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ON UPPER CHROMATIC NUMBER FOR SQS(10) AND SQS(16)

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A mixed hypergraph is characterized by the fact that it possesses anti-edges as well as edges. In a colouring of a mixed hypergraph, every anti-edge has at least two vertices of the same colour and every edge has at least two vertices coloured differently. The upper chromatic number $\bar{\chi}$ is the maximal number of colours for which there exists a colouring using all the colours. The concepts of mixed hypergraph and upper chromatic number are applied to STS and SQS. In fact it is possible to consider a Steiner system as a mixed hypergraph when all the blocks are anti-edges (Co-STSs, Co-SQSs) or at the same time edges and anti-edges (BSTSs, BSQSs). In this paper the necessary conditions in order to colour Co-STSs, BSTSs and Co-SQSs, BSQSs are given, and the values of upper chromatic number for Co-SQS(10), BSQS(10) and for BSQSs(16), obtained from a doubling construction, are determined.

1. Introduction.

The concepts of mixed hypergraph, strict colouring and upper chromatic number were introduced by V. Voloshin in 1993 [8], [9]. A mixed hypergraph H is the pair $(X, \mathcal{S})$, where X is a finite set of vertices and $\mathcal{S}$ a family of subsets of X with $\mathcal{S} = \mathcal{A} \bigcup \mathcal{E}$, where $\mathcal{A}$ and $\mathcal{E}$ are two subfamilies of $\mathcal{S}$. The elements of $\mathcal{A}$ are called anti-edges, while the elements of $\mathcal{E}$ are called edges. A vertex colouring of H is characterized by the fact that any edge has at least two vertices coloured differently and any anti-edge has at least two vertices of the same

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colour. If $\mathcal{E}$ is empty then H will be called co-hypergraph; if $\mathcal{A}$ is empty then H will be called hypergraph.

In this paper we follow the same symbology used in [8], [9].

Definition 1. *A strict colouring of a mixed hypergraph $H = (X, \mathcal{S})$ is a mapping $X \to \{1, 2, \cdots, k\}$, with $k \geq 1$, which respects the following conditions:*

1) *any anti-edge has at least two vertices of the same colour;*
2) *any edge has at least two vertices coloured differently;*
3) *the number of used colours is exactly k;*
4) *all the vertices are coloured.*

Definition 2. *The maximal [minimal] k, for which there exists a strict colouring of a mixed hypergraph H with k colours, is called upper [lower] chromatic number and it is denoted by $\bar{\chi}(H)[\chi(H)]$.*

If there exists no colouring of mixed hypergraph H, then H is called non colourable and $\chi(H) = \bar{\chi}(H) = 0$.

For a mixed hypergraph H, a set $L \subseteq X$ is called co-stable if it contains no anti-edges, and the co-stability number $\alpha_A(H)$ is the maximal cardinality of a co-stable set of H.

In this paper the concepts of strict colouring and upper chromatic number are applied to the Steiner systems. In particular we will study two different kinds of STSs and SQSs. The Co-STSs and Co-SQSs (Co-Steiner Triple Systems, Co-Steiner Quadruple Systems) where all the blocks are anti-edges and BSTSs and BSQSs (Bi-Steiner-Triple-Systems, Bi-Steiner-Quadruple-Systems) where all the blocks are at the same time edges and anti-edges. Necessary conditions are determined connected with the existence of colourings for Co-STSs, BSTSs and Co-SQSs, BSQSs. The exact value of the upper chromatic number is determined for BSQS(10) and Co-SQS(10) and we characterize all the colourings which use $\bar{\chi}$ colours for these two kinds of systems. For BSQSs(16), the upper chromatic number is calculated only for the systems obtained from doubling constructions.

We will use the following result [9] in the next sections.

Theorem 1. (Voloshin). *For any mixed hypergraph H, $\bar{\chi}(H) \leq \alpha_A(H)$.*

1.1. Steiner Systems.

A hypergraph $(X,\mathcal{B})$ where X is a finite set of vertices, with $|X| = v$, and B is a family of subsets of X (called blocks), which respects the following conditions:

1. any t distinct vertices are contained in one and only one block;

2. each block contains exactly k vertices;

is called a Steiner System and it is denoted by S(t,k,v) or S(t,k,v)(X,B).

