LE MATEMATICHE
Vol. XLIX (1994) – Fasc. I, pp. 15-24

UNIQUENESS OF SOLUTIONS OF THE CAUCHY PROBLEMS FOR FIRST ORDER PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS

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Consider the Cauchy problem

$$\text{(a)} \qquad \begin{cases} D_x z_i(x, y) = f_i\big(x, y, z(x, y), z(\cdot), D_y z_i(x, y)\big), & i = 1, \dots, m \\ z(x, y) = \alpha(x, y) \text{ for } (x, y) \in \Omega_0 \end{cases}$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is given function defined on the initial set $\Omega_0 \subset \mathbb{R}^{1+n}$, $z(x, y) = (z_1(x, y), \dots, z_m(x, y))$, $z(\cdot) = (z_1(\cdot), \dots, z_m(\cdot))$ and $D_y z_i(x, y) = \big(D_{y_1} z_i(x, y), \dots, D_{y_n} z_i(x, y)\big)$, $i = 1, \dots, m$.

We formulate a criterion of uniqueness of solutions of (a) using the comparison function of the Kamke type. This will be a generalization of classical results concerning first order equations with partial derivatives. We prove that the uniqueness criteria of Perron and Kamke type for differential-function problems are equivalent if given functions are continuous.

1. Introduction.

Consider the Cauchy problem

$$(1) \qquad \begin{cases} D_x z_i(x, y) = f_i(x, y, z(x, y), z(\cdot), D_y z_i(x, y), & i = 1, \dots, m \\ z(x, y) = \alpha(x, y) \quad \text{for} \quad (x, y) \in \Omega_0 \end{cases}$$

Entrato in Redazione il 20 settembre 1993.

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is given function defined on the initial set Ω_0,

$$z(x, y) = (z_1(x, y), \ldots, z_m(x, y)) \ , \ z(\cdot) = (z_1(\cdot), \ldots, z_m(\cdot))$$

and

$$D_y z_i(x, y) = \left(D_{y_1} z_i(x, y), \ldots, D_{y_n} z_i(x, y)\right), \quad i = 1, \ldots, m.$$

We formulate a criterion of uniqueness of solutions of (1) using the comparison function of the Kamke type.

Equations with partial derivatives of the first order have the following properties: the problem of existence of their solutions is equivalent with the problem of solving of certain system of ordinary differential equations. The investigations of properties of partial differential equations is connected with investigations of ordinary differential equations and inequalities. Such problem as: estimations of solutions of partial equations, estimations of domain of solutions, criterions of stability, conditions for uniqueness, estimation of the difference between solution and approximate solution are classical examples ([1], [2],[4],[8]).

Consider the initial-value problem

$$\text{(2)} \qquad \begin{cases} D_x z(x, y) = F(x, y, z(x, y), D_y z(x, y)), (x, y) \in H \\ z(x_0, y) = \omega(y) \quad \text{for} \quad y \in I_0 \end{cases}$$

where

$$H = \big\{(x, y) : x \in [x_0, x_0 + a), \ y = (y_1, \ldots, y_n), \\ |y_i - y_i^{(0)}| \le b_i - M_i(x - x_0), \ i = 1, \ldots, n\big\}$$

$a, b_i > 0, b_i \ge aM_i, i = 1, \ldots, n, I_0 = \{(x, y) \in H : x = x_0\}$.

Assume that $F : H \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and that for some function σ the following inequality

$$\text{(3)} \quad |F(x, y, p, q) - F(x, y, \bar{p}, \bar{q})| \le \sigma(x - x_0, |p - \bar{p}|) + \sum_{i=1}^{n} M_i |q_i - \bar{q}_i|$$

is true on $H \times \mathbb{R}^{1+n}$. If σ is a Perron type comparison function then there exists at most one solution of the problem (2) which is continuous, possesses first order partial derivatives in H and total derivative in $\partial H \cap ((x_0, x_0 + a) \times \mathbb{R}^n)$. If σ is Kamke's type comparison function and (3) is fulfilled in $(H \setminus I_0) \times \mathbb{R} \times \mathbb{R}^n$ then Cauchy problem (2) admits at most one solution in the class of functions satisfying the above conditions and possessing a continuous partial derivative with respect to x in I_0 (see [8], Chapter VII).

