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FOURIER-MUKAI TRANSFORMS OF LINE BUNDLES ON DERIVED EQUIVALENT ABELIAN VARIETIES

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We study the Fourier-Mukai functor $D(Y) \to D(X)$ induced by the universal family on a fine moduli space Y for simple semihomogeneous vector bundles on an abelian variety X. The main result is that the Fourier-Mukai transform of a very negative line bundle on Y is ample if and only if the bundles parametrized by Y are nef.

1. Introduction

In connection with their work on Generic Vanishing and related topics [10], Pareschi and Popa raise the following question: Let Y be a fine moduli space of stable sheaves on a smooth projective variety X, and fix a universal family $\mathscr{E}$ on $X \times Y$. Let $\mathscr{L}$ be an ample line bundle on Y, let $d = \dim Y$ and put

$$\mathscr{G}_n = R^d p_{1*}\left(p_2^*(\mathscr{L}^{-n}) \otimes \mathscr{E}\right) \tag{1.1}$$

where p_i denote the projections from $X \times Y$. Under suitable hypotheses (for instance as in Example 2.4), the sheaf $\mathscr{G}_n$ is locally free for n sufficiently large; it is the Fourier-Mukai transform of $\mathscr{L}^{-n}$ with kernel $\mathscr{E}$. What can be said about these bundles? In particular, are they stable, and are they ample?

In this text we work out the following special case:

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Theorem 1.1. *Let X be an abelian variety and Y a fine moduli space of simple semihomogeneous bundles on X. Fix an ample line bundle $\mathscr{L}$ on Y and define $\mathscr{G}_n$ by* (1.1)*. Then, for sufficiently large n, the following holds:*

1. *$\mathscr{G}_n$ is a simple semihomogeneous bundle. In particular it is stable.*
2. *$\mathscr{G}_n$ is ample if and only if the bundles $\mathscr{E}|_{X\times\{y\}}$ parametrized by Y are nef.*

The definition of semihomogeneous bundles is recalled in Section 2. Our viewpoint is that their moduli spaces are of the simplest possible form: In the mostly expository sections 3 and 4 we recall results of Mukai and Orlov showing that any moduli space Y of semihomogeneous bundles on X is again an abelian variety of the same dimension as X, and the Fourier-Mukai functor associated to a universal family is an equivalence $D(X) \cong D(Y)$ of derived categories. Conversely, any Fourier-Mukai equivalence between abelian varieties, with locally free kernel, is of this form: The Fourier-Mukai kernel gives Y the structure of a moduli space for semihomogeneous vector bundles on X.

Any line bundle is semihomogeneous, and the prototype for a moduli space of semihomogeneous sheaves is the dual abelian variety $Y = \widehat{X}$, equipped with the normalized Poincaré bundle $\mathscr{P}$. In this case, the above theorem is well known; in fact the bundle $\mathscr{G}_n$ is stable and ample already for $n = 1$. This can be deduced from a result of Mukai [5], saying that the pullback of $\mathscr{G}_1$ under the canonical isogeny

$$\phi_{\mathscr{L}}\colon \widehat{X} \to X, \quad \xi \mapsto T_\xi^*(\mathscr{L}) \otimes \mathscr{L}^\vee$$

(viewing X as the dual of $\widehat{X}$), is just

$$H^0(\widehat{X}, \mathscr{L}) \otimes_k \mathscr{L},$$

i.e. a direct sum of a suitable number of copies of $\mathscr{L}$ itself.

One motivation for studying bundles of the form $\mathscr{G}_n$ is the rôle they play in Hacon's and Pareschi-Popa's approach to Generic Vanishing: Returning to the general case, with Y an arbitrary moduli space of sheaves on a smooth projective variety X, we say that a sheaf $\mathscr{F}$ on X satisfies *Generic Vanishing* with respect to the universal family $\mathscr{E}$ if, for each i, the closed set

$$\left\{ y \in Y \,\middle|\, H^i(X, \mathscr{F} \otimes \mathscr{E}_y) \neq 0 \right\}$$

has codimension at least i. A criterion of Pareschi and Popa, generalizing work of Hacon, says that the bundle $\mathscr{G}_n$ can be used to detect Generic Vanishing. Namely, $\mathscr{F}$ satisfies Generic Vanishing if and only if

$$H^i(X, \mathscr{F} \otimes \mathscr{G}_n) = 0$$

for all $i > 0$. Here, it is enough to test with a bundle $\mathscr{G}_n$ associated to a fixed ample line bundle $\mathscr{L}$ and a fixed, but large, integer n. This criterion, together with Mukai's description of $\mathscr{G}_n$ in the case of an abelian variety and its dual, has been used by Hacon [1] and Pareschi-Popa [10] to generalize the Green-Lazarsfeld Generic Vanishing theorem. The upshot is that a good understanding of the bundles $\mathscr{G}_n$, and in particular their positivity properties, is required to make the Generic Vanishing criterion effective.

