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ON STABILITY OF STOCHASTIC MULTIOBJECTIVE PROGRAMMING PROBLEMS WITH RANDOM COEFFICIENTS IN THE OBJECTIVE FUNCTIONS

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In this paper, we present quantitative analysis of the stability set of the first kind of stochastic multiobjective programming problems with random coefficients in the objective functions. Using Kuhn - Tucker conditions and some statistical theorems, the authors determine the set of all random parameters that make the solution of our concerned problem be stable. An illustrative example be given to clarify the development theory in this paper.

1. Introduction.

A family of subsets A of a set X is called a σ-algebra [1] if:

(i) $X \in A$

(ii) if $E \in A$, then $X - E \in A$, and

(iii) if $E_i \in A$, $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} E_i \in A$.

(X, A) is called measurable space, and A are the measurable sets.

A function $f : X \to Y$ from a measurable space X into a topological space Y is called measurable if $f^{-1}(U)$ is a measurable set for every open set $U \subset Y$. For a measurable random mapping [3]

$$Y : (X, A, P_A) \to (R^\ell, L^\ell, P),$$

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where X is a measurable space, A is σ-algebra on X, P_A is a probability measure defined on A, L^ℓ is Borel σ-algebra and P is the probability measure induced by Y in L^ℓ.

We will make use of the following probability Theorem [2].

If $p(x_1, , x_2, \dots, x_n)$ is the probability density function of the vector

$$(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n),$$

then the probability of the point $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ failing in the half-space $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n < x$ is

$$\phi(x) = \int\cdots\int_{x_1+x_2+\cdots+x_n<x} p(x_1, x_2, \dots, x_n)\, dx_1 dx_2 \dots dx_n .$$

For $n = 2$

$$\phi(x) = \iint_{x_1+x_2<x} p(x_1, x_2)\, dx_1 dx_2 = \int dx_1 \int_{-\infty}^{x-x_1} p(x_1, x_2) dx_2 .$$

If the variables ε_1 and ε_2 are independent then $p(x_1, x_2) = p_1(x_1) p_2(x_2)$ and consequently

$$\phi(x) = \int dx_1 \int_{-\infty}^{x-x_1} p_1(x_1) p_2(x_2)\, dx_2 = \int_{-\infty}^{x} dz \int p_1(x_1) p_2(z - x_1)\, dx_1\},$$

then the integral $\int p_1(x_1)\ p_2(z - x_1)\, dx_1$ is the density function of the sum $\varepsilon_1 + \varepsilon_2$.

The solvability and the stability of the solution of stochastic multiobjective programming problems are defined and analyzed qualitatively in [7]. In this paper we determine quantitatively the set of random vector parameters that make the efficient solution of stochastic multiobjective programming problems with random coefficients in the objective functions is stable.

2. Formulation of the problem.

Let us consider the following stochastic multiobjective programming problem:

$$\text{VOP}(\lambda) \begin{cases} \min f_j(x, \lambda), \ \ j = 1, 2, \dots, m, \\ \text{subject to} \\ M = \{x \in \mathbb{R}^n : g_r(x) \le 0, \ r = 1, 2, \dots, t\}, \end{cases}$$

where $f_j(x, \lambda)$, $j = 1, 2, \ldots, m$, are real valued functions and convex on M, $g_r(x)$, $r = 1, 2, \ldots, t$, are real valued convex functions on $\mathbb{R}^n$ and λ is a random vector parameters in a measurable set $\Lambda \subset \mathbb{R}^\ell$.

Let us consider the case:

$$f_j(x, \lambda) = \lambda_j f_j(x), \quad j = 1, 2, \ldots, n,$$

then the nonnegative weighted sum technique is used to find the corresponding scalar optimization problem which takes the form:

$$P(\lambda) \begin{cases} \min \sum_{j=1}^{n} \omega_j \lambda_j f_j(x) \\ \text{subject to} \\ x \in M, \ \omega \in \Gamma = \{\omega \in \mathbb{R}^n : \omega_j \geq 0, \ j = 1, 2 \ldots, n, \ \sum_{j=1}^{n} \omega_j = 1\}. \end{cases}$$

It is well known from [3], [6] that the optimal solution $\bar{x}$ of problem $P(\lambda)$ is an efficient solution of VOP(λ) if and only if the following conditions are valid:

(1) $P(\lambda)$ has a unique solution, or

(2) all the weights of the weighting problem concerned are strictly positive.

