1$.

Let

$$(1) \qquad 0 \to \oplus R(-j)^{q_{nj}} \to \cdots \to \oplus R(-j)^{q_{2j}} \to \\ \to \oplus R(-j)^{q_{1j}} \to R \to R/I \to 0$$

be a minimal graded free resolution of R/I.

If V is a variety, we say that V is arithmetically Cohen-Macaulay (ACM for short), if its homogeneous coordinate ring $R/I(V)$ is Cohen-Macaulay.

If W is a k-vector space and $u_1, u_2, \dots, u_t$ are elements of W we will indicate with $\mathcal{L}(u_1, u_2, \dots, u_t)$ the subspace of W generated by these elements.

Some known result

Now we collect some known result about the maximum Betti numbers of homogeneous ideals and ideals defining a variety of points in $\mathbb{P}^3$ with a given Hilbert

Function, these maximum numbers are given in Theorem 1.9 and Remark 1.12 respectively.

If m, k are non negative integers we will use the notation: $\binom{m}{0} = 1$ for any $m \geq 0$ and $\binom{m}{k} = 0$ if $m < k$.

Lemma 1.2. *Let m, i be positive integers. Then m can be uniquely written as*

$$m = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j}$$

where $m(i) > m(i-1) > \cdots > m(j) \geq j \geq 1$.

Proof. See [11] Lemma 4.1. □

Definition 1.3. The unique expression $m = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j}$ where $m(i) > m(i-1) > \cdots > m(j) \geq j \geq 1$ is called *the binomial expansion of m in base i* and it is denoted by m_i.

Definition 1.4. Let m, i be positive integers and let $m_i = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j}$. Then we define:

$$(m_i)^+_+ = \binom{m(i)+1}{i+1} + \binom{m(i-1)+1}{i} + \cdots + \binom{m(j)+1}{j+1}.$$

We have $(m_i)^+_+ \geq m$, in fact:

$$(m_i)^+_+ = \sum_{t=j}^{i} \binom{m(t)+1}{t+1} =$$

$$= \sum_{t=j}^{i} \left[\binom{m(t)}{t+1} + \binom{m(t)}{t} \right] =$$

$$= \sum_{t=j}^{i} \binom{m(t)}{t+1} + m \geq m.$$

Definition 1.5. In $R = k[X_1, \ldots, X_{n+1}]$, we say that

$$X_1^{a_1} X_2^{a_2} \ldots X_{n+1}^{a_{n+1}} \leq_{\text{lex}} X_1^{b_1} X_2^{b_2} \cdots X_{n+1}^{b_{n+1}}$$

in the lexicographic ordering if either $a_1 < b_1$ or there exists an index i such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ and $a_i < b_i$.

Definition 1.6. Let $T_d \subset R$ be the set of monomials of degree d, a subset M of T_d is called a *lexicographic segment* (for short *lex–segment*), if there exist $t_1, t_2 \in T_d$ such that $t \in M$ iff $t_1 \leq t \leq t_2$.

Definition 1.7. J is called a *lexicographic segment ideal* (for short *lex–segment ideal*) if one can pick a set of generators such that the generators of minimum degree d_0 form a lexicographic segment beginning with $X_1^{d_0}, X_1^{d_0-1}X_2, \dots$ and the generators of J in degree $d \geq d_0$ form a lexicographic segment starting with the largest monomial of degree d not divisible by a generator of lower degree.

Definition 1.8. The rank of the i–th module in the resolution (1) is called the i–th *Betti number* of I and is denoted by $q_i(R/I)$; the graded Betti number of I, $q_{ij}(R/I)$, is defined to be the rank of the degree j graded R–module $\oplus R(-j)^{q_{ij}}$ for each i, j. Notice

$$q_i(R/I) = \sum_j q_{ij}(R/I).$$

Theorem 1.9. *Let $J \subset R$ be a lex–segment ideal with a given Hilbert Function H and let $I \subset R$ be any other homogeneous ideal with $H_{(R/J)} = H_{(R/I)} = H$. Then*

$$q_{ij}(R/J) \geq q_{ij}(R/I) \qquad \forall i, j.$$

Proof. See [4] Theorem 6. □

Theorem 1.10. *Let H be a Hilbert Function and $J \subset R$ the unique lex–segment ideal such that $H_{(R/J)} = H$; then*

$$q_{1j}(R/J) = (H(j-1)_{j-1})_+^+ - H(j).$$

Proof. See [11] Proposition 6.7. □

From [5] we know that monomial ideals in $\overline{R} = k[X_1, \dots, X_n]$ can be lifted to ideals of distinct points in $\mathbb{P}^n$ with the same Betti numbers, then we obtain the following result:

Theorem 1.11. *Let X be a set of distinct points in $\mathbb{P}^n$ with Hilbert Function $H(X, i)$ and $J \subset \overline{R}$ be the lex–segment ideal such that $H_{(\overline{R}/J)}(i) = \Delta H(X, i)$, then*

$$q_{ij}(R/I(X)) \leq q_{ij}(\overline{R}/J) \qquad \forall i, j.$$

Proof. Let (1) be the minimal graded free resolution of $A = R/I(X)$, we can suppose that X_{n+1} be a regular element in A and we can consider the ring

$$A/(X_{n+1}) = R/(I(X), X_{n+1}) = \frac{\overline{R}}{(I(X), X_{n+1})/(X_{n+1})}.$$

We put $\overline{I} = (I(X), X_{n+1})/(X_{n+1})$, it is easy to see that a minimal graded free resolution of $\overline{R}/\overline{I}$ in $\overline{R}$ is

$$0 \to \oplus\overline{R}(-j)^{q_{nj}} \to \ldots \to \oplus\overline{R}(-j)^{q_{2j}} \to \oplus\overline{R}(-j)^{q_{1j}} \to \overline{R} \to \overline{R}/\overline{I} \to 0,$$

so $H_{(\overline{R}/\overline{I})}(i) = \Delta H(X, i)$.