If $t = 2$ and $k = 3$ a Steiner System is called Steiner Triple System and it is denoted by STS(v) or STS(X,B); if $t = 3$ and $k = 4$ it is called Steiner Quadruple System and it is denoted by SQS(v) or SQS(X,B). A Steiner System with $|X| = v$ is said to be of order v.

In 1960 Hanani [3] proved that a necessary and sufficient condition for the existence of SQS(v) is $v \equiv 2$ or 4 (mod 6), while it is well known that STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6).

Let us recall some other definitions [2]:

Definition 3. *For a Steiner System of order v, a subset S of X is called a subsystem, if the blocks determined from any t distinct vertices of S are always contained entirely in S. $|S| = s$ is the order of the subsystem.*

Definition 4. *In a Steiner System a set $T \subseteq X$ such that for every $b \in B$, $b \cap T \neq 0$ is called trasversal set. If the complementary of T is also a trasversal set, then T is called blocking set.*

It is important to show these two theorems from [2] and [5]:

Theorem 2. (Gionfriddo-Lo Faro). *In SQS(X,B) with $|X| = v$, if $H \subseteq X$, $|H| = h$, T_h is the set of blocks included in H, and T_{v-h} is the set of blocks included in $X - H$, then*

$$|T_h| - |T_{v-h}| = r_0 - \binom{v-h}{1} r_1 + \binom{v-h}{2} r_2 - \binom{v-h}{3}$$

where

$$r_i = \binom{v-i}{3-i} \Big/ \binom{4-i}{3-i}$$

for $0 \leq i \leq 2$ and $|T_h| - |T_{v-h}| = 0$ if $h = \frac{v}{2}$.

Theorem 3. (Milazzo-Tuza). *Let $t > 2$ be an odd integer. Then, in every Steiner system $S(t, t+1, v)$, the stability number is at most $\frac{v}{2}$.*

Doubling Construction for SQS Let us consider a construction which allows us to obtain an SQS($2v$) from two SQSs(v) [4]. Let us consider the two systems SQS(X,A) and SQS(Y,B) with $|X| = |Y| = v$ and $X \cap Y = 0$.

Let us define

$$F = [F_1, F_2, \dots, F_{v-1}] \text{ and } G = [G_1, G_2, \dots, G_{v-1}]$$

as two 1-factorization of K_v on X and Y and α as a permutation on the set $[1,\ 2, \dots, v-1]$.

Define the collection of quadruples C on $Q = X \cup Y$ in this way:

1. Any quadruple belonging to A and B belongs to C;
2. If $x_1, x_2 \in X$ and $x_1, x_2 \in F_i$ then $\{x_1, x_2, y_1, y_2\} \in C$ if and only if y_1, $y_2 \in G_j$ and $i\alpha = j$.

Then (Q,C) is a SQS($2v$).

2. Strict Colouring for Co-STS and BSTS.

Let us consider a STS(X,B): it is a co-hypergraph if all its blocks are considered as anti-edges. We call it Co-STS (Co-Steiner Triple System). In [5] is proved that:

Theorem 4. *If in Co-STS(X,B) $|X| = v \leq 2^k - 1$ with $k \in N$, then $\bar{\chi} \leq k$.*

Corollary 1. *If in Co-STS(X,B) $\bar{\chi} = k$, then*

1. $v = 2^k - 1$.
2. *For any colouring of Co-STS with k colours the colouring classes have cardinality*

$$2^0,\ 2^1, \dots,\ 2^{k-1}$$

and are all co-stable sets.

Let $\mathcal{P}$ be a strict colouring of a Co-STS(v), with h colours, and X_i the set of vertices coloured with the colour i, $|X_i| = n_i$ for $1 \leq i \leq h$, and let us define the set of vertices $S_i = \bigcup_{j=1}^{i} X_j$ for $1 \leq i \leq h$, where $|S_i| = s_i$, in this set the number of pairs of vertices coloured with two different colours is:

$$c_i = \binom{s_i}{2} - \sum_{j=1}^{i} \binom{n_j}{2}.$$

In $\mathcal{P}$, every block can be coloured in only two ways:

i) two vertices coloured with one colour and the other vertex coloured with a different colour;

ii) all the vertices coloured with the same colour.