In this paper we extend the above theorems on the case first order partial differential-functional problems. This will be a generalization of the results published in [3], [5], [9], [11] (see also ([10]) as well as of classical results concerning first order equations with partial derivatives. We prove that the uniqueness criteria of Perron and Kamke type for differential-functional problems are equivalent if given functions are continuous.

2. Uniqueness of solution of the Cauchy problem.

Suppose that $f = (f_1, \dots, f_m) : H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) : H_0 \to \mathbb{R}^m$ where H is the Haar pyramid defined in n. 1 and H_0 is an initial set

$$H_0 = \left\{(x, y) : x \in [x_0 - \tau_0; x_0], |y_i - y_i^{(0)}| \le b_i,\ i = 1, \dots, n\right\},$$

$\tau_0 \ge 0$.

We consider the Cauchy problem

$$(4)\qquad \begin{cases} D_x z_i(x, y) = f_i\left(x, y, z(x, y), z(\cdot), D_y z_i(x, y)\right), & (x, y) \in H, \\ & i = 1, \dots, m \\ z_i(x, y) = \alpha_i(x, y) \quad \text{for} \quad (x, y) \in H_0,\ i = 1, \dots, m \end{cases}$$

where $z(\cdot) = (z_1(\cdot), \dots, z_m(\cdot))$.

The uniqueness of solutions of the Cauchy problem with $f = (f_1, \dots, f_m)$ independent on the functional argument was considered in many papers. In [1], [4],[8] we can find theorems on the uniqueness with linear and with nonlinear comparison functions.

Applications of the theory of differential and differential-functional inequalities to question like: estimate of the solution of (4), estimate of the difference between two solutions, uniqueness criteria of Perron type were considered in [3], [5], [10], [11].

Let us denote

$$H_x = \{(\xi, \eta) \in H_0 \cup H : \xi \le x\}, \qquad x \in [x_0, x_0 + a),$$
$$S_t = \{y : (x_0 + t, y) \in H\}, \qquad t \in [0, a)$$

and I_0 is defined in 1. We assume that the Cauchy problem (4) is of Volterra type, i.e.

$$(5)\qquad \begin{cases} f(x, y, p, u(\cdot), q) = f(x, y, p, v(\cdot), q) \quad \text{if } u, v \in C(H_0 \cup H, \mathbb{R}^m) \\ u(\xi, \eta) = v(\xi, \eta) \quad \text{for} \quad (\xi, \eta) \in H_x. \end{cases}$$

Let $\|w\| = \max_{1\le i\le m} |w_i|$ for $w = (w_1, \dots, w_m) \in \mathbb{R}^m$ and $\|z\|_x = \sup\{\|z(\xi,\eta)\| : (\xi,\eta) \in H_x\}$ for $z \in C(H_0 \cup H, \mathbb{R}^m)$.

A function $z : H_0 \cup H \to \mathbb{R}^m$ will be called the function of class D_0 in $H_0 \cup H$ if

(i) z is continuous on $H_0 \cup H$ and possesses the total differential on $S = \partial H \cap \big((x_0, x_0 + a) \times \mathbb{R}^n\big)$;

(ii) z has first derivatives with regard to (x, y) on H and the derivative $D_x z$ is continuous on I_0.

Assumption H_1. Suppose that

1) the function $f : H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies the Volterra condition (5);

2) $\sigma : (0, a) \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, the function $\sigma_0(t, w) = \sigma(t, w, w)$ is continuous on $(0, a) \times \mathbb{R}_+$ and $\sigma_0(t, 0) = 0$ for $t \in (0, a)$;

3) for each $\gamma \in (0, a)$ the function $w(t) = 0$, $t \in (0, \gamma)$ is the unique solution of the problem

$$w'(t) = \sigma_0\,(t, w(t)) \text{ for } t \in (0, \gamma) \text{ and } \lim_{t\to 0+} w(t) = \lim_{t\to 0+} \frac{w(t)}{t} = 0;$$

4) the estimation

$$\|f(x, y, p, u(\cdot), q) - f(x, y, \bar p, v(\cdot), \bar q)\| \le \le \sigma(x - x_0, \|p - \bar p\|, \|u - v\|_x) + \sum_{k=1}^{n} M_k |q_k - \bar q_k| \tag{6}$$

holds for $(x, y) \in H \setminus I_0$, $u, v \in C(H_0 \cup H, \mathbb{R}^m)$, $p, \bar p \in \mathbb{R}^m$, $q, \bar q \in \mathbb{R}^n$.

