LE MATEMATICHE
Vol. L (1995) – Fasc. I, pp. 163-172

$\mathscr{L}^{2,\lambda}$ REGULARITY FOR SECOND ORDER NON LINEAR NON VARIATIONAL ELLIPTIC SYSTEMS

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Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, we prove that the $n \times n$ matrix $H(u) = \{D_i D_j u\}$ belongs to the space $\mathscr{L}^{2,\lambda}_{\text{loc}}(\Omega, \mathbb{R}^{n^2 N})$($N$ integer ≥ 1) if $u \in H^2(\Omega, \mathbb{R}^N)$ is solution in Ω to the system

$$a\,(x, u, Du, H(u)) = f,$$

if $f \in \mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$, $0 < \lambda < n\left(1 - \frac{2}{q}\right)$ $(q > 2)$, and if the vector of $\mathbb{R}^N a(x, u, p, \xi)$ is elliptic, that is, satisfies the condition (A).

1. Introduction.

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, let $x = (x_1, x_2, \dots, x_n)$ denote a point of Ω and N be an integer ≥ 1. If u is a function defined on Ω in $\mathbb{R}^N$, we shall set:

$$Du = (D_1 u, D_2 u, \dots, D_n u), \qquad H(u) = \{D_{ij} u\}, \qquad i, j = 1, 2, \dots, n,$$

where $D_i u = \dfrac{\partial u}{\partial x_i}$, $D_{ij} u = D_i D_j u$.

$Du \in \mathbb{R}^{nN}$ and $H(u)$ is a $n \times n$ matrix of elements of $\mathbb{R}^N$, that is an element of $\mathbb{R}^{n^2 N}$. We shall denote by $p = (p_1, p_2, \dots, p_n)$, $p_i \in \mathbb{R}^N$, a generic vector of $\mathbb{R}^{nN}$ and by $\xi = \{\xi_{ij}\}$, $i, j = 1, 2, \dots, n$, $\xi_{ij} \in \mathbb{R}^N$, a generic element of $\mathbb{R}^{n^2 N}$.

Entrato in Redazione l'11 luglio 1995.

In Ω we shall study the following second order non linear non variational system

$$a\,(x, u, Du, H(u)) = f \tag{1.1}$$

where $f \in L^2(\Omega, \mathbb{R}^N)$ and $a(x, u, p, \xi) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N} \to \mathbb{R}^N$ is a mapping measurable in x, of class C^0 in (u, p, ξ), satisfying these conditions

$$a(x, u, p, 0) = 0, \tag{1.2}$$

(A) *there exist three positive constants α, γ and δ, with $\gamma + \delta < 1$, such that, for almost every $x \in \Omega$, $\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \xi, \tau \in \mathbb{R}^{n^2N}$, it results* (1)

$$\left\| \sum_i \tau_{ii} - \alpha\,[a(x, u, p, \tau + \xi) - a(x, u, p, \xi)] \right\|^2 \leq \tag{1.3}$$

$$\leq \gamma \|\tau\|^2 + \delta \left\| \sum_i \tau_{ii} \right\|^2 .$$

We shall show that if $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution in Ω to the system (1.1) and $f \in \mathcal{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$, $0 < \lambda < n\left(1 - \frac{2}{q}\right)$ (2), then $H(u) \in \mathcal{L}^{2,\lambda}_{\text{loc}}(\Omega, \mathbb{R}^{n^2N})$.

Hence we shall generalize the analogous result already got by S. Campanato for quasi-basic systems in [2] and [3] (see also [6] and [7]).

In [5] and [10] have been studied second order non linear non variational parabolic systems and, under condition (A), obtained results of partial Hölder continuity and differentiability of the solutions. For these systems it's open the $\mathcal{L}^{2,\lambda}$ regularity problem.

2. Preliminary results.

In this paragraph we shall recall some results that will be useful in the following.

Set

$$H(\Omega, \mathbb{R}^N) = H^2(\Omega, \mathbb{R}^N) \cap H^1_0(\Omega, \mathbb{R}^N),$$

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(1) The symbols $(\cdot \mid \cdot)_K$ and $\|\cdot\|_K$ will denote the scalar product and the norm in $\mathbb{R}^K$. We shall omit the index K wherever there is no ambiguity.