Proposition 1. *If $\mathcal{P}$ is a strict colouring for Co-STS(X, B), then*

1) $(c_h\ /\ 2)$ *blocks are coloured as in i);*
2) $|B| - (c_h\ /\ 2)$ *blocks are coloured as in ii);*
3) *all the c_i for $2 \leq i \leq h$ have to be even.*

Proof. 1) c_h is the number of pairs of vertices coloured with different colours and in every block coloured as in *i)* there are two different pairs of vertices coloured with different colours.
For 2) and 3) the proofs are evident. □

Lemma 1. *If $\mathcal{P}$ is a strict colouring for Co-STS, then the inequalities*

$$s_i(s_i - 1) \leq 3 \sum_{j=1}^{i} n_j(n_j - 1) \tag{1}$$

are true for $1 \leq i \leq h$.

Proof. The set $\mathcal{S}_i$ contains no more than $|B_i| = [s_i(s_i - 1)\ /\ 6]$ blocks and at least $(c_i\ /\ 2)$ blocks. In fact we have $|B_i|$ blocks inside $\mathcal{S}_i$ if it is a subsystem, and any pair of vertices of $\mathcal{S}_i$ coloured with two different colours is inside a block which is contained in $\mathcal{S}_i$. So $|B_i| \geq (c_i\ /\ 2)$ and (1) is demonstrated. □

Let us consider STSs(v) where all the blocks are at the same time edges and anti-edges and let us call these mixed hypergraphs BSTSs(v) (Bi-Steiner Triple Systems). It is evident that, in these systems all the blocks must be coloured as in *i)* and that, Theorem 4 and its corollary, 1) and 3) of Proposition 1, Lemma 1 are true also for BSTSs. So, we have the following result.

Lemma 2. *If $\mathcal{P}$ is a strict colouring for BSTS(v), then*

$$v(v - 1) = 3 \sum_{i=1}^{h} n_i(n_i - 1).$$

Proof. This is evident because $|B| = (c_h\ /\ 2)$. □

3. Strict Colourings for Co-SQS and BSQS.

Let us look at a SQS(X,B) as a co-hypergraph or a mixed hypergraph. As for STSs it is possible to define Co-SQSs and BSQSs (Co-Steiner Quadruple Systems, and Bi-Steiner Quadruple Systems) when all the blocks of B are anti-edges or at the same time edges and anti-edges.

Let $\mathcal{P}$ be a strict colouring, with h colours, for Co-SQSs or BSQSs, with X_i, n_i, $\mathcal{S}_i$, s_i defined as in the previous section. In $\mathcal{S}_i$, for $3 \leq i \leq h$, the number of triples of vertices coloured with different colours is:

$$c'_i = \binom{s_i}{3} - \left[\sum_{j=1}^{i}\binom{n_j}{3} + \sum_{j=1}^{i}\binom{n_j}{2}(s_i - n_j)\right].$$

Let us consider a BSQS(v); $\mathcal{P}$ colours every block of BSQS in one of these three possible ways:

i') Three vertices are coloured with one colour and the other vertex is coloured with another one.

ii') Two vertices are coloured with one colour and the other two vertices are coloured with another one.

iii') Two vertices are coloured with one colour and the other two vertices are coloured with two different colours different from the first.

Proposition 2. *If $\mathcal{P}$ is a strict colouring for BSQS(X, B), then*

1) $\sum_{i=1}^{h}\binom{n_i}{3}$ *blocks are coloured as in i');*
2) $(c'_h / 2)$ *blocks are coloured as in iii');*
3) $|B| - \frac{c'_h}{2} - \sum_{i=1}^{h}\binom{n_i}{3}$ *blocks are coloured as in ii');*
4) *all the c'_i have to be even.*

Proof. 1) The proof is evident from the definitions of SQS(v) and of strict colouring for BSQS(v).
2) c'_h is the number of triples of vertices coloured with three different colours in $\mathcal{P}$. These kinds of triples are inside type iii') blocks and in these blocks there are only two different triples of vertices coloured with three different colours.
For 3) and 4) the proofs are evident. □

Lemma 3. *If $\mathcal{P}$ is a strict colouring for BSQS(v), then*

$$\sum_{i=1}^{h}\binom{n_i}{2}(v - n_i) \geq \frac{1}{2}\binom{v}{3} + \sum_{i=1}^{h}\binom{n_i}{3}.$$

Proof. The inequality is true because in every strict colouring $\mathcal{P}$ of BSQS(v), from Proposition 2, we have that

$$|B| - \frac{c'_h}{2} - \sum_{i=1}^{h}\binom{n_i}{3} \geq 0. \qquad \square$$

In the demonstrations of next sections the following lemma will be useful.