Using the Theorem 1.9, we get

$$q_{ij}(\overline{R}/J) \geq q_{ij}(\overline{R}/\overline{I}) \qquad \forall i, j,$$

but

$$q_{ij}(R/I(X)) = q_{ij}(\overline{R}/\overline{I}),$$

this gives the conclusion. □

Remark 1.12. When $n = 3$ we know that $I(X)$ has a minimal graded free resolution:

$$0 \to \oplus R(-i)^{\gamma_i} \to \oplus R(-i)^{\beta_i} \to \oplus R(-i)^{\alpha_i} \to I(X) \to 0$$

where α_i, β_i and γ_i are the graded Betti numbers and it is easy to prove that they are linked with the relation

$$-\alpha_i + \beta_i - \gamma_i = \Delta^4 H(X, i); \tag{2}$$

particularly:

$$\alpha_i = \dim_k([I(X)]_i/R_1[I(X)]_{i-1}) =$$
$$= \dim_k[I(X)]_i - \dim_k R_1[I(X)]_{i-1}.$$

By the Theorem 1.11 we have:

$$\alpha_{i+1} \leq (\Delta H(X, i)_i)_+^+ - \Delta H(X, i+1) \qquad \forall i \in \mathbb{N}.$$

We note that $-\Delta^2 H(X, i+1) \leq (\Delta H(X, i)_i)_+^+ - \Delta H(X, i+1)$, in fact:

$$-\Delta^2 H(X, i+1) = -\Delta H(X, i+1) + \Delta H(X, i) \leq$$
$$\leq -\Delta H(X, i+1) + (\Delta H(X, i)_i)_+^+.$$

In the following we denote $I = I(X)$.

An algebraic result

Afterwards we will use the following:

Lemma 1.13. *Let $S = 0$, $S \in R = k[X_1, \dots, X_{n+1}]$, be the equation of a hypersurface $\Sigma \subset \mathbb{P}^n$ of degree s and L_1, L_2 two linear forms that define the hyperplanes π_1, π_2 in $\mathbb{P}^n$, such that:*

$$\begin{cases} \pi_1 \cap \pi_2 = r \\ r \not\subset \Sigma, \end{cases}$$

then (L_1, L_2, S) is a regular sequence.

Proof. Obviously L_1 is regular in R and L_2 in $R/(L_1)$; the ideal (L_1, L_2) is prime, so S is regular in $R/(L_1, L_2)$ iff $S \notin (L_1, L_2)$ and it is true because $r \subset \pi_1, \pi_2$ but $r \not\subset \Sigma$. □

2. Points on an irreducible quadric.

Let X be a set of distinct points in $\mathbb{P}^3$ on an irreducible quadric surface Σ defined by $Q = 0$, from [9] Theorem 2.2, we have

$$\Delta H(X, i) = \begin{cases} 2i+1 & 0 \leq i \leq b-1 \\ s_i & b \leq i \leq n \\ 0 & i \geq n+1 \end{cases}$$

such that $2b \geq s_b \geq s_{b+1} \geq \dots \geq s_n \geq 1$, $b \geq 2$.

We have

$$\Delta^2 H(X, i) = \begin{cases} 1 & i = 0 \\ 2 & 1 \leq i \leq b-1 \\ s_b - 2b + 1 & i = b \\ s_i - s_{i-1} & b+1 \leq i \leq n \\ -s_n & i = n+1 \\ 0 & i \geq n+2 \end{cases}$$

$$\Delta^4 H(X,i) = \begin{cases} 1 & i=0 \\ 0 & i=1 \\ -1 & i=2 \\ 0 & 3 \le i \le b-1 \\ s_b - 2b - 1 & i=b \\ s_{b+1} - 3s_b + 4b & i=b+1 \\ s_{b+2} - 3s_{b+1} + s_b - 2b + 1 & i=b+2 \\ s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b+3 \le i \le n \\ -3s_n + 3s_{n-1} - s_{n-2} & i=n+1 \\ 3s_n - s_{n-1} & i=n+2 \\ -s_n & i=n+3 \\ 0 & i \ge n+4. \end{cases}$$

Let α_i be the minimal number of generators of degree i of the ideal I, then:

$\alpha_2 = 1$
$\alpha_b = 2b + 1 - s_b$ (by the hypothesis on s_b, it is $\alpha_b \ge 1$),
$\alpha_i = 0 \qquad i \ne 2, i < b$ or $i \ge n+2$,
$\alpha_{b+1} \ge \max(0, -s_{b+1} + 3s_b - 4b)$.

In the case $b = 2$ by abuse of notation we will denote again $\alpha_2 = 1$ and by α_b we mean the dimension of the k-vector space I_2 minus one.

Furthermore it is easy to compute that

$$H(X,i) = \begin{cases} \binom{i+3}{3} & i = 0,1 \\ (i+1)^2 & 2 \le i \le b-1 \\ b^2 + \sum_{j=b}^{i} s_j & b \le i \le n \\ b^2 + \sum_{j=b}^{n} s_j & i \ge n. \end{cases}$$

Theorem 2.1. *Let X be a finite set of distinct points on an irreducible quadric Σ. Let now h be an integer such that $b+1 \le h \le n+1$. With the above notations and $s_{n+1} = 0$, then*

$$\alpha_h \le s_{h-1} - s_h = -\Delta^2 H(X,h).$$

Proof. For all $b-1 \le i \le h$

$$\dim I_i = \binom{i+3}{3} - H(X,i) = \binom{i+3}{3} - b^2 - \sum_{j=b}^{i} s_j.$$

and

$$\alpha_i = \dim I_i - \dim R_1 I_{i-1}.$$

We wish to find a lower bound for $\dim R_1 I_{h-1}$. For this aim, let π_1, π_2 be two planes defined by two linear forms L_1 and L_2 such that $\pi_i \cap X = \emptyset$ for $i = 1, 2$ and the line $\pi_1 \cap \pi_2$ is not contained in Σ, so Q, L_1, L_2 form a regular sequence in $k[X_1, \dots, X_4]$.

