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A NOTE ON LINEAR SUBSPACES OF DETERMINANTAL VARIETIES

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We use a geometrical approach to the study of the vector subspaces M in a determinantal variety in $\mathrm{Hom}(V, W)$ and derive conditions on $\dim(M)$ for M to belong to certain distinguished classes of subspaces.

1. Introduction.

The general classification problem for vector subspaces $M \subseteq \mathrm{Hom}(V, W)$ consists in describing the set of equivalence classes of such spaces for the equivalence relation $M \sim M'$ iff there exist some isomorphisms $\varphi : V \to V$ and $\psi : W \to W$ such that $M' = \psi M \varphi$. This problem dates back at least to Kronecker who solved it for pencils of matrices, i.e. $\dim M = 2$ (see [5]). We will not consider this problem but we will focus our attention to those subspaces M subject to a rank condition, that is $rk\varphi \leq r$ for each $\varphi \in M$, where r is a fixed positive integer with $r < \max(\dim V, \dim W)$. This problem has been considered by some authors (see e.g. [1],[2],[4] and [3] for an algebro-geometrical approach) and a satisfactory classification up to $r \leq 3$ ([1] and [3]) is now known. The general situation is not known and it seems difficult to understand (see the discussion at the end of [3]). In this work we will prove some results of a general nature: we will give a condition on $\dim M$ which force M to belong to a

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distinguished family of low rank subspaces, so called compression spaces, and a necessary condition on $\dim M$ for those M which are primitive spaces (see definitions in the next section).

2. Definitions and preliminaries.

Let V and W be finite dimensional vector spaces on an algebraically closed field of characteristic zero. Let $v = \dim V$ $w = \dim W$. Let r be a positive integer such that $r < \max(v, w)$ and $M \subseteq \mathrm{Hom}(V, W)$ a vector subspace such that $\max\{rk\varphi | \varphi \in M\} = r$. We will denote by M_r the open set $M_r = \{\varphi \in M | rk\varphi = r\}$ and we will call r the rank of M.

Definition 1. *We will call M maximal of rank$= r$ if it is not contained in a strictly larger subspace $M' \subseteq \mathrm{Hom}(V, W)$ which has the property $rk\varphi' \leq r$ for each $\varphi' \in M'$.*

Definition 2. *A rank$= r$ subspace M is called a compression space if there exist vector subspaces $N \subseteq V$ and $I \subseteq W$ such that* $\mathrm{codim}(N) + \dim(I) = r$ *and M is contained in the subspace* $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I) = \{\varphi \in M/\varphi(N) \subseteq I\}$.

Remarks.
(a) It is not difficult to verify that the space $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I)$ is maximal. These spaces arise naturally as the most simple ones subject to a rank condition.
(b) For the space $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I)$ we have:

$$N = \sum_{\varphi \in Mr} \ker \varphi$$

$$I = \bigcap_{\varphi \in Mr} \mathrm{Im}\, \varphi\,.$$

Definition 3. *Let M be a rank$=r$ space for which the following properties hold:*
(a) $N = \sum_{\varphi \in M_r} \ker \varphi = V$ *and* $I = \bigcap_{\varphi \in M_r} \mathrm{Im}\, \varphi = (0)$
(b) $\bigcap \ker \varphi = (0)$ *and* $\sum \mathrm{Im}\, \varphi = W$.

Then we will call M a primitive space.

The primitive spaces are the real core of the classification problem (see [1] and [3] for further information). The first example of a primitive space is given by the space M of all 3x3 antisymmetric matrices, which is maximal of rank =2. In general is not difficult to prove that all the spaces of antisymmetric matrices of odd order are primitive and maximal.
The main technical result we will apply in this work is the following:

Proposition 1. *Let X be a smooth irreducible quasi-projective variety defined on a field k algebraically closed and of $char(k) = 0$. Let F, G be two vector bundles on X, and V, W two k-vector spaces of global sections $V \subseteq H^0(F)$, $W \subseteq H^0(G)$ which globally generate F and G respectively.*
Then the natural multiplication map

$$\mu : V \otimes W \to H^0(F \otimes G)$$

has rank:

$$rk\mu \geq \dim V \cdot rkG + \dim W \cdot rkF - rkF \cdot rkG\,.$$

Remark. In [6] the result above is proved under the hypotheses that X is smooth irreducible and projective and V, W generically generate the bundles. The formulation we have given here is an easy consequence of the original one.