Definition 1. x^0 is said to be an efficient solution corresponding $\lambda^0 \in \Lambda$ if and only if

$$P\left\{\lambda^0 : \text{the system } f_j(x, \lambda^0) - f_j(x^0, \lambda^0) < 0 \text{ has no solution } x \in M\right\} = 1\,,$$

where P is the probability measure induced by Y [3], [6].

Definition 2. The solvability set [5] of $P(\lambda)$ is denoted by B and is defined by

$$B = \left\{\lambda \in \Lambda \subset \mathbb{R}^\ell / P(\lambda) \text{ is solvable }\right\}.$$

3. The stability set of the first kind.

Definition 3. Suppose $\bar{\lambda} \in B$ with a corresponding efficient solution $\bar{x}$, then the stability set of the first kind is denoted by $S(\bar{x})$ and is defined by

$$S(\bar{x}) = \left\{\lambda \in B / P\left\{\lambda : f_j(x, \lambda) - f_j(\bar{x}, \lambda) < 0 \right.\right. \\ \left.\left. \text{has no solution } x \in M\right\} = 1, \ j = 1, 2, \ldots, m\right\},$$

where P is the probability measure induced by a random mapping Y.

Now, we will determine the set $S(\bar{x})$ as follows.

Since $\bar{x}$ is an efficient solution, then there exist $\omega_j \geq 0$, $j = 1, 2, \dots, m$, $\sum_{j=1}^{m} \omega_j = 1$ such that $\sum_{j=1}^{m} \omega_j f_j(\bar{x}, \lambda) \leq \sum_{j=1}^{m} \omega_j f_j(x, \lambda)$.
Therefore, we can write $S(\bar{x})$ as:

$$S(\bar{x}) = \left\{\lambda \in B/P\left\{\lambda : \sum_{j=1}^{m} \omega_j \lambda_j f_j(\bar{x}) \leq \sum_{j=1}^{m} \omega_j \lambda_j f_j(x)\right\} = 1\right\}.$$

From necessary Kuhn - Tucker Theorem [4], we obtain the following system:

$$\begin{aligned}
&\lambda_1 \omega_1 \nabla f_1(\bar{x}) + \lambda_2 \omega_2 \nabla f_2(\bar{x}) + \cdots + \lambda_m \omega_m \nabla f_m(\bar{x}) + \\
&+ \bar{u}_1 \nabla g_1(\bar{x}) + \cdots + \bar{u}_t \nabla g_t(\bar{x}) \geq 0, \\
&\bar{x}[\lambda_1 \omega_1 \nabla f_1(\bar{x}) + \lambda_2 \omega_2 \nabla f_2(\bar{x}) + \cdots + \lambda_m \omega_m \nabla f_m(\bar{x}) + \\
&+ \bar{u}_1 \nabla g_1(\bar{x}) + \cdots + \bar{u}_t \nabla g_t(\bar{x})] = 0, \\
&g_r(\bar{x}) \leq 0, \quad r = 1, 2, \dots, t, \\
&\bar{u}_r g_r(\bar{x}) = 0, \quad r = 1, 2, \dots, t,
\end{aligned}$$

