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FIXED POINT RESULTS ON ABSTRACT ORDERED SETS

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The basic Zermelo-Bourbaki fixed point principle is being enlarged from a technical viewpoint. Some aspects involving Manka's fixed point result are also discussed.

0. Introduction.

Let E be a nonempty set and $\leq$ an *ordering* (i.e., a reflexive, anti-symmetric and transitive relation) over E. Let also $T : E \to E$ be a mapping. A point $x \in E$ will be said to be *fixed* under T when $x = Tx$; the set of all such points will be denoted fix (T). Now, a basic problem involving these elements is to indicate sufficient conditions under which fix(T) be nonempty and - eventually - cofinal in $(E, \leq)$. The prototype of all these is the 1949 one due to Bourbaki [3] which, in turn, refines the classical 1904 Zermelo's construction [9]. It is our main aim in the present exposition to enlarge this result. (Note that, any such device is technical in nature; because the statements in question are ultimately equivalent with the Axiom of Choice).

The basic strategies of attacking these problems will be discussed in Section 2. Their results are shown to refine some contributions in this area due to Pasini

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[5], Abian [1], Pelczar [6] and others. All preliminary material was collected in Section 1.

Further extensions of the obtained results are given in Section 3; these involve the Abian-Brown coincidence theorem [2], from the methodological perspective in Manka [4]. Some other aspects will be discussed elsewhere.

1. Preliminaries.

Let $(E, \leq)$ be a partially ordered set. For each part X of E, let $\mathrm{ubd}(X)$ stand for the set of all *upper bounds* of X; and $\sup(X)$, the *least upper bound* of X. The subset X of E will be said to be *well ordered* when each (nonempty) part of it admits a first element. Of course, X must be *totally ordered* in this case. (That is, for each $x, y \in X$, either $x \leq y$ or $y \leq x$; this will be also referred to as X being a *chain*).

Letting the selfmap T of E, denote

$$\mathrm{in}(T) = \{x \in E : x \leq Tx\}.$$

The following "conditional" fixed point result established in 1949 by Bourbaki [3] will be in effect for us.

Theorem 1. *Assume that*

C_1) *T is progressive (in $(T) = X$),*

and let $a \in E$ be arbitrary fixed. Then, a well ordered part $B = B(a)$ of E may be found so as

(1.1) *$a \in B$, $T(B) \subseteq B$ and $\sup(X) \in B$ whenever X is a (nonempty) part of B which admits a supremum (in E)*

(1.2) *$u, v \in B \implies u \geq v$ or $Tu \leq v$ (that is, $x \mapsto Tx$ is the immediate successor mapping of B); hence, in particular, T is increasing over B*

(1.3) *if $b = \sup(B)$ exists (in E) then $b \in B \cap \mathrm{fix}(T)$; conversely, if $b \in B$ is fixed under T then $b = \sup(B)$ (or, equivalently, b is the last element in B).*

Actually, B is the intersection of all (nonempty) parts Y of E fulfilling (1.1) above (with Y in place of B); see the quoted paper for details. The argument - refining the 1904 one due to Zermelo [9] - does not use the Axiom of Choice.

We are now in position to state a first answer to the question in the introductory part. Namely, we have (cf. Abian and Brown [2]):

Corollary 1. *Let the mapping* $T : E \to E$ *be as in* C_1). *And, in addition, assume*

C_2) *each well ordered part of* $(E, \leq)$ *has a supremum (in* E*).*

Then, $\text{fix}(T)$ *is nonempty and cofinal in* $(E, \leq)$ *(for each* $x \in E$ *there exists* $y \in \text{fix}(T)$ *with* $x \leq y$*).*

The proof consists in applying Theorem 1 to the subset B above. Of course, by the invariance property (1.1), condition C_2) may be sharpened to

$C_{2'}$) *each* T*-invariant well ordered part of* $(E, \leq)$ *has a supremum (in* E*).*

On the other hand, C_2) is trivially fulfilled when "well ordered" is replaced by "totally ordered". Note that such a fixed point result does not use the Axiom of Choice.