The first part of the theorem (with weaker hypotheses) is obtained as Corollary 4.2, as an immediate consequence of the results of Mukai and Orlov. Theorem 7.2 is a more precise version of the second part: We prove that the bundle $\mathscr{G}_n$ always satisfies an index theorem, and its index can be computed. By demanding its index to be zero, we arrive at the criterion for ampleness, stated as Corollary 7.3.

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2. Terminology

Throughout, let X denote an abelian variety of dimension g over an algebraically closed field k of characteristic zero. We write $\widehat{X}$ for the dual abelian variety. The normalized Poincaré line bundle on $X \times \widehat{X}$ is denoted $\mathscr{P}$.

For each point $x \in X$, we write $T_x \colon X \to X$ for translation by x. A line bundle is *homogeneous* if it is invariant under all translations. Via the Poincaré bundle, points in $\widehat{X}$ correspond to homogeneous line bundles on X and vice versa. We denote points in $\widehat{X}$ by Greek letters $\xi, \zeta, \ldots$, and we use the same symbols for the corresponding homogeneous line bundles on X.

If $\mathscr{L}$ is an arbitrary line bundle on X, we write $K(\mathscr{L}) \subseteq X$ for the subgroup of points $x \in X$ satisfying $T_x^*(\mathscr{L}) \cong \mathscr{L}$.

We use the words vector bundle and line bundle as synonyms for locally free sheaf and invertible sheaf. By stability, we mean Gieseker-stability with respect to any fixed polarization, the choice of which will not matter to us.

Definition 2.1 (Mukai [4, 6])**.** A coherent sheaf $\mathscr{E}$ on X is *semihomogeneous* if the locus

$$\Gamma(\mathscr{E}) = \{(x, \xi) \in X \times \widehat{X} \,\big|\, T_x^*(\mathscr{E}) \cong \mathscr{E} \otimes \xi\}$$

has dimension g.

If $\mathscr{E}$ locally free, then it is semihomogeneous if and only if the following condition holds: For each $x \in X$, there is a $\xi \in \widehat{X}$, such that

$$T_x^*(\mathscr{E}) \cong \mathscr{E} \otimes \xi.$$

The equivalence with the definition given above follows from noting that the kernel of the first projection $p_1 \colon \Gamma(\mathscr{E}) \to X$ is finite: In fact, its kernel is contained in the group of r-torsion points $\widehat{X}_r$, where r is the rank of $\mathscr{E}$.

Next we set up notation for the Fourier-Mukai transform. We write $D(X)$ for the bounded derived category of a variety X, equipped with the autofunctors $\mathscr{C} \mapsto \mathscr{C}[i]$ that shift a complex $\mathscr{C}$ the specified number i steps to the left. We view a sheaf as a complex concentrated in degree zero; thus $D(X)$ contains the category of coherent sheaves. Let X and Y be two varieties (both will be abelian varieties in our context), and let

$$X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$$

denote the projections. To any coherent sheaf $\mathscr{E}$ (or, more generally, any bounded complex) on the product $X \times Y$, we associate a pair of functors between the derived categories of X and Y:

Definition 2.2. The *Fourier-Mukai functors* with kernel $\mathscr{E}$ are the two functors

$$\Phi_{\mathscr{E}} \colon D(X) \to D(Y), \qquad \Phi_{\mathscr{E}}(-) = Rp_{2*}(p_1^*(-) \overset{L}{\otimes} \mathscr{E})$$
$$\Psi_{\mathscr{E}} \colon D(Y) \to D(X), \qquad \Psi_{\mathscr{E}}(-) = Rp_{1*}(p_2^*(-) \overset{L}{\otimes} \mathscr{E}).$$

We write $\Phi^i_{\mathscr{E}}(-)$ and $\Psi^i_{\mathscr{E}}(-)$ for the i'th cohomology sheaf of $\Phi_{\mathscr{E}}(-)$ and $\Psi_{\mathscr{E}}(-)$.

Definition 2.3. Given a triple $(X, Y, \mathscr{E})$ as above and a coherent sheaf $\mathscr{F}$ on X, we say that

1. $\mathscr{F}$ satisfies the *index theorem* (IT) with respect to $\mathscr{E}$ if there exists an integer i_0 such that

   $$H^i(X, \mathscr{F} \otimes \mathscr{E}|_{X \times \{y\}}) = 0 \quad \text{for all } i \neq i_0 \text{ and all } y \in Y.$$

2. $\mathscr{F}$ satisfies the *weak index theorem* (WIT) with respect to $\mathscr{E}$ if there exists an integer i_0 such that

   $$\Phi^i_{\mathscr{E}}(\mathscr{F}) = 0 \quad \text{for all } i \neq i_0.$$

Similarly, the vanishing of $H^i(Y,(-)\otimes\mathscr{E}|_{\{x\}\times Y})$ and $\Psi^i_{\mathscr{E}}(-)$ defines the properties IT and WIT for sheaves on X.