If $(\alpha, \beta) \subset \mathbb{R}$ and $w : (\alpha, \beta) \to \mathbb{R}$ then $D^+ w(t) (D_- w(t))$ is the right-hand upper (the left-hand lower) Dini's derivative of w at the point $t \in (\alpha, \beta)$.

In the proof of uniqueness theorem we shall use the following lemma.

Lemma 1. *Suppose that*

i) $\sigma_0 \in C((0, a) \times \mathbb{R}_+, \mathbb{R}_+)$ *and* σ_0 *satisfies the condition* 3) *of Assumption* H_1;

ii) $\varphi \in C([0, a), \mathbb{R}_+)$, φ *is nondecreasing and* $\varphi(0) = 0$, $D^+\varphi(0) = 0$;

iii) $I_+ = \{t \in (0, a) : \varphi(t) > 0$, *there exists* $\varepsilon > 0$ *such that for each* $\tau \in (t - \varepsilon, t)\ \varphi(\tau) < \varphi(t)\}$ *and for* $t \in I_+$ *we have* $D_-\varphi(t) \le \sigma(t, \varphi(t))$.

Under these assumptions $\varphi(t) = 0$ *for* $t \in [0, a)$.

The proof of this lemma is similar to the proof of the third comparison theorem in [8].

Now we prove

Theorem 1. *If Assumption* H_1 *is satisfied then the solution of class* D_0 *on* $H_0 \cup H$ *of the problem* (4) *is unique.*

Proof. Let u, v be solutions of class D_0 on $H_0 \cup H$ of (4), $u = (u_1, \dots, u_m)$, $v = (v_1, \dots, v_m)$. Denote $r(x, y) = (r_1(x, y), \dots, r_m(x, y)) = u(x, y) - v(x, y)$, $(x, y) \in H$,

$$\varphi_0^{(i)}(t) = \max_{y \in S_t} |r_i(x_0 + t, y)|, \ t \in [0, a), \ i = 1, \dots, m,$$

$$\varphi^{(i)}(t) = \max \left\{ \varphi_0^{(i)}(\tau) : \tau \in [0, t] \right\}, \ t \in [0, a), \ i = 1, \dots, m, \tag{7}$$

$$\varphi(t) = \max_{1 \le i \le m} \varphi^{(i)}(t), \ t \in [0, a).$$

Using Lemma 1 we prove that $\varphi(t) = 0$ for $t \in [0, a)$. It follows from (7) that $\varphi \in C([0, a), \mathbb{R}_+)$ and that $\varphi(0) = 0$. There exists i, $1 \le i \le m$ such that $\varphi(0) = \varphi^{(i)}(0)$. Then $D^+\varphi(0) \le D^+\varphi^{(i)}(0)$ and $D^+\varphi^{(i)}(0) \le D^+\varphi_0^{(i)}(0)$. We prove that $D^+\varphi_0^{(i)}(0) = 0$. It follows from (7) that $\varphi_0^{(i)}(0) = 0$. Let $\{t_k\}$ be a sequence such that $0 < t_k < a$, $k = 1, 2, \dots$,

$$\lim_{k \to +\infty} t_k = 0 \quad \text{and} \quad \lim_{k \to +\infty} \frac{\varphi_0^{(i)}(t_k)}{t_k} = D^+\varphi_0^{(i)}(0)\,.$$

Let $\{y^{(k)}\}$ be a sequence such that $y^{(k)} \in S_{t_k}$, $k = 1, 2, \dots$,

$$\varphi_0^{(i)}(t_k) = |u_i(x_0 + t_k, y^{(k)}) - v_i(x_0 + t_k, y^{(k)})|\,, \quad k = 1, 2, \dots$$

and

$$\lim_{k \to +\infty} y^{(k)} = y_0 \in I_0\,.$$

Then we have

$$\frac{\varphi_0^{(i)}(t_k)}{t_k} = \frac{1}{t_k} \left| u_i(x_0 + t_k, y^{(k)}) - v_i(x_0 + t_k, y^{(k)}) \right| =$$
$$= \left| D_x u_i(x_k, y^{(k)}) - D_x v_i(x_k, y^{(k)}) \right|$$