Lemma 4. (Tuza, private communication, 1994). *If $\mathcal{P}$ is a strict colouring of BSQS(v), then*

$$|B_i| = \frac{1}{4}\binom{s_i}{3} \leq \frac{s_i - 2}{2}\sum_{i=1}^{h}\binom{n_i}{2} - 2\sum_{i=1}^{h}\binom{n_i}{3}$$

for $3 \leq i \leq h$.

Proof. In the first i colouring classes there are no more than $|B_i| = \frac{1}{4}\binom{s_i}{3}$ blocks and there are $\binom{n_i}{2}$ monochromatic pairs of vertices of colour i; each pair is contained in no more than $[(s_i - 2) / 2]$ blocks. No block is allowed to be monochromatic, therefore the number of blocks meeting the colour class i in more than two points is precisely $\binom{n_i}{3}$ and every block with three monochromatic vertices contains three different monochromatic pairs of vertices. □

In the case of strict colourings $\mathcal{P}$ for Co-SQSs, it is possible to colour the blocks as in i'), ii'), iii'), but we have to consider the possibility of colouring a block with only one colour. For Co-SQSs we can determine the number of blocks coloured as in iii'), this number is the same as in Proposition 2, and all the c_i', for $3 \leq i \leq h$, have to be even.

Lemma 5. *If $\mathcal{P}$ is a strict colouring for Co-SQS, then*

$$\sum_{j=1}^{i}\binom{n_j}{3} + 2\sum_{j=1}^{i}\binom{n_j}{2}(s_i - n_j) \geq \binom{s_i}{3} \tag{2}$$

for $1 \leq i \leq h$.

The set S_i contains no more than $\frac{1}{4}\binom{s_i}{3}$ blocks, and at least $(c_i' / 2)$ blocks. These latter blocks are all coloured with three colours and we have to consider the blocks where three vertices are coloured with the same colour and also the monochromatic blocks, so

$$\frac{1}{4}\binom{s_i}{3} \geq \frac{c_i'}{2} + \frac{1}{4}\sum_{j=1}^{i}\binom{n_i}{3}$$

and this inequality is equivalent to (2). □

Because of a next demonstration it is necessary to show the following lemma proved in [5].

Lemma 6. *If $\mathcal{P}$ is a strict colouring for Co-SQS(X, B), then*

$$|B| \leq \sum_{i=1}^{h} \binom{n_i}{2} \frac{v-2}{2} - \sum_{i=1}^{h} \frac{5}{4} \binom{n_i}{3}.$$

4. Upper Chromatic Number for Co-SQS(10) and BSQS(10).

In [5] it was determined that for Co-SQS(8) $\bar{\chi} = 4$, for BSQS(8) $\bar{\chi} = 3$ and for Co-SQS(16) $\bar{\chi} \leq 5$. In this section a characterization of all the possible strict colourings with $\bar{\chi}$ colours, for Co-SQS(10) and BSQS(10), will be shown.

In [9], two strict colourings of a mixed hypergraph $H(X, \mathcal{S})$ are different if two vertices of X have the same colour for one of these strict colourings and different colours for the other one. For a colourable H it is possible to consider the set of all different strict colourings with h colours and to partition this set in classes in such a way that in any class the strict colourings have the same cardinalities for all the colouring classes. Let us indicate any class of these partition with the h-upla (n_1, n_2, ... , n_h), where n_i, for $1 \leq i \leq h$ (and $n_i \leq n_{i+1}$) are the cardinalities of the colouring classes.

The system SQS(10), represented in *Tab. 1*, is a unique system; in fact there are not other non isomorphic systems [4].