Let $A_i \subset I_i$ be the k–vector space generated by the forms of I_i that do not belong to $R_{i-2}Q$ and let $B_i \subset A_i$ be the greatest subspace that does not contain forms that are divisible by L_1, then it is easy to check that

$$\text{(3)} \qquad \dim A_i = \binom{i+3}{3} - b^2 - \sum_{j=b}^{i} s_j - \binom{i+1}{3} = $$
$$= (i+1)^2 - b^2 - \sum_{j=b}^{i} s_j$$

and

$$\dim B_i = \dim A_i - \dim A_{i-1} \qquad i \ge b.$$

Moreover, setting $W \subset R_1 I_{h-1}$ the vector space generated by $R_{h-2}Q$, $L_1 A_{h-1}$ and $L_2 B_{h-1}$, we have:

$$\text{(4)} \quad \dim W = \binom{h+1}{3} + \dim A_{h-1} + \dim B_{h-1} =$$
$$= \binom{h+1}{3} + \dim A_{h-1} + \dim A_{h-1} - \dim A_{h-2} =$$
$$= \binom{h+1}{3} + 2h^2 - 2b^2 - 2\sum_{j=b}^{h-1} s_j - (h-1)^2 + b^2 + \sum_{j=b}^{h-2} s_j =$$
$$= \binom{h+1}{3} + 2h^2 - b^2 - \sum_{j=b}^{h-1} s_j - s_{h-1} - h^2 + 2h - 1 =$$
$$= \binom{h+1}{3} + h^2 + 2h - 1 - b^2 - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$

In fact, let $F \in A_{h-1}$, $G \in B_{h-1}$, $A \in R_{h-2}$ be such that in $k[X_1, \dots, X_4]$

$$AQ + L_1F + L_2G = 0. \tag{5}$$

In $k[X_1, \dots, X_4]/(Q, L_1)$, it results $L_2G = 0$.

Since Q, L_1, L_2 are a regular sequence in $k[X_1, \dots, X_4]$, then $G = 0$ in $k[X_1, \dots, X_4]/(Q, L_1)$; it means that in $k[X_1, \dots, X_4]$

$$G = HQ + KL_1 \qquad \text{for some } H \text{ and } K.$$

Since X is contained in the surfaces defined by the forms G and Q it follows that K vanishes in all points of X, then $K \in I_{h-2}$ and $G \in (L_1 I_{h-2} \oplus R_{h-3}Q)$, which implies $G = 0$ because $G \in B_{h-1}$. So (5) becomes:

$$AQ + L_1F = 0;$$

since Q does not divide F, then $A = 0$ and $F = 0$. This proves (4).

Now let P_3 and P_4 be the points in which Σ intersects the line $\pi_1 \cap \pi_2$; for the generic choice of this line and for the irreducibility of Σ, we can suppose that there exists a form $H \in A_{h-1}$ such that P_3 and P_4 do not lie on the surface Ψ defined by H. Let us choose L_3 and L_4 two linear forms defining, respectively, two planes π_3, π_4 such that

$$\begin{cases} P_3 \in \pi_3 \\ P_4 \notin \pi_3 \end{cases} \quad \text{and} \quad \begin{cases} P_3 \notin \pi_4 \\ P_4 \in \pi_4. \end{cases}$$

We are now going to prove that L_3H, L_4H and the elements of W are linearly independent; in fact, let $A \in R_{h-2}$, $F \in A_{h-1}$, $G \in B_{h-1}$, $a, b \in k$ be such that in $k[X_1, \dots, X_4]$:

$$AQ + L_1F + L_2G + aL_3H + bL_4H = 0. \tag{6}$$

Let us compute (6) in P_3; we have

$$bL_4(P_3)H(P_3) = 0;$$

but $L_4(P_3) \neq 0$, $H(P_3) \neq 0$, thus $b = 0$. Similarly, computing (6) in P_4 we have $a = 0$. So (6) becomes

$$AQ + L_1F + L_2G = 0$$

that is (5); then $A = 0$, $F = 0$, $G = 0$.

Hence

$$\alpha_h \leq \dim I_h - \dim W - 2 =$$

$$= \binom{h+3}{3} - b^2 - \sum_{j=b}^{h} s_j - \binom{h+1}{3} - h^2 - 2h + 1 + b^2 +$$

$$+ \sum_{j=b}^{h-1} s_j + s_{h-1} - 2 =$$

$$= (h+1)^2 - s_h - h^2 - 2h - 1 + s_{h-1} =$$

$$= h^2 + 2h + 1 - s_h - h^2 - 2h - 1 + s_{h-1} =$$

$$= s_{h-1} - s_h,$$

that is the conclusion. □

Example 2.2. These bounds are sharp; in fact, by [6] Proposition 2.3 and [7] Remark 2.12 we have that for any set X of points on an irreducible quadric, ACM as subscheme of the quadric, is $\alpha_i = -\Delta^2 H(X, i)$ if $i = b+1, \dots, n$.

3. Points on an irreducible cubic surface.

Let X be a set of points in $\mathbb{P}^3$ on an irreducible surface Σ of degree 3, from [10] Theorem 2.1, we have

$$\Delta H(X, i) = \begin{cases} \binom{i+2}{2} & 0 \leq i \leq 2 \\ 3i & 3 \leq i \leq b-1 \\ s_i & b \leq i \leq n \\ 0 & i \geq n+1 \end{cases}$$

where

$s_b \leq 3b - 1,$

$s_b + 1 \geq s_{b+1} \geq s_{b+2} \geq \dots \geq s_n \geq 1$ and $b \geq 3$.