(2) q is the real number that appears in the Theorem 3.1.

we remember that, if $\partial\Omega$ is of class C^2, the space $H(\Omega, \mathbb{R}^N)$ is an Hilbert space with the norm:

$$\|u\|_{H(\Omega,\mathbb{R}^N)} = \left(\int_\Omega \|H(u)\|^2 dx\right)^{\frac{1}{2}}$$

and we have the following result of Miranda-Talenti ([11] and [12]):

Lemma 2.1. *If Ω is of class C^2 and convex, then, $\forall u \in H(\Omega, \mathbb{R}^N)$, the following estimate holds:*

$$\int_\Omega \|H(u)\|^2 dx \le \int_\Omega \|\Delta u\|^2 dx. \tag{2.1}$$

Now we prove the following existence theorem:

Theorem 2.1. *If Ω is of class C^2 and convex, if the vector $a(x, u, p, \xi)$ is measurable in x, of class C^0 in (u, p, ξ) and satisfies the conditions* (1.2) *and* (A), *then, $\forall F \in L^2(\Omega, \mathbb{R}^N)$ and $\forall u \in H^2(\Omega, \mathbb{R}^N)$, there exists an unique solution to the Dirichlet problem*

$$\begin{cases} w \in H(\Omega, \mathbb{R}^N) \\ a(x, u, Du, H(u) + H(w)) = F(x) \quad \text{in } \Omega \end{cases} \tag{2.2}$$

and we have the estimate

$$\int_\Omega \|H(w)\|^2 dx \le \frac{\alpha^2}{[1 - \sqrt{\gamma + \delta}]^2} \int_\Omega \|F(x) - a(x, u, Du, H(u))\|^2 dx. \tag{2.3}$$

Proof. Let us fix $F \in L^2(\Omega, \mathbb{R}^N)$ and $u \in H^2(\Omega, \mathbb{R}^N)$; let us prove that the problem (2.2) has one and only one solution w and this solution satisfies the estimate (2.3).

The hypothesis, supposed on $a(x, u, p, \xi)$, assure that

$$\|a(x, u, p, \xi)\| \le c\|\xi\|, \tag{2.4}$$

$\forall u \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^{nN}$, $\forall \xi \in \mathbb{R}^{n^2N}$ and for a.e. $x \in \Omega$; then the operator $A(w) = a(x, u, Du, H(u) + H(w))$ maps every $w \in H(\Omega, \mathbb{R}^N)$ in an element of $L^2(\Omega, \mathbb{R}^N)$:

$$A(w) : H(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N).$$

The operator of Laplace Δw is an isomorphism $H(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$.

We prove that $A(w)$ is near the operator Δw, that is: there exist two positive constants α and k, with $0 < k < 1$, such that $\forall w_1, w_2 \in H(\Omega, \mathbb{R}^N)$ it results:

$$\|\Delta w_1 - \Delta w_2 - \alpha[A(w_1) - A(w_2)]\|_{L^2(\Omega,\mathbb{R}^N)} \leq k\|\Delta w_1 - \Delta w_2\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{2.5}$$

Let w_1 and w_2 belong to $H(\Omega, \mathbb{R}^N)$, in virtue of the (1.3) we have:

$$\begin{aligned}
&\|\Delta w_1 - \Delta w_2 - \alpha[A(w_1) - A(w_2)]\|^2 = \\
&= \|\Delta(w_1 - w_2) - \alpha[a(x, u, Du, H(w_1 - w_2) + H(u) + H(w_2)) - \\
&\qquad - a(x, u, Du, H(u) + H(w_2))]\|^2 \leq \\
&\leq \gamma\|H(w_1 - w_2)\|^2 + \delta\|\Delta(w_1 - w_2)\|^2,
\end{aligned}$$

from which and from Lemma 2.1, it follows:

$$\begin{aligned}
&\int_\Omega \|\Delta w_1 - \Delta w_2 - \alpha[A(w_1) - A(w_2)]\|^2 dx \leq \\
&\leq \int_\Omega (\gamma\|H(w_1 - w_2)\|^2 + \delta\|\Delta(w_1 - w_2)\|^2) dx \leq \\
&\leq (\gamma + \delta) \int_\Omega \|\Delta(w_1 - w_2)\|^2 dx
\end{aligned}$$

that assures the (2.5) with $k = \sqrt{\gamma + \delta} < 1$.