Suppose that the kernel $\mathscr{E}$ of the Fourier-Mukai functor is a Y-flat coherent sheaf. Then the base change theorem in cohomology shows that IT implies WIT. The integer i_0 in the definition will be referred to as the $\mathscr{E}$-index of $\mathscr{F}$, denoted $i_{\mathscr{E}}(\mathscr{F})$.

Example 2.4. Let X and Y be projective varieties, and let $\mathscr{E}$ be a vector bundle on $X\times Y$. Assume that Y has a dualizing sheaf ω_Y. If $\mathscr{L}$ is an ample line bundle on Y and n is sufficiently large, then $\mathscr{L}^{-n}$ satisfies IT with respect to $\mathscr{E}$, and its $\mathscr{E}$-index is $d=\dim Y$. This follows from Serre's theorems: We have

$$H^i(Y,\mathscr{L}^{-n}\otimes\mathscr{E}|_{\{x\}\times Y})\cong H^{d-i}(Y,\mathscr{L}^{n}\otimes\mathscr{E}|^{\vee}_{\{x\}\times Y}\otimes\omega_Y)^{\vee}$$

and the latter vanishes for n sufficiently large if $i\neq d$. The bound on n can be made independent of x, by using that the vanishing of the cohomology vector spaces above is an open condition on $x\in X$.

Definition 2.5. Let $\mathscr{F}$ be a coherent sheaf on X satisfying WIT with respect to $\mathscr{E}$, and let i_0 denote its $\mathscr{E}$-index. The *Fourier-Mukai transform* of $\mathscr{F}$ with respect to $\mathscr{E}$ is the coherent sheaf $\Phi^{i_0}_{\mathscr{E}}(\mathscr{F})$.

When we do not specify the kernel explicitly, we will mean the Fourier-Mukai functor with respect to the Poincaré line bundle on $X\times\widehat{X}$. Thus, in this case, we will write Φ and Ψ with no subscript, and, if $\mathscr{F}$ satisfies WIT, its index (i.e. its $\mathscr{P}$-index) is denoted $i(\mathscr{F})$. In this case we also use the notation

$$\widehat{\mathscr{F}}=\Phi^{i(\mathscr{F})}(\mathscr{F})$$

for the Fourier-Mukai transform.

3. Moduli spaces of semihomogeneous bundles

Let M be the (quasi-projective) moduli space of stable vector bundles on X. The following results are due to Mukai [4]:

1. Every semihomogeneous bundle is semistable, and every simple semihomogeneous bundle is stable, with respect to any polarization.

2. For every simple (in particular, every stable) vector bundle $\mathscr{E}$ on X, we have

   $$\dim_k \operatorname{Ext}^1_X(\mathscr{E},\mathscr{E})\geq g$$

   with equality if and only if $\mathscr{E}$ is semihomogeneous.

By (1), simple semihomogeneous bundles are parametrized by a certain locus in M. Since the tangent space to any bundle $\mathscr{E} \in M$ is canonically isomorphic to $\operatorname{Ext}^1_X(\mathscr{E},\mathscr{E})$, point (2) gives a geometric characterization of this locus, as the points in M with tangent space of minimal dimension g. For these results we refer to the very readable original paper of Mukai. Here we essentially only make a remark:

Proposition 3.1. *Let $\mathscr{E}$ be a stable bundle, and let $Y \subseteq M$ be the connected component containing $\mathscr{E}$. Then the following are equivalent.*

1. *$\mathscr{E}$ is semihomogeneous*
2. *The tangent space to Y at $\mathscr{E}$ has dimension g.*
3. *Y is an abelian variety isogeneous to X.*

Proof. The equivalence of (1) and (2) is Mukai's theorem. Furthermore, it is obvious that (3) implies (2). We next show that (1) implies (3).

Let $\mathscr{Q}$ be the determinant of $\mathscr{E}$. We can form a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\tau"] \arrow[dr, "r_{\widehat{X}}"'] & Y \arrow[d, "\delta"] \\ & \widehat{X} \end{tikzcd}$$

where the twisting map τ sends a homogeneous line bundle $\xi \in \widehat{X}$ to $\mathscr{E} \otimes \xi$ and the determinant map δ sends a sheaf $\mathscr{F} \in Y$ to the homogeneous line bundle $\det(\mathscr{F}) \otimes \mathscr{Q}^{-1}$. The composition is multiplication by r, since $\det(\mathscr{E} \otimes \xi) \cong \mathscr{Q} \otimes \xi^r$.

Since the composed map $r_{\widehat{X}}$ is finite, so is τ, and hence its image is g-dimensional. The bundles parametrized by the image of τ are clearly semihomogeneous, hence, using the equivalence (1) $\Longleftrightarrow$ (2), we conclude that Y is nonsingular and g-dimensional at all points of $\tau(\widehat{X})$. Since Y is connected this implies that $\tau(\widehat{X}) = Y$, and so Y is a nonsingular g-dimensional variety.