Hence

$$\Delta^2 H(X, i) = \begin{cases} i+1 & i = 0, 1, 2 \\ 3 & 3 \leq i \leq b-1 \\ s_b - 3b + 3 & i = b \\ s_i - s_{i-1} & b+1 \leq i \leq n \\ -s_n & i = n+1 \\ 0 & i \geq n+2 \end{cases}$$

$$\Delta^4 H(X,i) = \begin{cases} 1 & i=0 \\ 0 & i=1,2 \\ -1 & i=3 \\ 0 & 4 \le i \le b-1 \\ s_b - 3b & i=b \\ s_{b+1} - 3s_b + 6b - 3 & i=b+1 \\ s_{b+2} - 3s_{b+1} + 3s_b - 3b + 3 & i=b+2 \\ s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b+3 \le i \le n \\ -3s_n + 3s_{n-1} - s_{n-2} & i=n+1 \\ 3s_n - s_{n-1} & i=n+2 \\ -s_n & i=n+3 \\ 0 & i \ge n+4. \end{cases}$$

Let α_i be the minimal number of generators of degree i of the ideal I, then:

$\alpha_3 = 1$ and we will call S the form defining Σ,

$\alpha_b = 3b - s_b$ ($\alpha_b \ge 1$ by the hypothesis on s_b) and we will call these forms $F_1, \dots, F_{\alpha_b}$,

$\alpha_i = 0 \qquad i \ne 3, i < b$ or $i \ge n+2$,

$\alpha_{b+1} \ge \max(0, -s_{b+1} + 3s_b - 6b + 3)$.

If $b = 3$ we will use the same notation as the previous section.

Furthermore it is easy to compute

$$H(X,i) = \begin{cases} \binom{i+3}{3} & i = 0,1,2 \\ \frac{3i^2+3i}{2} + 1 & 3 \le i \le b-1 \\ \frac{3b^2-3b}{2} + 1 + \sum_{j=b}^{i} s_j & b \le i \le n \\ \frac{3b^2-3b}{2} + 1 + \sum_{j=b}^{n} s_j & i \ge n. \end{cases}$$

Theorem 3.1. *Let X be a finite set of distinct points on an irreducible cubic surface Σ.*

With the previous notations, we have:

$$\alpha_{b+1} \le \max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)).$$

Proof. We have:

$$\dim I_{b+1} = \dim R_{b+1} - H(X, b+1) =$$
$$= \binom{b+4}{3} - \frac{3b^2 - 3b}{2} - 1 - s_b - s_{b+1}$$

and

$$\alpha_{b+1} = \dim I_{b+1} - \dim R_1 I_b.$$

We note that

$$\max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) = -\Delta^4 H(X, b+1)$$

iff

$$s_b = 3b - 1,$$

so we can distinguish two cases: i) $s_b < 3b - 1$, ii) $s_b = 3b - 1$.

If $s_b < 3b - 1$, then

$$\alpha_b = 3b - s_b > 3b - 3b + 1$$

and we can consider two forms F_1, F_2 of degree b which define, respectively, two surfaces Φ_1 and Φ_2.

We work as in Theorem 2.1. Let r be a line in $\mathbb{P}^3$ such that:

$$\begin{aligned} &r \cap \Sigma = \{P_1, P_2, P_3\} \\ &r \cap \Sigma \cap \Phi_1 = \emptyset \\ &r \cap \Sigma \cap \Phi_2 = \{P_2\} \end{aligned} \tag{7}$$

and let L_1, L_2, L_3 and L_4 be linear forms defining, respectively, the planes π_1, π_2, π_3 and π_4 such that π_1 and π_2 contain r but not the points of X and π_3 and π_4 such that

$$\begin{cases} P_1 \in \pi_3 \\ P_2 \notin \pi_3 \end{cases} \quad \text{and} \quad \begin{cases} P_1 \notin \pi_4 \\ P_2 \in \pi_4, \end{cases} \tag{8}$$

hence $P_3 \notin \pi_3$ and $P_3 \notin \pi_4$. Notice that S, L_1, L_2 form a regular sequence in $k[X_1, \ldots, X_4]$. The elements given by the $\binom{b+1}{3}$ generators of the vector subspace $R_{b-2}S$ of the forms of degree $b+1$ containing S and by the $2\alpha_b$ forms of type L_1F_i, L_2F_i, are linearly independent. In fact, let $A \in R_{b-2}, \overline{F}, \overline{\overline{F}} \in \mathscr{L}(F_1, \dots, F_{\alpha_b})$ be such that:

$$AS + L_1\overline{F} + L_2\overline{\overline{F}} = 0, \tag{9}$$

we have $L_2\overline{\overline{F}} = 0$ in $k[X_1, \dots, X_4]/(S, L_1)$. Since S, L_1, L_2 are a regular sequence in $k[X_1, \dots, X_4]$, it follows that $\overline{\overline{F}} = 0$ in $k[X_1, \dots, X_4]/(S, L_1)$; so

$$\overline{\overline{F}} = SH + L_1K \qquad \text{for some } H \text{ and } K.$$

Since X is contained in the surfaces defined by the forms $\overline{\overline{F}}$ and S, it follows that X must be contained in the surface defined by KL_1. Since $X \cap \pi_1 = \emptyset$, then the form K vanishes in X; but $\deg K = b - 1$, thus $K = 0$. In fact, if $K \neq 0$ we get $K \in (S)$ and so $\overline{\overline{F}} \in (S)$ and this is false; thus $\overline{\overline{F}} = 0$ and (9) becomes

$$AS + L_1\overline{F} = 0$$

then $A = 0$ and $\overline{F} = 0$. This proves that the previous $\binom{b+1}{3} + 2\alpha_b$ elements of R_1I_b are linearly independent.

We claim that if we add to these elements the forms L_3F_1, L_3F_2 and L_4F_1, we still get independent elements; in fact, let $A \in R_{b-2}, \overline{F}, \overline{\overline{F}} \in \mathscr{L}(F_1, \dots, F_{\alpha_b})$, $a, b, c \in k$ be such that in $k[X_1, \dots, X_4]$:

$$AS + L_1\overline{F} + L_2\overline{\overline{F}} + aL_3F_1 + bL_3F_2 + cL_4F_1 = 0. \tag{10}$$

Let us compute (10), respectively, in P_1, P_2, P_3; we have:

$$cL_4(P_1)F_1(P_1) = 0$$
$$aL_3(P_2)F_1(P_2) = 0$$
$$aL_3(P_3)F_1(P_3) + bL_3(P_3)F_2(P_3) + cL_4(P_3)F_1(P_3) = 0$$

and, by (7) and (8), $c = 0$, $a = 0$ and $b = 0$.