Hence, the nearness of the operator $A(w)$ with respect to the operator Δw has been showed, then, in virtue of the Theorems 1 and 2 in [8], A is a bijection $H(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$, that is, the Dirichlet problem (2.2) has an unique solution $w \in H(\Omega, \mathbb{R}^N)$ and the following estimate holds:

$$\|\Delta w\|_{L^2(\Omega,\mathbb{R}^N)} \leq \frac{\alpha}{1 - \sqrt{\gamma + \delta}} \|F(x) - a(x, u, Du, H(u))\|_{L^2(\Omega,\mathbb{R}^N)}. \tag{2.6}$$

Finally the (2.3) is consequence of the (2.6) and the Lemma 2.1. □

3. $H^{2,q}_{\text{loc}}$ regularity result.

Let $a^*(x, \xi)$ be a mapping defined on $\Omega \times \mathbb{R}^{n^2N}$ in $\mathbb{R}^N$, measurable in x, of class C^0 in ξ, verifying these conditions:

$$a^*(x, 0) = 0, \tag{3.1}$$

(3.2) *there exist three positive constants* α, γ *and* δ, *with* $\gamma+\delta<1$, *such that, for almost every* $x\in\Omega$, $\forall\xi,\tau\in\mathbb{R}^{n^2N}$, *it results*:

$$\left\|\sum_i \tau_{ii}-\alpha\left[a^*(x,\tau+\xi)-a^*(x,\xi)\right]\right\|^2\le\gamma\|\tau\|^2+\delta\left\|\sum_i\tau_{ii}\right\|^2.$$

For the solutions $v\in H^2(\Omega,\mathbb{R}^N)$ to the system

$$a^*(x,H(v))=0 \tag{3.3}$$

the following Caccioppoli estimate holds:

Lemma 3.1. *If* $v\in H^2(\Omega,\mathbb{R}^N)$ *is a solution in* Ω *to the system* (3.3), *then* $\forall B(x^0,2\sigma)=B(2\sigma)\subset\subset\Omega$ (3) *we have:*

$$\int_{B(\sigma)}\|H(v)\|^2dx\le c\,\sigma^{-2}\int_{B(2\sigma)}\|Dv-(Dv)_{2\sigma}\|^2dx \text{ (4)}, \tag{3.4}$$

where the constant c *does not depend on* x^0 *and* σ, *but depends on* γ, δ *and* n.

Proof. The (3.4) is immediate consequence of Lemma 3.1 in [4], with

$$b(x,u,p)=0,\quad a(x,u,p,\xi)=a^*(x,\xi).\qquad\square$$

Now, let us fix an arbitrary $u\in H^1(\Omega,\mathbb{R}^N)$ and investigate the system

$$a\,(x,u,Du,H(v))=0, \tag{3.5}$$

where $a(x,u,p,\xi)$ is a mapping defined on $\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\times\mathbb{R}^{n^2N}$ in $\mathbb{R}^N$, measurable in x, of class C^0 in (u,p,ξ), satisfying the conditions (1.2) and (A).

We show for the solutions v to this system the following $H^{2,q}_{\text{loc}}$ regularity result.

Theorem 3.1. *If* $v\in H^2(\Omega,\mathbb{R}^N)$ *is a solution in* Ω *to the system* (3.5), *then there exists* $q>2$ *such that* $H(v)\in L^q_{\text{loc}}(\Omega,\mathbb{R}^{n^2N})$ *and*, $\forall B(2\sigma)\subset\subset\Omega$, *we have:*

$$\left(⨍_{B(\sigma)}\|H(v)\|^q dx\right)^{\frac{2}{q}}\le c⨍_{B(2\sigma)}\|H(v)\|^2dx, \tag{3.6}$$

where the constant c *does not depend on* x^0 *and* σ, *but only depends on* γ, δ *and* n.

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(3) $B(x^0,\rho)=B(\rho)=\{x\in\mathbb{R}^n:\|x-x^0\|<\rho\}$, $x^0=(x_1^0,x_2^0,\dots,x_n^0)\in\mathbb{R}^n$, $\rho>0$.
(4) If $f\in L^1(B(\rho),\mathbb{R}^h)$, we set : $f_\rho=\frac{1}{\text{meas}\,B(\rho)}\int_{B(\rho)}f\,dx=⨍_{B(\rho)}f\,dx$.