Now let $Y(\mathscr{Q}) \subset Y$ be the subscheme $\delta^{-1}(0)$, i.e. the locus in Y parametrizing bundles with fixed determinant $\mathscr{Q}$. Then there is a Cartesian diagram

$$\begin{tikzcd} \widehat{X} \times Y(\mathscr{Q}) \arrow[r] \arrow[d] & Y \arrow[d, "\delta"] \\ \widehat{X} \arrow[r, "r_{\widehat{X}}"] & \widehat{X} \end{tikzcd}$$

where the left map is projection onto the first factor and the top map sends a pair $(\xi, \mathscr{F})$ to the tensor product $\mathscr{F} \otimes \xi$. In particular, the determinant map δ is locally trivial in the étale topology. Since Y is nonsingular, this implies that δ is étale. A variety admitting an étale map to an abelian variety is itself an abelian variety [7, Section 18], so we are done. □

4. Derived equivalent abelian varieties as moduli spaces

Let X and Y be abelian varieties, and suppose there exists a derived equivalence

$$D(X) \xrightarrow{\sim} D(Y).$$

By results of Orlov [9], any such equivalence is a Fourier-Mukai transform $\Phi_{\mathscr{E}}$, with kernel a sheaf $\mathscr{E}$ (i.e. a complex concentrated in one degree, although not necessarily degree zero) on the product $X \times Y$. Moreover, this sheaf is semihomogeneous. The semihomogeneity is perhaps only almost explicit in Orlov's work — so here is a short account:

Associated to $\mathscr{E}$, Orlov constructs an isomorphism

$$f\colon X \times \widehat{X} \xrightarrow{\sim} Y \times \widehat{Y},$$

which on points is given by

$$\begin{gathered} f(x,\xi) = (y,\zeta) \\ \Updownarrow \\ T^*_{(x,y)}(\mathscr{E}) \cong \mathscr{E} \otimes (p_1^*(\xi^{-1}) \otimes p_2^*(\zeta)). \end{gathered}$$

This says that a quadruple (x,ξ,y,ζ) belongs to the graph of f if and only if (x,y,ξ^{-1},ζ) belongs to $\Gamma(\mathscr{E})$, with notation as in Section 2. Since the graph of f is $2g$-dimensional, so is $\Gamma(\mathscr{E})$, which shows that $\mathscr{E}$ is semihomogeneous.

For simplicity, we will assume that $\mathscr{E}$ is also locally free.

Proposition 4.1. *Let X and Y be abelian varieties and $\mathscr{E}$ a vector bundle on their product $X \times Y$. Then the following are equivalent.*

1. *The Fourier-Mukai transform $\Phi_{\mathscr{E}}\colon D(X) \to D(Y)$ with kernel $\mathscr{E}$ is an equivalence.*

2. *The variety Y, equipped with the family $\mathscr{E}$, is a fine moduli space of simple semihomogeneous vector bundles on X.*

Proof. By a criterion of Bondal and Orlov [2, Corollary 7.5], the functor $\Phi_{\mathscr{E}}$ is fully faithful if and only if

$$\operatorname{Hom}(\mathscr{E}_y,\mathscr{E}_y) = k \qquad \text{for all } y \tag{4.1}$$

$$\operatorname{Ext}^i(\mathscr{E}_y,\mathscr{E}_{y'}) = 0 \qquad \text{for all } i \text{ and } y \neq y', \tag{4.2}$$

where $\mathscr{E}_y = \mathscr{E}|_{X\times\{y\}}$. We also need the fact that, by the triviality of the canonical bundles on X and Y, the functor $\Phi_{\mathscr{E}}$ is fully faithful if and only if it is an equivalence [2, Corollary 7.8].

First assume that Y, equipped with $\mathscr{E}$, is a moduli space of simple semihomogeneous vector bundles on X. Then (4.1) is satisfied since the fibres $\mathscr{E}_y$ are simple, and (4.2) follows from Mukai's work on homogeneous and semihomogeneous vector bundles — see Lemma 4.8 in Orlov's paper [9]. So $\Phi_{\mathscr{E}}$ is fully faithful and hence an equivalence.

Conversely, assume $\Phi_{\mathscr{E}}$ is an equivalence. Since (4.1) is satisfied, the fibres $\mathscr{E}_y$ are simple, and they are semihomogeneous since $\mathscr{E}$ is a semihomogeneous bundle on $X \times Y$. Since $\mathscr{E}$ is locally free, it is flat over Y, and so induces a morphism

$$f \colon Y \to M$$

to the moduli space M of stable sheaves on X. By Proposition 3.1, this map f hits a component M' of M which is an abelian g-dimensional variety. Since (4.2) is satisfied for $i = 0$, all distinct fibres $\mathscr{E}_y$ and $\mathscr{E}_{y'}$ are non isomorphic, which says that f has degree 1. Thus f is an isomorphism, being a degree 1 map between abelian varieties of the same dimension. □