Hence (10) becomes

$$AS + L_1\overline{F} + L_2\overline{\overline{F}} = 0,$$

then $A = 0$, $\overline{F} = 0$ and $\overline{\overline{F}} = 0$.

Therefore

$$\dim R_1I_b \geq \binom{b+1}{3} + 2\alpha_b + 3 =$$

$$= \binom{b+1}{3} + 6b - 2s_b + 3$$

and, consequently,

$$\alpha_{b+1} \leq \binom{b+4}{3} - \frac{3b^2-3b}{2} - 1 - s_b - s_{b+1} - \binom{b+1}{3} - 6b + 2s_b - 3 =$$

$$= \frac{3b^2+9b+6}{2} + 1 - \frac{3b^2-3b}{2} - 4 + s_b - s_{b+1} - 6b =$$

$$= 6b + 3 - 3 + s_b - s_{b+1} - 6b =$$

$$= s_b - s_{b+1}$$

that concludes the first case.

If $s_b = 3b - 1$ then $\alpha_b = 1$ and, since S is irreducible, $\beta_{b+1} = 0$; so, by (2),

$$\alpha_{b+1} = -\Delta^4 H(X, b+1)$$

(obviously $\gamma_{b+1} = 0$). □

Corollary 3.2. *Let X be a finite set of distinct points on an irreducible cubic surface Σ.*

With the previous notations, if $s_b < 3b - 1$ then

$$s_{b+1} \leq s_b.$$

Proof. Under these hypotheses

$$\max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) = -\Delta^2 H(X, b+1)$$

so, by the previous Theorem,

$$0 \leq \alpha_{b+1} \leq -s_{b+1} + s_b. \quad \square$$

This Corollary improves the result given in [10] Theorem 2.1.

Theorem 3.3. *Let X be a finite set of distinct points on an irreducible cubic surface Σ. Let now h be an integer such that $b + 1 \leq h \leq n + 1$. With the above notations and $s_{n+1} = 0$, then*

$$\alpha_h \leq \begin{cases} \max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) & h = b+1 \\ -\Delta^2 H(X, h) & h > b+1. \end{cases}$$

Proof. For $h = b + 1$ see Theorem 3.1, so we assume $h > b + 1$. Let H_1, $H_2 \in (I_{h-1} - R_{h-4}S)$ be two forms defining surfaces Ψ_1 and Ψ_2.

We can choose a line r in $\mathbb{P}^3$ such that:

$$r \cap \Sigma = \{P_1, P_2, P_3\}$$

$$r \cap \Sigma \cap \Psi_1 = \emptyset$$

$$r \cap \Sigma \cap \Psi_2 = \{P_2\};$$

let us introduce the linear forms L_1, L_2, L_3 and L_4 as before. Notice that S, L_1, L_2 form a regular sequence in $k[X_1, \dots, X_4]$.

For all $b - 1 \leq i \leq h$, we have

$$\dim I_i = \binom{i+3}{3} - H(X, i) =$$

$$= \binom{i+3}{3} - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j.$$

Now we proceed as in the quadric case. Let $A_i \subset I_i$ be the k–vector space generated by the forms of I_i that do not belong to $R_{i-3}S$ and let $B_i \subset A_i$ be the greatest subspace that does not contain forms that are divisible by L_1, then it is easy to check that

$$\dim A_i = \binom{i+3}{3} - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j - \binom{i}{3} =$$

$$= \frac{3i^2 + 3i}{2} + 1 - \frac{3b^2 - 3b}{2} - 1 - \sum_{j=b}^{i} s_j =$$

$$= \frac{3i^2 + 3i}{2} - \frac{3b^2 - 3b}{2} - \sum_{j=b}^{i} s_j$$

and

$$\dim B_i = \dim A_i - \dim A_{i-1} = 3i - s_i \qquad i \geq b.$$

Moreover, denoting $W \subset R_1 I_{h-1}$ the vector space generated by $R_{h-3}S$, $L_1 A_{h-1}$ and $L_2 B_{h-1}$, we get as in the previous Theorem:

$$\dim W = \binom{h}{3} + \dim A_{h-1} + \dim B_{h-1} =$$

$$= \binom{h}{3} + \frac{3(h-1)^2 + 9(h-1)}{2} - \frac{3b^2 + 3b}{2} - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$

Again we can add the forms L_3H_1, L_3H_2, L_4H_1 still obtaining independent elements; so:

$$\begin{aligned}\alpha_h \leq & \dim I_h - \dim W - 3 = \\ = & s_{h-1} - s_h . \end{aligned} \qquad \square$$

Example 3.4. We give now an example of a set of points in which these bounds are sharp. In fact, let Σ be an irreducible cubic surface and let r be a line contained in Σ; let us consider 5 points in r and other 15 generic points in Σ, we call X the set consisting of these points. Thus the Hilbert Function of X is given by:

	0	1	2	3	4	5	6	7	8	...
H	1	4	10	19	20	20	20	20	20	...
ΔH	1	3	6	9	1	0	0	0	0	...
$\Delta^2 H$	1	2	3	3	-8	-1	0	0	0	...
$\Delta^3 H$	1	1	1	0	-11	7	1	0	0	...
$\Delta^4 H$	1	0	0	-1	-11	18	-6	-1	0	...

In this case $b = n = 4$, $\alpha_3 = 1$, $\alpha_4 = 11$ and $\alpha_5 = 1$ because the generators of degree lesser than 5 must vanish in all points of r, hence $\alpha_5 = -\Delta^2 H(X, 5)$.