Proof. For every fixed $u \in H^1(\Omega, \mathbb{R}^N)$, this function:

$$a^*(x, \xi) = a(x, u(x), Du(x), \xi) : \Omega \times \mathbb{R}^{n^2N} \to \mathbb{R}^N, \tag{3.7}$$

in virtue of hypothesis about $a(x, u, p, \xi)$, is measurable in x, of class C^0 in ξ, and satisfies the conditions (3.1) and (3.2) (with the same constants α, γ and δ of (A)).

On the other hand, fixed $u \in H^1(\Omega, \mathbb{R}^N)$, if $v \in H^2(\Omega, \mathbb{R}^N)$ is a solution in Ω to the system (3.5), it will be also solution in Ω to the system (3.3) (with $a^*(x, \xi) = a(x, u(x), Du(x), \xi)$), then the Lemma 3.1 assures that for all $B(2\sigma) \subset\subset \Omega$

$$\int_{B(\sigma)} \|H(v)\|^2 dx \leq c(\gamma, \delta, n)\sigma^{-2} \int_{B(2\sigma)} \|Dv - (Dv)_{2\sigma}\|^2 dx \tag{3.8}$$

and, hence, using the theorem of Poincarè (see [1], Chap. I, Theorem 3.IV), we have:

$$⨍_{B(\sigma)} \|H(v)\|^2 dx \leq c(\gamma, \delta, n) \left(⨍_{B(2\sigma)} \|H(v)\|^{\frac{2n}{n+2}} dx \right)^{(n+2)/n},$$

from which, by virtue of a well-known Lemma of Gehring-Giaquinta-G.Modica (see [1], Chap. II, Lemma 10.I) ([5]), the thesis follows. □

From the Theorem 3.1 follows this fundamental interior estimate.

Theorem 3.2. *If $v \in H^2(\Omega, \mathbb{R}^N)$ is a solution in Ω to the system (3.5), then there exists $q > 2$, such that $\forall B(\sigma) \subset\subset \Omega$ and $\forall t \in (0, 1)$, we have:*

$$\int_{B(t\sigma)} \|H(v)\|^2 dx \leq c\, t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(v)\|^2 dx, \tag{3.9}$$

where the constant c does not depend on x^0, σ and t, but depends only on γ, δ and n.

Proof. Let us fix $B(\sigma) \subset\subset \Omega$ and $t \in \left(0, \frac{1}{2}\right)$, the Theorem 3.1 ensures the existence of $q > 2$ such that

$$H(v) \in L^q\left(B(\sigma), \mathbb{R}^{n^2N}\right) \subset L^2\left(B(\sigma), \mathbb{R}^{n^2N}\right)$$

([5]) Written with $U = \|H(v)\|^{\frac{2n}{n+2}}$, $G = 0$ and $r = \frac{n+2}{n}$. See also [9], Proposition 5.1.

and

$$(3.10)\qquad \left(\fint_{B(\frac{\sigma}{2})} \|H(v)\|^q dx\right)^{\frac{2}{q}} \le c \fint_{B(\sigma)} \|H(v)\|^2 dx,$$

where the constant c does not depend on x^0 and σ, but depends only on γ, δ and n.

In virtue of Hölder inequality, we get:

$$(3.11)\qquad \int_{B(t\sigma)} \|H(v)\|^2 dx \le [\omega(n)]^{1-\frac{2}{q}} (t\sigma)^{n(1-\frac{2}{q})} \left(\int_{B(t\sigma)} \|H(v)\|^q dx\right)^{\frac{2}{q}} \le$$

$$\le \omega(n) t^{n(1-\frac{2}{q})} \sigma^n \left(\fint_{B(\frac{\sigma}{2})} \|H(v)\|^q dx\right)^{\frac{2}{q}} \ (^6).$$

Let us estimate the right hand side of (3.11) by (3.10)

$$(3.12)\qquad \omega(n) t^{n(1-\frac{2}{q})} \sigma^n \left(\fint_{B(\frac{\sigma}{2})} \|H(v)\|^q dx\right)^{\frac{2}{q}} \le c\, t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(v)\|^2 dx.$$

The (3.9), with $t \in \left(0, \frac{1}{2}\right)$, is obvious consequence of (3.11) and (3.12).

Finally the (3.9) is trivially true if $t \in \left[\frac{1}{2}, 1\right)$. □

4. $\mathcal{L}^{2,\lambda}$ regularity for the matrix $H(u)$.