Corollary 4.2. *Let Y be a fine moduli space for simple semihomogeneous vector bundles on X, with a fixed universal family $\mathscr{E}$. Let $\mathscr{L}$ be a line bundle on Y satisfying IT with respect to $\mathscr{E}$. Then the Fourier-Mukai transform $\mathscr{G}$ of $\mathscr{L}$ with respect to $\mathscr{E}$ is a simple semihomogeneous vector bundle on X. In particular it is stable.*

Proof. By the proposition, the functor $\Phi_{\mathscr{E}}$ is an equivalence of categories, which implies that $\Psi_{\mathscr{E}}$ is an equivalence also [2, Remark 7.7]. Thus

$$\operatorname{Ext}^i_Y(\mathscr{L},\mathscr{L}) \cong \operatorname{Ext}^i_X(\mathscr{G},\mathscr{G})$$

for all i. In particular, for $i = 0$ we get that $\mathscr{G}$ is simple, and for $i = 1$ we get $\dim \operatorname{Ext}^1_X(\mathscr{G},\mathscr{G}) = g$, which implies $\mathscr{G}$ is semihomogeneous, by Proposition 3.1. Finally, we apply Mukai's result, quoted in the previous section, to conclude that $\mathscr{G}$ is stable. □

The first part of Theorem 1.1 follows, since the very negative line bundle $\mathscr{L}^{-n}$ considered there satisfies IT, by Example 2.4.

5. Index theorems

Recall that a line bundle on an abelian variety is *degenerate* if its Euler characteristic is zero; otherwise it is *nondegenerate*. By Mumford's vanishing theorem [7, Section 16], every nondegenerate line bundle $\mathscr{L}$ satisfies IT (with respect to the Poincaré bundle). In this section we show that degenerate line bundles satisfy WIT. This is probably well known, but I do not know of a suitable reference.

The starting point is the following construction by Kempf [8]: Let $\mathscr{L}$ be a degenerate line bundle on X. Let $Y \subseteq X$ be the identity component of $K(\mathscr{L})$, with reduced structure, and let

$$\pi \colon X \to X/Y$$

be the quotient. Then there exist a nondegenerate line bundle $\mathscr{M}$ on X/Y and a homogeneous line bundle $\xi \in \widehat{X}$ such that

$$\mathscr{L} \cong \pi^*(\mathscr{M}) \otimes \xi. \tag{5.1}$$

Proposition 5.1. *Let $\mathscr{L}$ be a degenerate line bundle, and write $\mathscr{L} = \pi^*(\mathscr{M}) \otimes \xi$ with $\mathscr{M}$ nondegenerate, as above. Then $\mathscr{L}$ satisfies WIT with index*

$$i(\mathscr{L}) = \dim K(\mathscr{L}) + i(\mathscr{M})$$

and its Fourier-Mukai transform is

$$\widehat{\mathscr{L}} \cong T_\xi^*(\widehat{\pi}_*(\widehat{\mathscr{M}})).$$

Remark 5.2. Note that, since $\widehat{\pi}$ is an embedding, the expression $\widehat{\pi}_*(\widehat{\mathscr{M}})$ above simply means $\widehat{\mathscr{M}}$ considered as a torsion sheaf on $\widehat{X}$.

Proof. For any homogeneous line bundle ξ, there is a natural isomorphism [5]

$$\Phi((-) \otimes \xi) \cong T_\xi^*(\Phi(-)).$$

Furthermore, for an arbitrary homomorphism $\phi \colon X \to Z$ of abelian varieties, there is an isomorphism [11, Section 11.3]

$$\Phi \circ L\phi^*[d] \cong R\widehat{\phi}_* \circ \Phi$$

where $d = \dim X - \dim Z$. Note that Φ on the left hand side is the Fourier-Mukai functor with kernel the Poincaré bundle on $X \times \widehat{X}$, whereas the same symbol on the right hand side is the Fourier-Mukai functor with kernel the Poincaré bundle on $Z \times \widehat{Z}$.

Since π is flat and $\widehat{\pi}$ finite, we have $L\pi^* = \pi^*$ and $R\widehat{\pi}_* = \widehat{\pi}_*$. Thus we get, for every integer i,

$$\begin{aligned}\Phi^i(\pi^*(\mathscr{M}) \otimes \xi) &\cong T_\xi^*(\Phi^i(\pi^*(\mathscr{M}))) \\ &\cong T_\xi^*(\widehat{\pi}_*(\Phi^{i-d}(\mathscr{M}))),\end{aligned}$$

where $d = \dim Y$, which is also the dimension of $K(\mathscr{L})$. Since $\mathscr{M}$ is nondegenerate, it satisfies IT with some index $i(\mathscr{M})$. The claim follows. $\square$

Corollary 5.3. *Every line bundle $\mathscr{L}$ satisfies WIT. The index of a line bundle satisfies*

1. $i(\mathscr{L}^\vee) = g + \dim K(\mathscr{L}) - i(\mathscr{L})$,
2. $i(\mathscr{L} \otimes \xi) = i(\mathscr{L})$ *for all* $\xi \in \widehat{X}$,
3. $i(\mathscr{L}^n) = i(\mathscr{L})$ *for all* $n > 0$.