4. Points on an irreducible surface of degree 4.

Let X be a set of points in $\mathbb{P}^3$ on an irreducible surface Σ of degree 4, from [10] Theorem 2.1, we have

$$\Delta H(X, i) = \begin{cases} \binom{i+2}{2} & 0 \leq i \leq 3 \\ 4i - 2 & 4 \leq i \leq b - 1 \\ s_i & b \leq i \leq n \\ 0 & i \geq n + 1 \end{cases}$$

with

$s_b \leq 4b - 3$,
$s_{b+1} \leq s_b + 2$,
$s_{b+2} \leq s_{b+1} + 1$,
$s_{b+2} \geq s_{b+3} \geq \ldots \geq s_n \geq 1$ and $b \geq 4$.

We have

$$\Delta^2 H(X,i) = \begin{cases} i+1 & 0 \le i \le 3 \\ 4 & 4 \le i \le b-1 \\ s_b - 4b + 6 & i = b \\ s_i - s_{i-1} & b+1 \le i \le n \\ -s_n & i = n+1 \\ 0 & i \ge n+2 \end{cases}$$

$$\Delta^4 H(X,i) = \begin{cases} 1 & i = 0 \\ 0 & 1 \le i \le 3 \\ -1 & i = 4 \\ 0 & 5 \le i \le b-1 \\ s_b - 4b + 2 & i = b \\ s_{b+1} - 3s_b + 8b - 8 & i = b+1 \\ s_{b+2} - 3s_{b+1} + 3s_b - 4b + 6 & i = b+2 \\ s_i - 3s_{i-1} + 3s_{i-2} - s_{i-3} & b+3 \le i \le n \\ -3s_n + 3s_{n-1} - s_{n-2} & i = n+1 \\ 3s_n - s_{n-1} & i = n+2 \\ -s_n & i = n+3 \\ 0 & i \ge n+4. \end{cases}$$

Let α_i be the minimal number of generators of degree i of the ideal I, then:

$\alpha_4 = 1$ and we will call S the form defining Σ,
$\alpha_b = 4b - 2 - s_b$ ($\alpha_b \ge 1$ by the hypothesis on s_b) and we will call these forms $F_1, \dots, F_{\alpha_b}$,
$\alpha_i = 0$ for $i \ne 4$, $i < b$ or $i \ge n+2$,
$\alpha_{b+1} \ge \max(0, -s_{b+1} + 3s_b - 8b + 8)$.

If $b = 4$ we will use the same notation as the previous section.

Furthermore it is easy to compute

$$H(X,i) = \begin{cases} \binom{i+3}{3} & i = 0,1,2 \\ 2i^2 + 2 & 3 \le i \le b-1 \\ 2b^2 - 4b + 4 + \sum_{j=b}^{i} s_j & b \le i \le n \\ 2b^2 - 4b + 4 + \sum_{j=b}^{n} s_j & i \ge n. \end{cases}$$

Case $i = b + 1$

Theorem 4.1. *Let X be a finite set of distinct points on an irreducible surface Σ of degree 4.*

With the previous notations we have:

$$\alpha_{b+1} \leq \max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)).$$

Proof. We have:

$$\dim I_{b+1} = \dim R_{b+1} - H(X, b+1) = \\ = \binom{b+4}{3} - 2b^2 + 4b - 4 - s_b - s_{b+1}$$

and

$$\alpha_{b+1} = \dim I_{b+1} - \dim R_1 I_b.$$

We note that

$$\max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) = -\Delta^2 H(X, b+1)$$

iff

$$s_b \leq 4b - 4$$

so we can distinguish two cases: i) $s_b \leq 4b - 4$, ii) $s_b = 4b - 3$.

If $s_b \leq 4b - 4$, then

$$\alpha_b = 4b - 2 - s_b \geq 4b - 2 - 4b + 4 = 2,$$

so we can consider two forms F_1, F_2 which define, respectively, two surfaces Φ_1 and Φ_2 of degree b.

One can easily check that, if S is irreducible, there exists a line r in $\mathbb{P}^3$ such that:

$$\begin{aligned} & r \cap \Sigma = \{P_1, P_2, P_3, P_4\} \\ & r \cap \Sigma \cap \Phi_1 = \emptyset \\ (11) \qquad & r \cap \Sigma \cap \Phi_2 = \{P_1, P_2\} \\ & P_3 \not\equiv P_4. \end{aligned}$$

Let L_1, L_2, L_3 and L_4 be linear forms defining, respectively, the planes π_1, π_2, π_3 and π_4 such that π_1 and π_2 contain r but not the points of X and π_3 and π_4 such that

$$(12) \qquad \begin{cases} P_1 \in \pi_3 \\ P_2 \notin \pi_3 \end{cases} \quad \text{and} \quad \begin{cases} P_1 \notin \pi_4 \\ P_2 \in \pi_4, \end{cases}$$

so that P_3, $P_4 \notin \pi_3$ and P_3, $P_4 \notin \pi_4$. Again one shows that $\binom{b}{3} + 2\alpha_b$ elements given by the $\binom{b}{3}$ generators of the vector subspace $R_{b-3}S$ of the forms of degree $b+1$ containing S and by the $2\alpha_b$ forms of type L_1F_i, L_2F_i, are linearly independent.