Let $f : \Omega \to \mathbb{R}^N$ be a function of class $L^2(\Omega, \mathbb{R}^N)$ and $u \in H^2(\Omega, \mathbb{R}^N)$ a solution in Ω to the system:

$$(4.1)\qquad a(x, u, Du, H(u)) = f$$

where $a(x, u, p, \xi)$ is a mapping defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N}$ in $\mathbb{R}^N$, measurable in x, of class C^0 in (u, p, ξ), verifying the conditions (1.2) and (A).

Let us prove the following

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(6) $\omega(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$, where Γ is the factorial function.

Lemma 4.1. *For every $B(2\sigma) \subset\subset \Omega$ and $\forall t \in (0, 1)$, it results:*

$$\int_{B(t\sigma)} \|H(u)\|^2 dx \le c \left\{ t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(u)\|^2 dx + \int_{B(2\sigma)} \|f\|^2 dx \right\}, \tag{4.2}$$

where the constant c does not depend on x^0, σ and t, but depends only on α, γ, δ and n.

Proof. Let us fix $B(2\sigma) \subset\subset \Omega$ and in $B(2\sigma)$ write $u = v - w$, where w is the solution to the Dirichlet problem (see Theorem 2.1):

$$\begin{cases} w \in H(B(2\sigma), \mathbb{R}^N) \\ a(x, u, Du, H(u) + H(w)) = a(x, u, Du, H(u)) - f(x) \text{ in } B(2\sigma) \end{cases} \tag{4.3}$$

while $v = u + w \in H^2(B(2\sigma), \mathbb{R}^N)$ is solution to the system:

$$a(x, u, Du, H(v)) = 0 \quad \text{in } B(2\sigma).$$

For the Theorem 2.1, since $a(x, u, Du, H(u)) - f \in L^2(\Omega, \mathbb{R}^N)$, w exists, is unique and holds the estimate

$$\int_{B(2\sigma)} \|H(w)\|^2 dx \le \frac{\alpha^2}{[1 - \sqrt{\gamma + \delta}]^2} \int_{B(2\sigma)} \|f\|^2 dx. \tag{4.4}$$

On the other hand, in virtue of the Theorem 3.2, there exists $q > 2$ such that, $\forall t \in (0, 1)$, we have:

$$\int_{B(t\sigma)} \|H(v)\|^2 dx \le c\, t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(v)\|^2 dx, \tag{4.5}$$

where the constant c depends only on γ, δ and n.

Since $u = v - w$, from (4.4) and (4.5), it follows:

$$\begin{aligned} \int_{B(t\sigma)} \|H(u)\|^2 dx &\le 2 \int_{B(t\sigma)} \|H(v)\|^2 dx + 2 \int_{B(t\sigma)} \|H(w)\|^2 dx \le \\ &\le c(\gamma, \delta, n) t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(v)\|^2 dx + 2 \int_{B(2\sigma)} \|H(w)\|^2 dx \le \\ &\le c(\gamma, \delta, n) t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(u)\|^2 dx + c(\gamma, \delta, n) \int_{B(2\sigma)} \|H(w)\|^2 dx \le \\ &\le c(\gamma, \delta, n) t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(u)\|^2 dx + c(\alpha, \gamma, \delta, n) \int_{B(2\sigma)} \|f\|^2 dx \end{aligned}$$

and, hence, the (4.2). □

Now we can prove the $\mathcal{L}^{2,\lambda}$ regularity result for the matrix $H(u)$ of the solutions to the system (4.1).

Theorem 4.1. *If the vector* $a(x, u, p, \xi)$ *is measurable in* x, *of class* C^0 *in* (u, p, ξ) *and satisfies the conditions* (1.2) *and* (A), *if* $f \in \mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$, $0 < \lambda < n(1 - \frac{2}{q})$, *and if* $u \in H^2(\Omega, \mathbb{R}^N)$ *is solution in* Ω *to the system*

$$a(x, u, Du, H(u)) = f,$$

then

$$H(u) \in \mathscr{L}^{2,\lambda}_{\text{loc}}(\Omega, \mathbb{R}^{n^2 N}) \tag{4.6}$$

and, for every open subset $\Omega^* \subset\subset \Omega$, *we have the estimate:*

$$\|H(u)\|_{\mathscr{L}^{2,\lambda}(\Omega^*, \mathbb{R}^{n^2 N})} \le c\,\{\|H(u)\|_{L^2(\Omega, \mathbb{R}^{n^2 N})} + \|f\|_{\mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)}\},$$

where the constant c *depends on* $\lambda, \alpha, \gamma, \delta, n$ *and on the distance of* $\overline{\Omega}^*$ *from* $\partial\Omega$.