where $x_k \in (x_0, x_0 + t_k)$. Since $D_x u_i$ and $D_x v_i$ are continuous on I_0 and $\lim_{k\to+\infty} x_k = x_0$ it follows that

$$\begin{aligned} D^+\varphi_0^{(i)}(0) &= |D_x u_i(x_0, y_0) - D_x v_i(x_0, y_0)| = \\ &= \big| f_i\left(x_0, y_0, \alpha(x_0, y_0), u(\cdot), D_y\alpha_i(x_0, y_0)\right) - \\ &\quad - f_i\big(x_0, y_0, \alpha(x_0, y_0), v(\cdot), D_y\alpha_i(x_0, y_0)\big)\big| = 0. \end{aligned}$$

Then we have $D^+\alpha(0) = 0$.

Suppose that $t^* \in (0, a)$, $\varphi(t^*) > 0$ and there exists $\varepsilon > 0$ such that for each $\tau \in (t^* - \varepsilon, t^*)$ we have $\varphi(\tau) < \varphi(t^*)$. We prove that $D_-\varphi(t^*) \le \sigma(t^*, \varphi(t^*))$. There exist i_0, $1 \le i_0 \le m$ such that $\varphi(t^*) = \varphi^{(i_0)}(t^*)$. Then

$$D_-\varphi(t^*) \le D_-\varphi^{(i_0)}(t^*)$$

and

$$\varphi^{(i_0)}(\tau) \le \varphi(\tau) < \varphi(t^*) = \varphi^{(i_0)}(t^*) \quad \text{for} \quad \tau \in (t^* - \varepsilon, t^*).$$

It follows from the above estimation and from (7) that $\varphi_0^{(i_0)}(t^*) = \varphi^{(i_0)}(t^*)$. Denote $x^* = x_0 + t^*$ and let us take $y^* \in S_{t^*}$ such that $\varphi_0^{(i_0)}(t^*) = |r_{i_0}(x^*, y^*)|$. Let $(x^*, y^*) \in \operatorname{Int} H$. Then $D_y r_{i_0}(x^*, y^*) = 0$ and

$$D_-\varphi(t^*) \le D_-\varphi^{(i_0)}(t^*) = \liminf_{h\to 0-} \frac{\varphi^{(i_0)}(t^* + h) - \varphi^{(i_0)}(t^*)}{h} \le$$

$$\begin{aligned} \le \liminf_{h\to 0-} \frac{\varphi_0^{(i_0)}(t^* + h) - \varphi_0^{(i_0)}(t^*)}{h} &= D_-\varphi_0^{(i_0)}(t^*) \le \left|D_x r_{i_0}(x^*, y^*)\right| \le \\ &\le \big| f_{i_0}\big(x^*, y^*, u(x^*, y^*), u(\cdot), D_y u_{i_0}(x^*, y^*)\big) - \\ &\quad - f_{i_0}\big(x^*, y^*, v(x^*, y^*), v(\cdot), D_y v_{i_0}(x^*, y^*)\big)\big| \le \end{aligned}$$

$$\begin{aligned} \le \sigma(x - x_0, \|r(x^*, y^*)\|, \|u - v\|_{x^*}) + \sum_{k=1}^{n} M_k \left|D_{y_k} r_{i_0}(x^*, y^*)\right| = \\ = \sigma(t^*, \varphi(t^*), \|u - v\|_{x^*}) = \sigma_0(t^*, \varphi(t^*)) \,. \end{aligned}$$

Let $(x^*, y^*) \in \partial H \backslash H_0$. Then there exists a sequence $\{k_1, \dots, k_n\}$, $k_i \in \{1, \dots, n\}$ for $i = 1, \dots, n$ such that

$$y_{k_j}^* = y_{k_j}^{(0)} - b_{k_j} + M_{k_j}(x^* - x_0) \quad \text{for} \quad j = 1, \dots, s,$$

$$y^*_{k_j} = y^{(0)}_{k_j} + b_{k_j} - M_{k_j}(x^* - x_0) \quad \text{for} \quad j = s+1, \dots, p$$

and

$$\left| y^*_{k_j} - y^{(0)}_{k_j} \right| < b_{k_j} - M_{k_j}(x^* - x_0) \quad \text{for} \quad j = p+1, \dots, n\,,$$

where $s \geq 1$ or $p - s \geq 1$ and $k_j \neq k_i$ for $j \neq i$. For simplicity suppose that $k_j = j$, $j = 1, \dots, n$.

