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ABOUT THE SPLITTING FIELD FOR RATIONAL VALUED CHARACTERS

ION ARMEANU

The problem of finding minimal splitting fields for group characters is very old and important (see [4], Chapter 9). The most part of the papers on this subject are concerned with all irreducible characters of a group under certain conditions. It seems more difficult to obtain minimal splitting fields for only one character without strong conditions about the group. In this case, naturally, the number theoretical methods play an essential role. This paper concerns to prove that under certain circumstances if a rational character of a group has $\mathbb{Q}(2^{1/2}, i)$ as splitting field, then $\mathbb{Q}(i)$ or even $\mathbb{Q}(2^{1/2})$ are splitting fields too.

The notations and definitions will be those of [3].

Theorem. *Let G be a finite group of order $|G| = 2^a\ q_1^{b_1} \cdots q_s^{b_s}$ with $q_j \equiv 3$ (modulo 8), $j = 1, \dots, s$, prime numbers and let $\chi \in \mathrm{Irr}(G)$ an irreducible rational valued character such that $\mathbb{Q}(2^{1/2})$ is a splitting field for χ. Then:*

(1) *$\mathbb{Q}(i)$ is a splitting field for χ;*
(2) *if besides $\mathbb{R}$ is a splitting field for χ, then $\mathbb{Q}(2^{1/2})$ is also a splitting field for χ.*

Proof. (1) Let k be an algebraic number field, let p be a prime ideal and k_p the p-adic valuation over p. It is well known that if the Schur index $m_{k_p}(\chi) = 1$ for all p, then $m_k(\chi) = 1$. If $p \nmid |G|$ then $m_{k_p}(\chi) = 1$ (see [1], p. 186), so

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that we have to compute only Schur indices for the valuations over 2 and q_j, $j = 1, \dots, s$.

Evidently, 2 is associate in divisibility with $(1+i)^2$. Let be $q = 1+i$. Over 2 there is only one prime ideal in $\mathbb{Z}[i]$ so that $\overline{\mathbb{Q}(i)}_q$ can be excepted ($\bar{k}_p$ will denote the p-adic completion of k_p (see [2], p. 129)).

Since q_j, $j = 1, \dots, s$, are prime in $\mathbb{Z}[i]$, then $\overline{\mathbb{Q}(i)} = \mathbb{Q}_{q_j}(i)$. In the field $\mathbb{Q}_{q_j}$, -2 is a square, so that $\mathbb{Q}_{q_j}(i) = \mathbb{Q}_{q_j}(2^{1/2}, i)$ and it follows that $\overline{\mathbb{Q}(i)}$ is a splitting field for χ.

Since $\overline{\mathbb{Q}(i)}_\infty = \mathbb{C}$ it follows that in all complete fields over $\mathbb{Q}(i)$ (except one) the irreducible representations of G can be realized.

Let A be a simple component of the group algebra $\mathbb{Q}G$ corresponding to χ. Then $A = D_n$ a full matrix algebra over a division algebra D with centre $\mathbb{Q}(\chi)$. Then $m_{\mathbb{Q}}(\chi)$ is equal to the index of D. Thus, when $m_{\mathbb{Q}}(\chi) = 2$, D has degree 4 over its centre $\mathbb{Q}(\chi)$. The Brauer - Speiser Theorem (see [3]) states that $m_{\mathbb{Q}}(\chi) \leq 2$ whenever χ is real valued. Since χ is rational valued, then either $D = \mathbb{Q}$ or D is a rational quaternion algebra. If $D = \mathbb{Q}$ then $m_{\mathbb{Q}}(\chi) = 1$. Suppose now that D is a quaternion algebra and let $ax^2 + by^2 - abz^2$ be the corresponding quadratic form. This form has nontrivial roots in all complete fields over $\mathbb{Q}(i)$, except one. Since the product formula for the Hilbert symbol and Minkowski - Hasse Theorem (see [2]) are true for $\mathbb{Q}(i)$, then the quadratic form has nontrivial roots in $\overline{\mathbb{Q}(i)}_{(1+i)}$ and then it has nontrivial roots in $\mathbb{Q}(i)$ too, so that $\mathbb{Q}(i)$ is a splitting field for χ.

(2) Observe that $\mathbb{Q}_{q_j}(2^{1/2}) = \mathbb{Q}_{q_j}(2^{1/2}, i)$, $2 = (2^{1/2})^2$ and that over 2 there is only one prime ideal of $\mathbb{Z}[2^{1/2}]$. Then the proof follows analogous as the prove of (1). □

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