{ 1, 2, 4, 5 }	{ 1, 2, 3, 7 }	{ 1, 3, 5, 8 }
{ 2, 3, 5, 6 }	{ 2, 3, 4, 8 }	{ 2, 4, 6, 9 }
{ 3, 4, 6 ,7 }	{ 3, 4, 5, 9 }	{ 3, 5, 7, 0 }
{ 4, 5, 7, 8 }	{ 4, 5, 6, 0 }	{ 1, 4, 6, 8 }
{ 5, 6, 8, 9 }	{ 1, 5, 6, 7 }	{ 2, 5, 7, 9 }
{ 6, 7, 9, 0 }	{ 2, 6, 7, 8 }	{ 3, 6, 8, 0 }
{ 1, 7, 8, 0 }	{ 3, 7, 8, 9 }	{ 1, 4, 7, 9 }
{ 1, 2, 8, 9 }	{ 4, 8, 9, 0 }	{ 2, 5, 8, 0 }
{ 2, 3, 9, 0 }	{ 1, 5, 9, 0 }	{ 1, 3, 6, 9 }
{ 1, 3, 4, 0 }	{ 1, 2, 6, 0 }	{ 2, 4, 7, 0 }

Table 1

Proposition 3. *For Co-SQS(10) and BSQS(10) the co-stability number is* $\alpha_A = 5$.

Proof. It is easy to see in *Table* 1 that the set of vertices { 2, 3, 6, 8, 9 } is a blocking set, so from Theorem 3 we have that $\alpha_A = 5$. □

Theorem 5. *For Co-SQS(10),* $\bar{\chi} = 4$ *and any strict colouring with four colours belongs to* (1, 1, 4, 4) *or* (1, 1, 2, 6).

Proof. In SQS(10) there are blocking sets. Therefore Co-SQS(10) is definitely bi-colourable in such a way that every block has either three vertices coloured with the same colour and the other vertex coloured with the other colour, or two vertices coloured with one colour and the other two vertices coloured with the other colour. So for whichever vertex of Co-SQS(10), it is possible to colour it with a new colour, thus obtaining a new strict colouring and $\bar{\chi} \geq 3$. From Theorem 1, $\bar{\chi} \leq 5$. Let us suppose that $\bar{\chi} = 5$, so there exists at least one strict colouring $\overline{\mathcal{P}}$ with five colours. If in this colouring there is a colouring class with only one vertex x, then it is possible to study the derived system Co-STS(9) obtained from the blocks, where x is present in Co-SQS(10). If we colour Co-STS(9) with $\overline{\mathcal{P}}$, with the exception of x, we have a colouring of Co-STS(9) with four colours. However because, for this system $\bar{\chi} = 3$ [6], $\overline{\mathcal{P}}$ is not a strict colouring for Co-STS(9). At least one block of Co-STS(9) is coloured with three colours, and if we add x to this block we have a block coloured with four different colours in Co-SQS(10); so in $\overline{\mathcal{P}}$ all the colouring classes have at least two vertices. In a Co-SQS every pair of vertices is inside $[(v-2)/2]$ blocks, and in every block a monochromatic pair of vertices has to be present, so we find that the number of blocks of Co-SQS(10) is greater than $5 \cdot [(v-2)/2]$ and $\overline{\mathcal{P}}$ is not a strict colouring.

So $\bar{\chi} \leq 4$. If $\bar{\chi} = 4$, let us suppose that in every strict colouring with four colours every colouring class has at least two vertices. This colouring from Proposition 3 in [7] belongs to (2, 2, 3, 3), but it does not respect the inequality of Lemma 5 in Section 3. So if a strict colouring with four colours exists, it has a colouring class with one vertex. Let us call it x'. If we consider the derived system Co-STS(9) with respect to vertex x' from Propositions 3 and 4 of [6] we find that all the possible colourings belong to $\mathcal{P}' = (1, 1, 4, 4)$ or $\mathcal{P}'' = (1, 1, 2, 6)$. If we consider the sets of vertices of SQS(10) in *Table* 1 { 1 }, { 2 }, { 3, 4, 5, 7 }, { 0, 6, 8, 9 } as a colouring in $\mathcal{P}'$ and { 2 }, { 3 }, { 5, 6 }, { 1, 4, 7, 8, 9, 0 } as a colouring in $\mathcal{P}''$, then it is easy to see that these two colourings are strict colourings for Co-SQS(10) and $\bar{\chi} = 4$. □

Corollary 1. *In every strict colouring of Co-SQS(10), which belongs to* (1, 1, 2, 6), *there are three monochromatic blocks.*

Proof. From Proposition 2 the vertices of the first three colouring classes (1, 1, 2) are in a unique block. From Theorem 2 we have that $|T_6| - |T_4| = 2$, so the fourth colouring class contains three blocks. □

Theorem 6. *For BSQS(10),* $\bar{\chi} = 4$ *and all the strict colourings with four colours belong to* $(1, 1, 4, 4)$.