Let $\varphi_0^{(i_0)}(t^*) = r_{i_0}(x^*, y^*)$. Then we have

$$D_{y_j} r_{i_0}(x^*, y^*) \leq 0 \quad \text{for} \quad j = 1, \dots, s\,,$$
$$D_{y_j} r_{i_0}(x^*, y^*) \geq 0 \quad \text{for} \quad j = s+1, \dots, p$$

and

$$D_{y_j} r_{i_0}(x^*, y^*) = 0 \quad \text{for} \quad j = p+1, \dots, n\,.$$

Denote

$$m(t) = \big(x_0 + t,\, y_1^{(0)} - b_1 + M_1 t, \dots, y_s^{(0)} - b_s + M_s t,$$
$$y_{s+1}^{(0)} + b_{s+1} - M_{s+1} t, \dots, y_p^{(0)} = b_p - M_p t,\, y^*_{p+1}, \dots, y^*_n\big)$$

and $\tilde{r}(t) = r_{i_0}(m(t))$ for $t \in [0, t^*]$.

We have $\tilde{r} \in C([0, t^*], \mathbb{R})$, $\tilde{r}(\tau) \leq \varphi_0^{(i_0)}(\tau)$ for $\tau \in [0, t^*]$ and $\tilde{r}(t^*) = \varphi_0^{(i_0)}(t^*)$. Thus

$$\frac{\tilde{r}(\tau) - \tilde{r}(t^*)}{\tau - t^*} \geq \frac{\varphi_0^{(i_0)}(\tau) - \varphi_0^{(i_0)}(t^*)}{\tau - t^*}$$

for $\tau \in [0, t^*]$ and

$$D_- \tilde{r}(t^*) \geq D_- \varphi_0^{(i_0)}(t^*) \geq D_- \varphi^{(i_0)}(t^*) \geq D_- \varphi(t^*)\,.$$

Since

$$D_- \tilde{r}(t) = D_x r_{i_0}(m(t)) + \sum_{j=1}^{s} M_j D_{y_j} r_{i_0}(m(t)) - \sum_{j=s+1}^{p} M_j D_{y_j} r_{i_0}(m(t))$$

it follows that

$$D_- \tilde{r}(t^*) = D_x r_{i_0}(x^*, y^*) + \sum_{j=1}^{s} M_j D_{y_j} r_{i_0}(x^*, y^*) - \sum_{j=s+1}^{p} M_j D_{y_j} r_{i_0}(x^*, y^*) \leq$$

$$\leq \sigma(x^* - x_0, \|r(x^*, y^*)\|, \|u - v\|_{x^*}) + \sum_{j=1}^{n} M_j \left| D_{y_j} r_{i_0}(x^*, y^*) \right| +$$

$$+ \sum_{j=1}^{s} M_j D_{y_j} r_{i_0}(x^*, y^*) - \sum_{j=s+1}^{p} M_j D_{y_j} r_{i_0}(x^*, y^*) = \sigma(x^* - x_0, \varphi(t^*), \varphi(t^*)).$$

Thus $D_-\varphi(t^*) \leq \sigma_0(t^*, \dot{\varphi}(t^*))$.

In a similar way we obtain the above differential inequality in the case when $\varphi_0^{(i_0)}(t^*) = -r_{i_0}(x^*, y^*)$. From Lemma 1 it follows that $\varphi(t) = 0$ for $t \in [0, a)$ and Theorem 1 is proved. □

We have not assumed the continuity of f on $H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n$ in uniqueness Theorem 1. Suppose that f is continuous. We prove that the Kamke's type uniqueness theorem for (4) is equivalent to Perron's type in this case.