Now we can add to them the four forms L_3F_1, L_3F_2 L_4F_1 and L_4F_2 still getting independent elements; in fact, let $A \in R_{b-3}$, $\overline{F}, \overline{\overline{F}} \in \mathscr{L}(F_1, \dots, F_{\alpha_b})$, $a, b, c, d \in k$ be such that in $k[X_1, \dots, X_4]$:

$$(13) \qquad AS + L_1\overline{F} + L_2\overline{\overline{F}} + aL_3F_1 + bL_3F_2 + cL_4F_1 + dL_4F_2 = 0.$$

Let us compute (13), respectively, in P_1, P_2; we get

$cL_4(P_1)F_1(P_1) = 0$
$aL_3(P_2)F_1(P_2) = 0$

so, by (11) and (12), $c = 0$ and $a = 0$; computing (13) in P_3 and P_4 we obtain the following system:

$$(14) \qquad \begin{cases} bL_3(P_3)F_2(P_3) + dL_4(P_3)F_2(P_3) = 0 \\ bL_3(P_4)F_2(P_4) + dL_4(P_4)F_2(P_4) = 0, \end{cases}$$

since $F_2(P_3) \neq 0$ and $F_2(P_4) \neq 0$, then (14) becomes:

$$\begin{cases} bL_3(P_3) + dL_4(P_3) = 0 \\ bL_3(P_4) + dL_4(P_4) = 0, \end{cases}$$

since $L_3 \cap L_4$ and r are skew, then (14) has only the solution $b = 0$, $d = 0$. Hence (13) becomes

$$AS + L_1\overline{F} + L_2\overline{\overline{F}} = 0,$$

so again $A = 0$, $\overline{F} = 0$ and $\overline{\overline{F}} = 0$.

Therefore

$$\dim R_1 I_b \geq \binom{b}{3} + 2\alpha_b + 4 = \binom{b}{3} + 8b - 2s_b$$

and

$$\alpha_{b+1} \le \binom{b+4}{3} - 2b^2 + 4b - 4 - s_b - s_{b+1} - \binom{b}{3} - 8b + 2s_b =$$

$$= 2b^2 + 4b + 2 + 2 - 2b^2 - 4b - 4 + s_b - s_{b+1} =$$

$$= s_b - s_{b+1}$$

that concludes the first case.

If $s_b = 4b - 3$ then $\alpha_b = 1$ and, being S irreducible, $\beta_{b+1} = 0$; so, by (2),

$$\alpha_{b+1} = -\Delta^4 H(X, b+1)$$

(obviously $\gamma_{b+1} = 0$). □

Corollary 4.2. *Let X be a finite set of distinct points on an irreducible surface Σ of degree* 4.

With the previous notations, if $s_b < 4b - 3$ then

$$s_{b+1} \le s_b.$$

Proof. Under these hypotheses

$$\max(-\Delta^2 H(X, b+1), -\Delta^4 H(X, b+1)) = -\Delta^2 H(X, b+1)$$

so, by the previous Theorem,

$$0 \le \alpha_{b+1} \le -s_{b+1} + s_b. \qquad \square$$

Case $i = b + 2$

Theorem 4.3. *Let X be a finite set of distinct points on an irreducible surface Σ of degree* 4.

With the previous notations it follows:

$$\alpha_{b+2} \le \begin{cases} -\Delta^4 H(X, b+2) & \text{if } s_{b+1} = s_b + 2 \\ -\Delta^2 H(X, b+2) & \text{otherwise.} \end{cases}$$

Proof. We know that:

$$\dim I_{b+2} = \dim R_{b+2} - H(X, b+2) =$$

$$= \binom{b+5}{3} - 2b^2 + 4b - 4 - s_b - s_{b+1} - s_{b+2}$$

and

$$\alpha_{b+2} = \dim I_{b+2} - \dim R_1 I_{b+1}.$$

If $s_{b+1} \leq s_b + 1$ we can have two cases: either $s_b < 4b - 3$ then $\alpha_b \geq 2$, or $s_b = 4b - 3$ and $s_{b+1} \leq s_b + 1$ thus in this case $\alpha_b = 1$ but $\alpha_{b+1} \geq 1$; so, in both cases, we can find two surfaces Ψ_1 and Ψ_2 defined by H_1 and H_2 in $(I_{b+1} - R_{b-3}S)$ and a line r satisfying the conditions:

$$\begin{aligned} & r \cap \Sigma = \{P_1, P_2, P_3, P_4\} \\ & r \cap \Sigma \cap \Psi_1 = \emptyset \\ & r \cap \Sigma \cap \Psi_2 = \{P_1, P_2\} \\ & P_3 \not\equiv P_4. \end{aligned} \tag{15}$$

Let L_1, L_2, L_3, L_4 be the linear forms as in the previous case; if $t = \dim I_{b+1}$, let $\mathscr{B} = \{G_1, \ldots, G_t\}$ be a basis of I_{b+1}, we can suppose that $G_i \in R_{b-3}S$ for $i = 1, \ldots, \binom{b}{3}$ and $G_i = L_1 F_{i-\binom{b}{3}}$ for $i = \binom{b}{3} + 1, \ldots, \binom{b}{3} + \alpha_b$.

As in the previous case we can choose independent elements in I_{b+2} in the following way. Pick $\binom{b+1}{3}$ generators of $R_{b-2}S$, elements of type $L_1 G_i$ with $i > \binom{b}{3}$ (i.e. $G_i \in \mathscr{B}$ and S does not divide G_i; they are in number of $\dim I_{b+1} - \binom{b}{3}$), and elements of type $L_2 G_i$ with $i > \binom{b}{3} + \alpha_b$ (they are in number of $\dim I_{b+1} - \binom{b}{3} - \alpha_b$).

So we find

$$N = \binom{b+1}{3} + \dim I_{b+1} - \dim R_{b-3}S + \dim I_{b+1} - \dim R_{b-3}S - \alpha_b =$$

$$= \binom{b+1}{3} + 12b + 2 - s_b - 2s_{b+1}$$

surfaces in $R_1 I_{b+1}$ that are linearly independent.

As in the previous case, we can add the four forms $L_3 H_1$, $L_4 H_1$, $L_3 H_2$ and $L_4 H_2$; so

$$\alpha_{b+2} \leq \binom{b+5}{3} - 2b^2 + 4b - 4 - s_b - s_{b+1} - s_{b+2} - N - 4 =$$

$$= s_{b+1} - s_{b+2}$$

this concludes the proof if $s_{b+1} \leq s_b + 1$.