or

$$\begin{aligned}
&\lambda_1 \left[\bar{x}_1 \omega_1 \frac{\partial f_1}{\partial x_1}(\bar{x}) + \cdots + \bar{x}_n \omega_1 \frac{\partial f_1}{\partial x_n}(\bar{x})\right] + \\
&+ \lambda_2 \left[\bar{x}_1 \omega_2 \frac{\partial f_2}{\partial x_1}(\bar{x}) + \cdots + \bar{x}_n \omega_2 \frac{\partial f_2}{\partial x_n}(\bar{x})\right] + \cdots + \\
&+ \lambda_m \left[\bar{x}_1 \omega_m \frac{\partial f_m}{\partial x_1}(\bar{x}) + \cdots + \bar{x}_n \omega_m \frac{\partial f_m}{\partial x_n}(\bar{x})\right] = \\
&= -u_1 \left[\bar{x}_1 \frac{\partial g_1}{\partial x_1}(\bar{x}) + \cdots + \bar{x}_n \frac{\partial g_1}{\partial x_n}(\bar{x})\right] - \cdots - \\
&- u_t \left[\bar{x}_1 \frac{\partial g_t}{\partial x_1}(\bar{x}) + \cdots + \bar{x}_n \frac{\partial g_t}{\partial x_n}(\bar{x})\right]
\end{aligned}$$

and

$$\lambda_1\left[\omega_1\frac{\partial f_1}{\partial x_1}(\bar{x})+\cdots+\omega_1\frac{\partial f_1}{\partial x_n}(\bar{x})\right]+\cdots+$$
$$+\lambda_m\left[\omega_m\frac{\partial f_m}{\partial x_1}(\bar{x})+\cdots+\omega_m\frac{\partial f_m}{\partial x_n}(\bar{x})\right]\geq$$
$$\geq -u_1\left[\frac{\partial g_1}{\partial x_1}(\bar{x})+\cdots+\frac{\partial g_1}{\partial x_n}(\bar{x})\right]-\cdots-$$
$$-u_t\left[\frac{\partial g_t}{\partial x_1}(\bar{x})+\cdots+\frac{\partial g_t}{\partial x_n}(\bar{x})\right].$$

By adding the above two systems, we obtain the following system of inequalities which are linear in random unknowns λ_j, $j=1,2,\ldots,m$, and deterministic unknowns u_i, $i=1,2,\ldots,t$:

$$\lambda_1\left[(\bar{x}_1-1)\omega_1\frac{\partial f_1}{\partial x_1}(\bar{x})+\cdots+(\bar{x}_n-1)\omega_1\frac{\partial f_1}{\partial x_n}(\bar{x})\right]+\cdots+$$
$$+\lambda_m\left[(\bar{x}_1-1)\omega_m\frac{\partial f_m}{\partial x_1}(\bar{x})+\cdots+(\bar{x}_n-1)\omega_m\frac{\partial f_m}{\partial x_n}(\bar{x})\right]\leq$$
$$\leq u_1\left[(1-\bar{x}_1)\frac{\partial g_1}{\partial x_1}(\bar{x})+\cdots+(1-\bar{x}_n)\frac{\partial g_1}{\partial x_n}(\bar{x})\right]+\cdots+$$
$$+u_t\left[(1-\bar{x}_1)\frac{\partial g_t}{\partial x_1}(\bar{x})+\cdots+(1-\bar{x}_n)\frac{\partial g_t}{\partial x_n}(\bar{x})\right]$$

or

$$a_1\lambda_1+a_2\lambda_2+\ldots+a_m\lambda_m\leq\nu,$$

where

$$a_i=(\bar{x}_1-1)\omega_i\frac{\partial f_i}{\partial x_1}(\bar{x})+\cdots+(\bar{x}_n-1)\omega_i\frac{\partial f_i}{\partial x_n}(\bar{x}),\quad i=1,2,\ldots,m$$

and

$$\nu=\sum_{k=1}^{t}u_k\left[(1-\bar{x}_1)\frac{\partial g_k}{\partial x_1}(\bar{x})+\cdots+(1-\bar{x}_n)\frac{\partial g_k}{\partial x_n}(\bar{x})\right].$$

Since λ_i, $i=1,2,\ldots,m$, are random variables for which the probability density functions $d_i(y)$, $i=1,2,\ldots,m$, are known, then $a_i\lambda_i$, $i=1,2,\ldots,m$, are random variables and we can determine their densities. Also, we can find the probability of failing the point $(y_1,y_2,\ldots,y_m)$ in the half-space

$$a_1\lambda_1+a_2\lambda_2+\cdots+a_m\lambda_m\leq\nu$$

from the probability Theorem [2] (see the introduction) as follows:

$$\phi(\nu) = \int_{y_1+y_2+\cdots+y_m \leq \nu} \cdots \int p(y_1, y_2, \ldots, y_m)\, dy_1 dy_2 \ldots dy_m,$$

where y_i, $i = 1, 2, \ldots, m$, are the available values of the random variables $a_i\lambda_i$, $i = 1, 2, \ldots, m$.