Assumptions H_2. Suppose that

1) $\bar{\sigma} \in C([0, a) \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\bar{\sigma}$ is non-decreasing with respect to the last argument and $\bar{\sigma}(t, 0, 0) = 0$ for $t \in [0, a)$;

2) the function $w(t) = 0$ for $t \in [0, a)$ is the unique solution of the problem

$$w'(t) = \bar{\sigma}(t, w(t), w(t)), \; t \in (0, a), \; w(0) = 0. \tag{8}$$

A function $z : H_0 \cup H \to \mathbb{R}^m$ will be called the function of class D if z is continuous on $H_0 \cup H$ and possesses the total differential on S and has first derivatives with regard to (x, y) in Int H.

We start with the following theorem of Perron's type.

Theorem 2. (see [3]). *Suppose that*

i) $f : H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^m$, *$f$ satisfies the Volterra condition* (5).

ii) The function $\bar{\sigma}$ satisfies Assumption H_2 and the estimation

$$\|f(x, y, p, u(\cdot), q) - f(x, y, \bar{p}, v(\cdot), \bar{q})\| \leq \\ \leq \bar{\sigma}(x - x_0, \|p - \bar{p}\|, \|u - v\|_x) + \sum_{k=1}^{n} M_k |q_k - \bar{q}_k| \tag{9}$$

holds on $H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n$.

Under these assumptions the solution z of class D on $H_0 \cup H$ of the problem (4) *is unique.*

It is easy to see that the Perron's type comparison function satisfies the condition 3) of Assumption H_1. Suppose that f satisfies the inequality (6) and σ fulfils the condition 2) and 3) of Assumption H_1.
We prove that for continuous function f there exists a function $\bar{\sigma}$ such that the estimation (9) holds and $\bar{\sigma}$ satisfies Assumption H_2.

The following theorem is generalization of the results due to Olech [6] for ordinary differential equations.

Theorem 3. *If assumption* H_1 *is satisfied, the function* f *is continuous on* $H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n$ *and the function* σ *is non-decreasing with respect to last argument then there exists the function* $\bar{\sigma}$ *such that the assumption* $ii)$ *of Theorem 2 is satisfied.*

Proof. We define for $t \in [0, a)$, $s, r \in \mathbb{R}_+$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n_+$, $(x_0 + t, y) \in H$

$$h(t, y, s, r, \eta) = \sup\big\{\|f(x_0 + t, y, p, u(\cdot), q) - f(x_0 + t, y, \bar{p}, v(\cdot), \bar{q})\| : \\ \|p - \bar{p}\| = s,\ \|u - v\|_{x_0+t} \le r,\ |q_i - \bar{q}_i| \le \eta_i,\ i = 1, \dots, n\big\}$$

and

$$\bar{\sigma}(t, s, r) = \sup\big\{h(t, y, s, r, \theta) : y \in S_t\big\}$$

where $\theta = (0, \dots, 0) \in \mathbb{R}^n$.
Then $\bar{\sigma} \in C([0, a) \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\bar{\sigma}(t, 0, 0) = 0$ for $t \in [0, a)$, $\bar{\sigma}$ is non-decreasing with respect to the last argument and

$$\|f(x, y, p, u(\cdot), q) - f(x, y, \bar{p}, v(\cdot), \bar{q})\| \le \bar{\sigma}(x - x_0, \|p - \bar{p}\|, \|u - v\|_x) + \\ + \sum_{k=1}^{n} M_k |q_k - \bar{q}_k| \quad \text{on} \quad H \times \mathbb{R}^m \times C(H_0 \cup H, \mathbb{R}^m) \times \mathbb{R}^n.$$

We prove that the function $\varphi \equiv 0$ is the unique solution of (8). We have

$$\bar{\sigma}(t, s, r) \le \sup_{y \in S_t} \sup\big\{\sigma(t, \|p - \bar{p}\|, \|u - v\|_{x_0+t}) : \\ \|p - \bar{p}\| = s,\ \|u - v\|_{x_0+t}\big\} = \sigma(t, s, r).$$

If φ is a solution of (8) then $\varphi'(t) = \bar{\sigma}(t, \varphi(t), \varphi(t)) \le \sigma(t, \varphi(t), \varphi(t))$, $t \in (0, a)$.
Since $\varphi(0) = 0$ and $D^+\varphi(0) = 0$ it follows that φ satisfies all the assumptions of the third comparison theorem in [8] and thus $\varphi(t) = 0$ for $t \in [0, a)$. This completes the proof of Theorem 3. □

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