5. The Upper Chromatic Number For Particular BSQSs(16).

In [5], [7] it was proved that for Co-SQS(16) $\bar{\chi} \leq 5$, all the strict colourings with five colours are in $(1, 1, 2, 4, 8,)$ and always have monochromatic blocks.

In this section we find the upper chromatic number for BSQS(16) obtaine ' from doubling construction.

Theorem 7. *If BSQS(16) is obtained from a doubling construction, then* $\bar{\chi} = 3$ *or* $\bar{\chi} = 0$.

Proof. If BSQS(16) is not colourable, then $\bar{\chi} = 0$. Let us suppose BSQS(16) is colourable: so $\bar{\chi} \geq 3$. If a strict colouring with four colours exists, then let us consider the following cases.

Case $n_4 = 8$: the colouring class with eight vertices is a blocking set and the other vertices in the other three colouring classes also form a blocking set. If, in this last blocking set, we choose three vertices coloured with three different colours, these vertices belong to a block coloured with four colours, and this is absurd because it contrasts with the definition of strict colouring for BSQS.

Case $n_4 = 7$: because in (n_1, n_2, n_3, n_4) $n_i \leq n_{i+1}$ for $1 \leq i \leq 3$, then all the possible classes with $n_4 = 7$ are:

$$a = (1, 1, 7, 7); \quad b = (1, 2, 6, 7); \quad c = (1, 3, 5, 7);$$
$$d = (2, 2, 5, 7); \quad e = (1, 4, 4, 7); \quad f = (2, 3, 4, 7);$$
$$g = (3, 3, 3, 7).$$

The colourings in a, b, f, g are not strict colourings because they do not respect Lemma 4 in Section 3.

In the colourings which belong to c, c'_3 is not even, then for Proposition 2 these colourings are not strict colourings.

The colourings in d and in e are not strict colourings. In fact, let us consider $\mathcal{H}$ the vertex set which contains all the vertices of the colouring class with seven vertices, and $\mathcal{K}$ the complementary of $\mathcal{H}$. $T_{|\mathcal{H}|}$ is the set of blocks included in $\mathcal{H}$ and $T_{|\mathcal{K}|}$ the set of blocks included in $\mathcal{K}$. For Theorem 2

$$|T_9| - |T_7| = 7$$

but $\mathcal{H}$ is a stable set so $|T_{|\mathcal{K}|}| = 7$ and for Proposition 2 we have at least $(c'_3 / 2) = 10$ blocks inside $\mathcal{K}$, for the colourings in d, and $(c'_3 / 2) = 8$ blocks inside $\mathcal{K}$, for the colourings in e, and this is absurd.

Case $n_4 = 6$: the possible classes are:

$$h = (1, 3, 6, 6); \quad i = (2, 2, 6, 6); \quad j = (1, 4, 5, 6);$$
$$k = (2, 3, 5, 6); \quad l = (2, 4, 4, 6); \quad m = (3, 3, 4, 6).$$

The colourings in i and in m are not strict colourings because they do not respect Lemma 4 in Section 3.

The colourings in l are not strict colourings. In fact, if $\mathcal{H}$ is a vertex set which contains all the vertices of the colouring class with six vertices, and $\mathcal{K}$ is the complementary of $\mathcal{H}$, then

$$|T_{10}| - |T_6| = 15$$

and $|T_{|\mathcal{K}|}| = 15$. However for Proposition 2 we find that there are at least $(c'_3 / 2) = 16$ blocks inside to $T_{\mathcal{K}}$ and this is absurd.

The colourings in h and in j are not strict colourings. In fact, let us suppose that in these classes there is a strict colouring $\overline{\mathcal{P}}$ and let us consider the derived system BSTS(15) obtained from the blocks which contain the vertex belonging to the colouring class with cardinality one. If we colour the vertices of BSTS(15) with $\overline{\mathcal{P}}$, then this colouring will also be strict colouring for BSTS(15), but it does not respect the inequality of Lemma 1 in Section 2 and this is absurd.

For $n_4 = 6$ it remains to investigate only the colourings which belong to k.