Proof. When $\mathscr{L}$ is nondegenerate, these are standard facts. Otherwise, write $\mathscr{L}$ as $\pi^*(\mathscr{M}) \otimes \xi$ as before, and apply the nondegenerate case to $\mathscr{M}$. The claims then follow from the formula for $i(\mathscr{L})$ in the proposition, since $\dim K(\mathscr{L})$ is invariant under dualizing, twisting with homogeneous line bundles and positive tensor powers. □

As a further application of Proposition 5.1, we give a characterization of nef line bundles on abelian varieties.

Corollary 5.4. *The following are equivalent conditions on a line bundle $\mathscr{L}$.*

1. $\mathscr{L}$ *is nef*
2. $i(\mathscr{L}) = \dim K(\mathscr{L})$
3. *There exist an abelian subvariety $Y \subseteq X$ and an ample line bundle $\mathscr{M}$ on X/Y such that*
$$\mathscr{L} \cong \pi^*(\mathscr{M}) \otimes \xi$$
where $\pi \colon X \to X/Y$ is the quotient and $\xi \in \widehat{X}$ is a homogeneous line bundle.

Remark 5.5. In particular, on a simple abelian variety, a nef line bundle is either ample or algebraically equivalent to $\mathscr{O}_X$.

Proof. By the results of Kempf, we may write an arbitrary line bundle $\mathscr{L}$ as in (3), with Y the identity component of $K(\mathscr{L})$ and $\mathscr{M}$ nondegenerate, but not necessarily ample. Let (3') be the condition that $\mathscr{M}$ is ample, for this particular choice of Y. Clearly (3') implies (3), and (3) implies (1).

Condition (1) implies (3'): If $\pi^*(\mathscr{M}) \otimes \xi$ is nef, then also $\pi^*(\mathscr{M})$ is nef. But then $\mathscr{M}$ was nef to begin with [3, Example 1.4.4]. Since $\mathscr{M}$ is also nondegenerate, it is ample [3, Corollary 1.5.18].

Conditions (2) and (3') are equivalent: By Proposition 5.1, condition (2) holds if and only if $\mathscr{M}$ has index zero. But a nondegenerate line bundle has index zero if and only if it is ample. □

6. Semihomogeneous vector bundles

Let $\mathscr{E}$ be a simple semihomogeneous vector bundle on X. Recall the following:

1. There are an isogeny $f\colon Y \to X$ and a line bundle $\mathscr{L}$ on Y such that $f^*(\mathscr{E}) \cong \mathscr{L}^{\oplus r}$.
2. There are an isogeny $f\colon Y \to X$ and a line bundle $\mathscr{L}$ on Y such that $\mathscr{E} \cong f_*(\mathscr{L})$.

In fact, among simple vector bundles, the semihomogeneous ones are characterized by either of these two properties [4]. In this section we will use these facts to reduce many questions about $\mathscr{E}$ to the corresponding questions about its determinant line bundle $\mathscr{Q}$.

The notion of degeneracy can be extended plainly from line bundles to simple semihomogeneous vector bundles:

Definition 6.1. A simple semihomogeneous vector bundle $\mathscr{E}$ is *degenerate* if its Euler characteristic $\chi(\mathscr{E})$ is zero. Otherwise it is *nondegenerate*.

Proposition 6.2. *Let $\mathscr{E}$ be a simple semihomogeneous vector bundle on X, with determinant line bundle $\mathscr{Q}$.*

1. *$\mathscr{E}$ is nondegenerate if and only if $\mathscr{Q}$ is nondegenerate.*
2. *$\mathscr{E}$ is ample if and only if $\mathscr{Q}$ is ample.*
3. *$\mathscr{E}$ is nef if and only if $\mathscr{Q}$ is nef.*

Proof. Part (1) follows from Mukai's formula [4]

$$\chi(\mathscr{E}) = \chi(\mathscr{Q})/r^{g-1}$$

for the Euler characteristic, where r is the rank of $\mathscr{E}$.