By the same manner, we can obtain the set of random parameters corresponding another ω from the set of all weights that give the same efficient solution. Finally taking the union to obtain $S(\bar{x})$.

Example. Let us consider the following problem:

$$\begin{cases} \min\,[\lambda_1(x+1)\,,\ \lambda_2((x-3)^2+1)] \\ \text{subject to} \\ x \geq 0, \end{cases}$$

where the density functions of λ_1 and λ_2 are

$$\begin{aligned} d_1(z_1) &= 2z_1, \quad z_1 \in [0, 1], \\ d_2(z_2) &= 2z_2, \quad z_2 \in [0, 1]. \end{aligned}$$

Let $Y : ([0, 1], A, d) \to ([0, 1], A, p)$ be a measurable mapping such that $Y(\bar{A}) = \bar{A}^c$ for each $\bar{A} \in A$, and $p(\bar{A}^c) = d_i(\bar{A}^c)$, $i = 1, 2$. Assuming that $\bar{A}_1^c = \bar{\lambda}_1 = \bar{A}_2^c = \bar{\lambda}_2 = \frac{1}{2}$ and $\bar{\omega}_1 = \bar{\omega}_2 = 2$, then the efficient solutions corresponding to $(\bar{\lambda}, \bar{\omega})$ are the interval [1,3].

From Kuhn-Tucker Theorem we get

$$\begin{aligned} 2\lambda_1 + 4\lambda_2(\bar{x} - 3) - \bar{u} &\geq 0, \\ \bar{u}x &= 0 \\ \bar{x} &\geq 0 \\ \bar{u} &\geq 0. \end{aligned}$$

Let $\bar{x} = 3$, then $2\lambda_2 \leq 0$. Since $d_2(z_2) = 2z_2$, $z_2 \in [0, 1]$, then the density function of the random variables $2\lambda_2$ is

$$P_2(y) = \begin{cases} \frac{3}{2}y^3, & 0 < y \leq 1 \\ -\frac{2}{3}y^3 + 4y - \frac{8}{3}, & 1 < y \leq 2, \end{cases}$$

and hence the set of all random variables for which the densities are:

$$p_1(y) = 2y, \qquad 0 \leq y \leq 1$$

and

$$p_2(y) = \begin{cases} \frac{3}{2}y^3, & 0 \leq y \leq 1 \\ -\frac{2}{3}y^3 + 4y - \frac{8}{3}, & 1 < y \leq 2, \end{cases}$$

and $p_1(y) = p_2(y) = 1$, corresponding the efficient solution $\bar{x} = 3$.

REFERENCES

[1] E.K. Blum, *Numerical Analysis and Computation, Theory and practics,* Addison-Wesley Inc., 1972.

[2] B.V. Gnedenko, *The Theory of probability,* Mir Publishers, Moscow, 1975.

[3] J. Guddat - F. Guerra - K. Tanmer - K. Wendler, *Multiobjective and Stochastic Optimization Based on Parametric Optimization,* Akademic-Verlage, Berlin, 1985.

[4] O.L. Mangasarian, *Nonlinear Programming,* McGraw-Hill Inc., New York, London, 1969.

[5] M.S. Osman - A.H. El-Banna - I.A. Youness, *On a General Class of Parametric Convex Programming Problem,* Advances in Modelling and Simulation, Vol. 5, No. 1, 1985.

[6] C. Vira - Y.H. Yacov, *Multiobjective Decision Making, Theory and Methodology,* Series Vol. 8, North-Holand, Series Inc., New York, Amsterdam, 1983.

[7] E.A. Youness, *Study on Stochastic Multiobjective Programming Problems,* AMSE Review,17 (1991), pp. 51 - 56.

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