If $s_{b+1} = s_b + 2$, for the Corollary 4.2, $s_b = 4b - 3$ then $\alpha_b = 1$ and $\alpha_{b+1} = 0$.

Since Σ is irreducible and of degree 4, then $\beta_{b+2} = 0$ and $\gamma_{b+2} = 0$; thus

$$\alpha_{b+2} = -\Delta^4 H(X, b+2). \qquad \square$$

Corollary 4.4. *Let X be a finite set of distinct points on an irreducible surface Σ of degree 4.*

With the previous notations, if $s_{b+1} \leq s_b + 1$, then

$$s_{b+2} \leq s_{b+1}.$$

Proof. Applying the previous Theorem with $s_{b+1} \leq s_b + 1$

$$0 \leq \alpha_{b+2} \leq s_{b+1} - s_{b+2}. \qquad \square$$

Remark 4.5. Corollary 4.2 and Corollary 4.4 improve the results given in [10] for the Hilbert Function of a set of points lying on an irreducible surface.

The general case

Theorem 4.6. *Let X be a finite set of distinct points on an irreducible surface Σ of degree 4. Let now h be an integer such that $b + 3 \leq h \leq n + 1$. With the previous notations and $s_{n+1} = 0$, then*

$$\alpha_h \leq -\Delta^2 H(X, h) = s_{h-1} - s_h.$$

Proof. We know

$$\dim I_h = \binom{h+3}{3} - H(X, h) =$$

$$= \binom{h+3}{3} - 2b^2 + 4b - 4 - \sum_{j=b}^{h} s_j.$$

Let $H_1, H_2 \in (I_{h-1} - R_{h-4}S)$ be two forms defining surfaces Ψ_1 and Ψ_2 such that:

$$\begin{aligned} & r \cap \Sigma = \{P_1, P_2, P_3, P_4\} \\ & r \cap \Sigma \cap \Psi_1 = \emptyset \\ & r \cap \Sigma \cap \Psi_2 = \{P_1, P_2\} \qquad (16) \\ & P_3 \not\equiv P_4, \end{aligned}$$

let us introduce the linear forms L_1, L_2, L_3 and L_4 as before.

Let $A_h \subset I_h$ be the k-vector space generated by the forms of I_h that do not belong to $R_{h-4}S$ and let $B_h \subset A_h$ be the greatest subspace that does not contain forms that are divisible by L_1, then it is easy to check that

$$\dim A_h = \binom{h+3}{3} - 2b^2 + 4b - 4 - \sum_{j=b}^{h} s_j - \binom{h-1}{3} =$$

$$=2h^2 - 2b^2 + 4b - 2 - \sum_{j=b}^{h} s_j$$

and

$$\dim B_h = \dim A_h - \dim A_{h-1} =$$

$$=4h - 2 - s_h.$$

Moreover, if $W \subset R_1 I_{h-1}$ is the vector space generated by $R_{h-4}S$, $L_1 A_{h-1}$ and $L_2 B_{h-1}$, again we obtain:

$$\dim W = \binom{h-1}{3} + \dim A_{h-1} + \dim B_{h-1} =$$

$$=\binom{h-1}{3} + 2h^2 - 6 - 2b^2 + 4b - \sum_{j=b}^{h-1} s_j - s_{h-1}.$$

Adding, like before cases, the forms $L_3 H_1$, $L_3 H_2$, $L_4 H_1$ and $L_4 H_2$, we obtain:

$$\alpha_h \leq \dim I_h - \dim W - 4 = s_{h-1} - s_h . \quad \square$$

Example 4.7. We construct a set of points for which these bounds are sharp; let Σ be an irreducible surface of degree 4 containing a line r. Let us consider 6 points in r and other 29 generic points in Σ, we call X the set consisting of these points. Thus the Hilbert Function of X is given by:

	0	1	2	3	4	5	6	7	8	9	...
H	1	4	10	20	34	35	35	35	35	35	...
ΔH	1	3	6	10	14	1	0	0	0	0	...
$\Delta^2 H$	1	2	3	4	4	-13	-1	0	0	0	...
$\Delta^3 H$	1	1	1	1	0	-17	12	1	0	0	...
$\Delta^4 H$	1	0	0	0	-1	-17	29	-11	-1	0	...

In this case $b = n = 5$, $\alpha_4 = 1$, $\alpha_5 = 17$ and $\alpha_6 = 1$ because the generators of degree lesser than 6 must vanish in all points of r, hence $\alpha_6 = -\Delta^2 H(X, 6)$.

5. Applications.

Remark 5.1. Let $b + 1 \leq h \leq n + 1$, all bounds founded for α_h are valid just asking that there exists a form F in I_{h-1} defining a surface with no common components with a surface Σ of smaller degree containing X.

Example 5.2. Let Q the form defining Σ. The hypothesis that there exists a surface in I of degree $h - 1$ such that Q and F have no common components is necessary. Let us consider 10 points in generic position on a plane π and 3 generic points outside π; let X be the set of these 13 points, then the Hilbert Function of X is given by:

	0	1	2	3	4	5	6	7	...
H	1	4	9	13	13	13	13	13	...
ΔH	1	3	5	4	0	0	0	0	...
$\Delta^2 H$	1	2	2	-1	-4	0	0	0	...
$\Delta^3 H$	1	1	0	-3	-3	4	0	0	...
$\Delta^4 H$	1	0	-1	-3	0	7	-4	0	...

Hence $\alpha_2 = 1$, $\alpha_3 = 3$, $\alpha_4 \geq 3$ and $\alpha_4 = \beta_4$, but β_4 is equal to the number of syzyges of degree 3 for a set consisting of 3 generic points in $\mathbb{P}^3$, so $\alpha_4 = 5$, while, by Theorem 2.1, $\alpha_4 \leq 4$.

This happens because the quadric passing through these 13 points splits in π and in the plane defined by the 3 points outside π and all cubic surfaces passing through X contain π.

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Dipartimento di Matematica,
Università di Catania,
Viale A. Doria 6,
95125 Catania (ITALY)