To prove (2) and (3), choose an isogeny $f\colon Y \to X$ such that $f^*(\mathscr{E}) \cong \mathscr{L}^{\oplus r}$ for a line bundle $\mathscr{L}$ on Y. Then $f^*(\mathscr{Q}) \cong \mathscr{L}^r$. Since f is a finite map, a vector bundle on X is ample (nef) if and only if its pullback to Y is. Thus we have

$$\begin{aligned} \mathscr{E} \text{ ample (nef)} &\iff \mathscr{L}^{\oplus r} \text{ ample (nef)} \\ \mathscr{Q} \text{ ample (nef)} &\iff \mathscr{L}^{r} \text{ ample (nef)} \end{aligned}$$

and the two statements on the right are both equivalent to the ampleness (nefness) of $\mathscr{L}$. □

Proposition 6.3. *Let $\mathscr{E}$ be a simple semihomogeneous vector bundle. Then $\mathscr{E}$ satisfies WIT, and its index equals the index of its determinant line bundle. If $\mathscr{E}$ is nondegenerate, then it satisfies IT.*

Proof. Let

$$f\colon Y \to X$$

be an isogeny such that $f_*(\mathscr{L}) \cong \mathscr{E}$ for some line bundle $\mathscr{L}$ on Y. Being a line bundle, $\mathscr{L}$ satisfies WIT by Corollary 5.3. Furthermore we have [5, Section 3]

$$\Phi^i(f_*(\mathscr{L})) \cong \widehat{f}^*(\Phi^i(\mathscr{L})),$$

which implies that $\mathscr{E}$ satisfies WIT, and its index equals the index of $\mathscr{L}$.

Next we show that the index of $\mathscr{L}$ equals the index of the determinant $\mathscr{Q}$ of $\mathscr{E}$. Firstly, we have [4, Lemma 6.21]

$$f^*(\mathscr{Q}) \sim \mathscr{L}^d \quad \text{(algebraic equivalence)} \tag{6.1}$$

where d is the degree of f. By Corollary 5.3, it follows that $\mathscr{L}$ and $f^*(\mathscr{Q})$ have the same index. Moreover [5, Section 3], there are isomorphisms

$$\Phi^i(f^*(\mathscr{Q})) \cong \widehat{f}_*(\Phi^i(\mathscr{Q}))$$

for all i, so $f^*(\mathscr{Q})$ has the same index as $\mathscr{Q}$. This proves the first part.

Since the dual map $\widehat{f}$ is again an isogeny, we have that every $\zeta \in \widehat{Y}$ is a pullback $f^*(\xi)$ of some element $\xi \in \widehat{X}$. By the projection formula,

$$H^i(Y, \mathscr{L} \otimes \zeta) \cong H^i(X, f_*(\mathscr{L} \otimes \zeta)) \cong H^i(X, \mathscr{E} \otimes \xi).$$

Thus $\mathscr{L}$ satisfies IT if and only if $\mathscr{E}$ does. But, if $\mathscr{E}$ is nondegenerate, then so is $\mathscr{Q}$ by Proposition 6.2, and then (6.1) shows that

$$d^g \chi(\mathscr{L}) = \chi(\mathscr{L}^d) = d\chi(\mathscr{Q}) \neq 0,$$

so $\mathscr{L}$ is nondegenerate also. Thus $\mathscr{L}$ satisfies IT. By what we just said, this proves that $\mathscr{E}$ satisfies IT. □

7. Fourier-Mukai transforms of negative line bundles

Lemma 7.1. *Let Y be a fine moduli space of simple semihomogeneous vector bundles on X, and let $\mathscr{E}$ be a fixed universal family on $X \times Y$.*

1. *There exists an isogeny $f\colon \widehat{X} \to Y$, that sends an element $\xi \in \widehat{X}$ to the bundle $\mathscr{E}|_{X \times \{0\}} \otimes \xi$.*
2. *For every i, the sheaf $R^i p_{2*}(\mathscr{E})$ is zero if and only if $\Phi^i(\mathscr{E}|_{X \times \{0\}})$ is zero.*

Proof. The bundle

$$\mathscr{P} \otimes p_1^*(\mathscr{E}|_{X\times\{0\}})$$

on $X \times \widehat{X}$, considered as a family over $\widehat{X}$ of simple semihomogeneous bundles, induces a map $f\colon \widehat{X} \to Y$. This is the map required in the first part.

Moreover, by the universal property of Poincaré bundle, there exists a line bundle $\mathscr{L}$ on $\widehat{X}$ such that

$$(1_X \times f)^*(\mathscr{E}) \cong p_2^*(\mathscr{L}) \otimes \mathscr{P} \otimes p_1^*(\mathscr{E}|_{X\times\{0\}}).$$

Hence, using flat base change for higher push forward and the projection formula, we get

$$\begin{aligned} f^*(R^i p_{2*}(\mathscr{E})) &\cong R^i p_{2*}((1_X \times f)^*(\mathscr{E})) \\ &\cong R^i p_{2*}(p_2^*(\mathscr{L}) \otimes \mathscr{P} \otimes p_1^*(\mathscr{E}|_{X\times\{0\}})) \\ &\cong \mathscr{L} \otimes R^i p_{2*}(p_1^*(\mathscr{E}|_{X\times\{0\}}) \otimes \mathscr{P}) \\ &= \mathscr{L} \otimes \Phi^i(\mathscr{E}|_{X\times\{0\}}). \end{aligned}$$