3. Statements and proofs.

We will always suppose that $v = \dim V \geq \dim W = w$, without loss of generality.
The core of this work is the following result.

Theorem 1. *Let $M \subseteq \mathrm{Hom}(V, W)$ be a space of rank $= r$ and let $N = \sum_{\varphi \in Mr} \ker\varphi$ and $I = \bigcup_{\varphi \in Mr} \mathrm{Im}\,\varphi = (0)$ with $\mathrm{codim}(N) = a$ and $\dim(I) = b$.*
Then we have:

$$\dim M \leq vr - r(v - r) + a(w - r) + b(v - r)\,.$$

Proof. On $M_r \setminus \{0\}$ are defined two vector bundles F and G given respectively as $(\ker\Phi)^*$ and $\mathrm{coker}\,\Phi$, where $\Phi : V \otimes O_M \to W \otimes O_M$ is defined at the fiber level as $\Phi(\varphi)(v) = \varphi(v)$.
Since $\ker\varphi \subseteq N$ and $\mathrm{Im}\,\varphi \supseteq I$ for each $\varphi \in M_r$, we deduce that N^* globally

generates F and W/I globally generates G. Furthermore $N^* \subseteq H^0(F)$ and $W/I \subseteq H^0(G)$, which is another consequence of the fact that $N = \sum_{\varphi \in M_r} \ker \varphi$ and $I = \bigcap_{\varphi \in M_r} \operatorname{Im} \varphi$. At this point we can apply Proposition 1 to the multiplication map $\mu : N^* \otimes (W/I) \to H^0(F \otimes G)$ and get:

$$rk\mu \geq \dim N \cdot rkG + \dim W/I \cdot rkF - rkF \cdot rkG$$

$$= (v-a)(w-r) + (w-b)(v-r) - (v-r)(w-r)\,.$$

For any $\varphi \in M_r$ we can write the composition of surjective linear maps

$$V^* \otimes W \to N^* \otimes W/I \to \operatorname{Im} \mu \to F(\varphi) \otimes G(\varphi) =$$

$$= \ker \varphi \otimes (\operatorname{coker} \varphi)^* = \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi)$$

dualizing which we see that:

$$\sum_{\varphi \in Mr} \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi) \subseteq (\operatorname{Im} \mu)^* \subseteq \operatorname{Hom}(W, V)\,.$$

From the fact that $\operatorname{Im} \mu \subseteq H^0(F \otimes G)$ it follows that indeed:

$$\sum_{\varphi \in Mr} \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi) = (\operatorname{Im} \mu)^*.$$

Finally we apply the well known geometrical fact that the tangent space to a determinantal variety $H_r = \{\varphi \in \operatorname{Hom}(V, W) \mid rk\varphi \leq r\}$ at a point φ of rank r is the vector space

$$T_{Hr,\varphi} = \{\psi \in \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi) \mid \psi(\ker \varphi) \subseteq \operatorname{Im} \varphi\}.$$

If we identify $\operatorname{Hom}(W, V)$ with $\operatorname{Hom}(V, W)^*$ by the natural pairing $\langle \psi, \varphi \rangle = Tr(\psi\varphi)$, the annihilator of $T_{H_r,\varphi}$ is exactly

$$(T_{H_r,\varphi})^{\perp} = \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi).$$

Since $M \subseteq H_r$ is a linear subspace, then $M \subseteq T_{H_r,\varphi}$ for each $\varphi \in M_r$ and then $M^{\perp} \supseteq (T_{H_r,\varphi})^{\perp}$ for each $\varphi \in M_r$. In conclusion we have:

$$M^{\perp} \supseteq \sum_{\varphi \in Mr} \operatorname{Hom}(W/\operatorname{Im} \varphi, \ker \varphi) = (\operatorname{Im} \mu)^*.$$

It follows:

$$vw - \dim M = \dim M^{\perp} \geq \dim \operatorname{Im} \mu \geq$$

$$\geq (v-a)(w-r) + (w-b)(v-r) - (v-r)(w-r)$$

from which derives, after some calculation, the result:

$$\dim M \leq vr - r(v-r) + a(w-r) + b(v-r).$$

We deduce all the announced results of this work as corollaries of the theorem above.

At first we give an alternative proof of the first general result in the classification problem we are dealing with. The original proof is that of Flanders [4].