Proof. From Lemma 4.1 and in virtue of the hypothesis $f \in \mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$, it follows $\forall B(2\sigma) \subset\subset \Omega$ and $\forall t \in (0, 1)$:

$$\int_{B(t\sigma)} \|H(u)\|^2 dx \le c\, t^{n(1-\frac{2}{q})} \int_{B(\sigma)} \|H(u)\|^2 dx + c\sigma^\lambda \|f\|^2_{\mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)}. \tag{4.7}$$

Let Ω^* be an open subset $\subset\subset \Omega$ and d^* the distance of $\overline{\Omega}^*$ from $\partial\Omega$, the (4.7) is true $\forall B(x^0, 2\sigma)$, with $x^0 \in \overline{\Omega}^*$, $\sigma \in (0, \frac{d^*}{4}]$, and $\forall t \in (0, 1)$; then the Lemma 1.I of the Chap. I of [1] assures that $\forall x^0 \in \overline{\Omega}^*$ and $\forall t \in (0, 1)$, we have:

$$\int_{B(t\frac{d^*}{4})} \|H(u)\|^2 dx \le c\, t^\lambda \left\{ \int_\Omega \|H(u)\|^2 dx + \left(\frac{d^*}{4}\right)^\lambda \|f\|^2_{\mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)} \right\}, \tag{4.8}$$

where the constant c does not depend on x^0 and t.

The thesis follows immediately from the (4.8). □

Finally we can recall the same note of n. 4 in [2].

From (4.6) it follows:

$$Du \in \mathscr{L}^{2,2+\lambda}_{\text{loc}}(\Omega, \mathbb{R}^{nN}) \quad \text{and} \quad u \in \mathcal{L}^{2,4+\lambda}_{\text{loc}}(\Omega, \mathbb{R}^N),$$

then

Du is Hölder continuous in Ω if $n = 2$,

u is Hölder continuous in Ω if $n \le 4$.

REFERENCES

[1] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno,* Quaderni Scuola Normale Sup. Pisa, 1980.

[2] S. Campanato, *A Cordes Type Condition for non Linear non Variational Systems,* Rend. Acc. Naz. dei XL, 13 (1989), pp. 307–321.

[3] S. Campanato, *Non variational differential systems: A condition for local existence and uniqueness,* Proceedings of the Caccioppoli Conference, Ricerche di Matem., Suppl. vol. XL (1991), pp. 129–140.

[4] S. Campanato, $\mathscr{L}^{2,\lambda}$ *theory for non Linear non Variational Differential Systems,* Rend. Mat., (7) 10 (1990), pp. 531–549.

[5] S. Campanato, *Non variational basic parabolic systems of second order,* Rend. Mat. Acc. Lincei, (9) 2 (1991), pp. 129–136.

[6] S. Campanato, *Sistemi differenziali del 2° ordine di tipo ellittico,* Quaderno n.1 del Dottorato di Ricerca in Matematica, Dipart. di Matem. Univ. Catania, 1991.

[7] S. Campanato, *Sulla condizione di vicinanza tra operatori,* Quaderno n.2 del Dottorato di Ricerca in Matematica, Dipart. di Matem. Univ. Catania, 1993.

[8] S. Campanato, *On the Condition of Nearness Between Operators,* Ann. Mat. Pura Appl., (4) 167 (1994), pp. 243–256.

[9] M. Giaquinta - G. Modica, *Regularity results for some classes of higher order non linear elliptic systems,* J. Reine Angew. Math., 311/312 (1979), pp. 145–169.

[10] M. Marino - A. Maugeri, *Second order non linear non variational parabolic systems,* Rend. Mat., (7) 13 (1993), pp. 499–527.

[11] C. Miranda, *Su di una particolare equazione ellittica del 2° ordine a coefficienti discontinui,* Anal. Sti. Univ. ≪Al. I. Cura≫ Jasi Sert. Ib. Fiz., 11 (1965), pp. 209–215.

[12] G. Talenti, *Sopra una classe di equazioni ellittiche a coefficienti misurabili,* Ann. Mat. Pura Appl., 69 (1965), pp. 285–304.

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