The second part of the lemma follows. □

Theorem 7.2. *Let Y be a fine moduli space of simple semihomogeneous vector bundles on X, and let $\mathscr{E}$ be a fixed universal family on $X \times Y$. Also let $\mathscr{L}$ be an ample line bundle on Y. There exists an integer n_0 such that for all $n \geq n_0$,*

1. *the line bundle $\mathscr{L}^{-n}$ satisfies the index theorem with respect to $\mathscr{E}$, and its index is g;*
2. *the Fourier-Mukai transform $\mathscr{G}_n = \Psi^g_{\mathscr{E}}(\mathscr{L}^{-n})$ with respect to $\mathscr{E}$ is nondegenerate and its index is*

   $$i(\mathscr{G}_n) = i(\mathscr{Q}) - \dim K(\mathscr{Q})$$

   where $\mathscr{Q}$ is the determinant line bundle of any of the bundles $\mathscr{E}|_{X\times\{y\}}$ parametrized by Y.

Proof. The first part was established in Example 2.4. For the second part, we make use of the Fourier-Mukai equivalence induced by $\mathscr{E}$. As $\mathscr{L}^{-n}$ has index g, we have

$$\Psi_{\mathscr{E}}(\mathscr{L}^{-n})[g] \cong \mathscr{G}_n \tag{7.1}$$

(where, as usual, the bundle on the right hand side is considered as a complex concentrated in degree zero). Furthermore, by Proposition 6.3, the semihomogeneous vector bundle $\mathscr{E}|^\vee_{X\times\{0\}}$ satisfies WIT with index equal to the index of $\mathscr{Q}^\vee$, which is

$$i_0 = g + \dim K(\mathscr{Q}) - i(\mathscr{Q})$$

by Corollary 5.3. Now apply Lemma 7.1 to $\mathscr{E}^\vee$ to conclude that $\Phi^i_{\mathscr{E}^\vee}(\mathscr{O}_X) = R^i p_{2*}(\mathscr{E}^\vee)$ vanishes for all i except i_0. In other words,

$$\Phi_{\mathscr{E}^\vee}(\mathscr{O}_X) \cong \mathscr{F}[-i_0] \tag{7.2}$$

for some coherent sheaf $\mathscr{F}$. Since $\Phi_{\mathscr{E}^\vee}(-)$ and $\Psi_{\mathscr{E}}(-)[g]$ are quasi-inverse functors [2, Proposition 5.9], the isomorphisms (7.1) and (7.2) give

$$\begin{aligned} H^p(X,\mathscr{G}_n) &\cong \operatorname{Ext}^p_X(\mathscr{O}_X,\mathscr{G}_n) \\ &\cong \operatorname{Hom}_{D(X)}(\mathscr{O}_X,\mathscr{G}_n[p]) \\ &\cong \operatorname{Hom}_{D(Y)}(\Phi_{\mathscr{E}^\vee}(\mathscr{O}_X),\mathscr{L}^{-n}[p]) \\ &\cong \operatorname{Hom}_{D(Y)}(\mathscr{F}[-i_0],\mathscr{L}^{-n}[p]) \\ &\cong \operatorname{Ext}^{p+i_0}_Y(\mathscr{F},\mathscr{L}^{-n}) \\ &\cong H^{g-p-i_0}(Y,\mathscr{F}\otimes\mathscr{L}^n)^\vee, \end{aligned}$$

using Serre duality in the last step. If n is sufficiently large, the cohomology group in the last line vanishes if and only if p differs from

$$g - i_0 = i(\mathscr{Q}) - \dim K(\mathscr{Q}).$$

Thus we have proved that $H^p(X,\mathscr{G}_n)$ is nonzero if and only if p has this value. On the one hand, this shows that *if* $\mathscr{G}_n$ satisfies IT, then this p is its index. On the other hand, it also shows that $\mathscr{G}_n$ is nondegenerate, so it satisfies IT by Proposition 6.3, and we are done. □

The second part of Theorem 1.1 follows:

Corollary 7.3. *The vector bundle $\mathscr{G}_n$ is ample for n sufficiently large if and only if the bundles $\mathscr{E}|_{X\times\{y\}}$ parametrized by Y are nef (equivalently, have nef determinant).*

Proof. A line bundle is ample if and only if its is nondegenerate and has index 0. The same holds for any simple semihomogeneous vector bundle, since both conditions “ample” and “nondegenerate of index 0” can be tested on the determinant line bundle, by Proposition 6.2 and Proposition 6.3. Hence, by the theorem, the simple semihomogeneous vector bundle $\mathscr{G}_n$ is ample if and only if $i(\mathscr{Q}) = \dim K(\mathscr{Q})$, where $\mathscr{Q}$ is the determinant of $\mathscr{E}|_{X\times\{y\}}$. By Corollary 5.4 this is equivalent to $\mathscr{Q}$ being nef, which again is equivalent to nefness of $\mathscr{E}|_{X\times\{y\}}$, by Proposition 6.2. □

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