Case $n_4 = 5$: the possible classes are:

$$n = (1, 5, 5, 5); \quad p = (2, 4, 5, 5); \quad q = (3, 3, 5, 5);$$
$$r = (3, 4, 4, 5).$$

The colourings in n are not strict colourings because, for Proposition 2, c'_3 is not even.

The colourings in q, r are not strict colourings because they do not respect the inequality of Lemma 3 in Section 3.

For $n_4 = 5$ it remains to investigate the colourings in p.

Case $n_4 = 4$: in this case there is only the class

$$s = (4, 4, 4, 4)$$

and none of its colourings are strict colourings because they do not respect the inequality of Lemma 3 in Section 3.

For BSQS(16) it is necessary to prove that the colourings in k and p are not strict colourings. BSQS(16) is obtained from a doubling construction, so it contains the two subsystems BSQS(X',B') and BSQS(X'',B''), where $|X'| = |X''| = 8$ and $X' \cap X'' = 0$. For BSQS(8) we know [5] that $\bar{\chi} = 3$, $\alpha_A = 4$, and every colouring class in a strict colouring is a stable set.

Let us define n_i' and n_i'' for $1 \leq i \leq 4$ as the number of vertices of $BSQS'$ and $BSQS''$ coloured respectively with the colour (i). So, for the cardinalities of the colouring classes of the colourings in k or p, we have that $n_i = n_i' + n_i''$ for $1 \leq i \leq 4$.

Let us consider the colourings in k.

Because $n_i' \leq 4$ and $n_i'' \leq 4$ one can colour the vertices of $BSQS'(X', B')$ and $BSQS''(X'', B'')$ with a colouring which belongs to one of these three subclasses of k.

1) $BSQS', \; n_4' = 4, \; n_3' = 1, \; n_2' = 3, \; n_1' = 0,$
$BSQS'', \; n_4'' = 2, \; n_3'' = 4, \; n_2'' = 0, \; n_1'' = 2.$

Because BSQS(16) is obtained from a doubling construction, if we choose a vertex of colour (2) in X', then there exist, in every 1-factorization of K_8 on X', four 1-factors, in which there is a pair of vertices coloured with colours (2) and (4). These pairs of vertices can not make blocks with the pairs coloured with colours (1) and (3) in X''. However in every 1-factorization of K_8 on X'' the pairs of vertices coloured with colours (1) and (3) are in at least four 1-factors and this implies the existence of polychromatic blocks. So, the colourings in 1) are not strict colourings.

The colourings in

2) $BSQS', \; n_4' = 4, \; n_3' = 2, \; n_2' = 0, \; n_1' = 2,$
$BSQS'', \; n_4'' = 2, \; n_3'' = n_2'' = 3, \; n_1'' = 0,$

are not strict colourings. In fact if we consider a vertex of colour (1) in X', then in every 1-factorization of K_8 on X', four 1-factor with pairs coloured with colours (1) and (4) exist. These pairs can not form blocks with pairs coloured with (2) and (3) in X''. In every 1-factorization of X'', these pairs are in the 1-factor where the pair of vertices coloured with colour (4) is present. Therefore it is easy to prove that in every 1-factorization of X'' there are only three 1-factors where pairs coloured with (2) and (3) do not exist.

The colourings in

3) $BSQS', \; n_4' = 3, \; n_3' = 2, \; n_2' = 3, \; n_1' = 0,$
$BSQS'', \; n_4'' = 3, \; n_3'' = 3, \; n_2'' = 0, \; n_1'' = 2,$

are not strict colourings because they contain at least one monochromatic block. Let us choose a vertex of colour (1) in X'', then in every 1-factorization of K_8

on X'', three 1-factors, in which each contains the pair of vertices coloured with colours (1) and (4), exist. These pairs of vertices can not make blocks with pairs coloured with colours (2) and (3) in X'. So it is possible to match these three 1-factors of K_8 on X'' with one of these three possible 1-factors of K_8 on X' in which the vertices have the following colourings:

$$\text{two 1-factor} \begin{bmatrix} (3), & (4) \\ (3), & (4) \\ (2), & (4) \\ (2), & (2) \end{bmatrix} \quad \text{and one 1-factor} \begin{bmatrix} (3), & (4) \\ (3), & (4) \\ (2), & (4) \\ (2), & (2) \end{bmatrix}$$

$$\text{or} \begin{bmatrix} (3), & (3) \\ (4), & (4) \\ (2), & (2) \\ (2), & (4) \end{bmatrix} \quad \text{or} \begin{bmatrix} (3), & (3) \\ (4), & (2) \\ (4), & (2) \\ (4), & (2) \end{bmatrix}$$