Corollary 1. *If M is a rank= r space the* $\dim M \leq vr$. *If* $\dim M = vr$ *then if* $v > w$ *we have* $M = \mathrm{Hom}(V, I)$ *where* $I \subseteq W$ *is an* r*-dimensional subspace; if* $v = w$ *also the case* $M = \mathrm{Hom}(V/N, W)$ *with* $\mathrm{codim}\, N = r$ *can occur.*

Proof. We can suppose M maximal. If we pose $N = \sum_{\varphi \in M_r} \ker \varphi = V$, $I = \bigcap_{\varphi \in M_r} \mathrm{Im}\, \varphi$, $a = \mathrm{codim}(N)$ and $b = \dim(I)$ we see that $\mathrm{Hom}(V/N, W) \subseteq M$ and $\mathrm{Hom}(V, I) \subseteq M$. One can see this observing that any ψ such that $\ker \psi \supseteq N$ or $\mathrm{Im}\, s\psi \subseteq I$ has the property that $\lambda\psi + \varphi$ has rank not greater than r, for any $\varphi \in M_r$. By maximality it follows that such a ψ is already in M. We can write then:

$$\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I) \subseteq M.$$

An important consequence of this fact is that $\mathrm{codim}(N) + \dim(I) \leq r$ since the maximal rank in the space $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I)$ is exactly $\mathrm{codim}(N) + \dim(I)$.

If in the inequality of Theorem 1 we impose the constraint $a + b \leq r$, we see after some calculation that the maximum of the expression $vr - r(v-r) + a(w-r) + b(v-r)$ is vr. Then $\dim M \leq v \cdot r$ and equality holds only for $a + b = r$. This means that $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I) = M$ because $\mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I)$ is maximal. But in this case $\dim M = aw + vb - ab = vr$ if and only if $a = 0$ and $b = r$ or $v = w$ and $a = 0$, $b = r$ or $a = r$, $b = 0$, which ends the proof.

Corollary 2. *If* $\dim M > vr - v + r$ *then M is a compression space. If* $v > w$ *and* $\dim M \geq vr - v + r$ *then M is a compression space.*

Proof. M is a compression space if and only if there is a maximal space of the same rank $M' \supseteq M$ which is a compression space. Then there will be no loss of generality in assuming M maximal. If in the notation of the corollary 1 we have $a + b = r$ then we get, as in the proof of corollary 1, that $M = \mathrm{Hom}(V/N, W) + \mathrm{Hom}(V, I)$ since this latter is maximal of rank= r.

We suppose then $a + b \leq r - 1$ and calculate the maximum of the expression $vr - r(v-r) + a(w-r) + b(v-r)$ appearing in Theorem 1 under this constraint. This maximum turns out to be $vr - v + r$, and this value is achieved for $b = r - 1$ and $a = 0$ or, if $v = w, a + b = r - 1$. In any case we get that if $\dim M > vr - v + r$ then M is forced to be a compression space. Moreover if $v > w$ and $\dim M = vr - v + r$ we have that in this case $\dim I = r - 1$ and $N = V$. If we project M onto the space $M'' \subseteq \mathrm{Hom}(V, W/I)$ we see that this latter is composed of elements of rank ≤ 1. But this case is well understood: M'' is made of maps with image in a fixed 1-dimensional subspace $(w'') \subseteq W/I$, or with kernel containing a fixed $(v-1)$-dimensional subspace $V' \subseteq V$. This latter case is indeed excluded by the condition $a = 0$, so we are left with $M'' \subseteq \mathrm{Hom}(V, (w''))$, that is $M \subseteq \mathrm{Hom}(V, (w'') + I)$. But $\dim((w'') + I) = r$ which implies that M is a compression space.

Corollary 3. *Let M be a primitive space. Then* $\dim M \leq r^2$.

Proof. For a primitive space we have $a = b = 0$. From Theorem 1 we get the result.

Remarks and examples.
(a) The inequality of Corollary 3 is not sharp. We can improve it of a unity, but since we expect the sharp one to be $\sim r^2/2$ we have not done the effort of reporting this. Our expectation derives from the fact that the largest primitive spaces we know of are the spaces of anti-symmetric matrices of odd order $r+1$. Their dimension is $r(r+1)/2$.
(b) The result of Corollary 2 can be compared with the results in [2]. Those authors prove that if $\dim M > vr - v + w - r + 1$ then m is a compression space. Our inequality is a better one for r small compared with v and w.

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