In the other four 1-factors of K_8 in X' there are at least two which contain a pair of vertices coloured with colour (4). In the three 1-factors of K_8 on X'', in which the pair of vertices coloured with colours (1) and (4) is present, pairs of vertices coloured with colours (1) and (3) can not be present. So in these 1-factors, pairs of vertices coloured with colour (4) can not be present. Therefore in every colouring of 3) monochromatic blocks coloured with colour (4) exist.

Let us consider the colourings in p.

It is possible to colour the vertices of $BSQS'(X', B')$ and $BSQS''(X'', B'')$ with a colouring which belongs to one of these three subclasses of p:

The colourings in

1′) $BSQS'$ $n'_4 = 3,\ n'_3 = 1,\ n'_2 = 4,\ n'_1 = 0,$
$BSQS''$ $n''_4 = 2,\ n''_3 = 4,\ n''_2 = 0,\ n''_1 = 2;$

2′) $BSQS'$ $n'_4 = 4,\ n'_3 = 2,\ n'_2 = 0,\ n'_1 = 2,$
$BSQS''$ $n''_4 = 1,\ n''_3 = 3,\ n''_2 = 4,\ n''_1 = 0,$

are not strict colourings and it is possible to prove this with the same procedure used in colouring 1).

The colourings in

3′) $BSQS'$ $n'_4 = 3,\ n'_3 = 3,\ n'_2 = 0,\ n'_1 = 2,$
$BSQS''$ $n''_4 = 2,\ n''_3 = 2,\ n''_2 = 4,\ n''_1 = 0,$

are not strict colourings because if we choose a vertex of colour (4) in X'', there exist, in every 1-factorization of K_8 on X'', four 1-factors, in which there is the pair of vertices coloured with colours (4) and (2). These pairs of vertices can not make blocks with pairs coloured with colours (3) and (1) in X'. So, it is possible to match these four 1-factors with the only possible four 1-factors of K_8 on X' in which the vertices have the following colourings:

$$\text{three 1-factors} \begin{bmatrix} (1), (4) \\ (1), (4) \\ (3), (4) \\ (3), (3) \end{bmatrix} \quad \text{and one 1-factor} \begin{bmatrix} (1), (1) \\ (4), (3) \\ (4), (3) \\ (4), (3). \end{bmatrix}$$

In the other three 1-factors of K_8 on X' there is always a pair of vertices coloured with colour (4), then in the colourings of 3′) there are monochromatic blocks coloured with colour (4), and this contrasts with the definition of strict colouring for BSQS.

So, because for Co-SQS(16) $\bar{\chi} \leq 5$, and because in the strict colourings of Co-SQS(16) with five colours there are always monochromatic blocks [5],[7], then for BSQS(16) colourable $\bar{\chi} = 3$. □

In Theorem 7 it is possible to observe that every colouring with four colours, except colourings in k and p, is not strict colouring for any BSQS(16). It remains to be shown whether the colourings in k and p are (or are not) strict colourings for BSQS(16) not obtained by doubling construction.

6. Concluding Remarks.

In this paper conditions have been given for the existence of strict colourings for STSs and SQSs.

SQSs(v) has been examined in particular detail. For $v = 10$ the value of upper chromatic number is determined both for Co-SQS(10) and for BSQS(10), finding characterizations of the cardinalities of colouring classes for strict colourings with $\bar{\chi}$(Co-SQS(10)) and $\bar{\chi}$(BSQS(10)) colours.

For $v = 16$, $\bar{\chi}(BSQS(16)) = 0$ or 3 only in particular systems, that is those obtained from doubling construction from two SQSs(8). Therefore, it will be necessary to verify whether in general for BSQS(16) $\bar{\chi} \leq 4$.

An important problem will be to understand the behaviour of $\bar{\chi}(v)$ for general BSQS(v), and verify if for $v \to \infty$ BSQS(v) becomes non colourable for particular classes of SQSs.

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