Now, call the point $z \in E$, *maximal* when $z \leq w$ implies $z = w$. The set of all these elements will be denoted $\max(E, \leq)$. As a completion of the answer above, one has (cf. Abian [1]):

Corollary 2. *Let the partial order structure* $(E, \leq)$ *be such that*

C_3) *each well ordered part of* $(E, \leq)$ *is bounded from above.*

Then, for each progressive mapping $T : E \to E$, $\text{fix}(T)$ *is nonempty and cofinal in* $(E, \leq)$.

The argument runs essentially as follows. Apply the Axiom of Choice to establish, via C_3), that $\max(E, \leq)$ is nonempty and cofinal in $(E, \leq)$. This, combined with $\max(E, \leq) \subseteq \text{fix}(T)$ whenever T is progressive, gives the desired conclusion. (We refer to the quoted paper by Bourbaki for details). Of course, a sufficient condition for C_3) is, as above said, its variant with "well ordered" substituted by "totally ordered".

The basic assumption used in these corollaries was C_1). So, it would be interesting to determine what happens with the conclusion above in case C_1) is to be relaxed to

$C_{1'}$) T *is quasi-progressive (*$\text{in}(T) \neq \emptyset$*).*

In this direction, one has (cf. Turinici [8]):

Corollary 3. *Let the increasing selfmap* T *of* E *be quasi-progressive and the ambient order structure* $(E, \leq)$ *satisfy* C_2). *Then,* $\text{fix}(T)$ *is nonempty and cofinal in* $(G, \leq)$, *where* $G = \text{in}(T)$.

The proof is immediate by noting that conditions of Corollary 1 are fulfilled over $(G, \leq)$. It is to be underlined here that the Axiom of Choice was not effectively used.

2. Main results.

Let $(E, \leq)$ be a partially ordered set and $T : E \to E$, a progressive mapping. It was proved in Corollary 2 above that, whenever $C_3)$ is to be accepted, then $\mathrm{fix}(T)$ is nonempty and cofinal in $(E, \leq)$. In particular, a sufficient condition for $C_3)$ is $C_2)$. This, combined with the remark following Corollary 1, leads us to the following question: to what extent is the above conclusion true, in case we restrict our considerations to T-invariant (well ordered) parts? It is our aim in what follows to give a positive answer to this. Precisely, we have:

Theorem 2. *Let the partially ordered structure $(E, \leq)$ and the progressive map $T : E \to E$ be such that*

$C_{3'})$ *each T-invariant well ordered part of $(E, \leq)$ is bounded from above.*

Then, $\mathrm{fix}(T)$ *is nonempty and cofinal in $(E, \leq)$.*

Proof. Letting a be arbitrary fixed in E, denote by $\mathscr{B}(a)$ the class of all T-invariant well ordered parts of E, containing a. That $\mathscr{B}(a)$ is not empty, follows immediately from

$$A = \{T^n a;\ n = 0, 1, \ldots\} \quad \text{is an element of } \mathscr{B}(a). \tag{2.1}$$

Let us introduce an ordering $\leq$ over $\mathscr{B}(a)$, by the convention

$$X \leq Y \quad \text{iff} \quad X \quad \text{is a segment of} \quad Y.$$

The partial order structure $(\mathscr{B}(a), \leq)$ fulfils evidently $C_2)$; hence a fortiori, $C_3)$. (For, if $\mathscr{W}$ is a well ordered part of $\mathscr{B}(a)$, the set $Z = \cup\mathscr{W}$ is easily shown to be its supremum in $\mathscr{B}(a)$). So, by the Zorn maximality principle, we have that, for the given (by (2.1)) element A in $\mathscr{B}(a)$, there must be a maximal (modulo $\leq$) element B of $\mathscr{B}(a)$ with $A \leq B$. In view of the accepted hypothesis, B admits upper bounds; let z be one of these. We claim $z \in B$ (i.e., z is the last element of B). Suppose not; then, putting

$$B^* = B \cup \{T^n z;\ n = 0, 1, \ldots\},$$

it is clear that B^* is an element of $\mathscr{B}(a)$ with $B \leq B^*$, $B \neq B^*$. But this contradicts the maximality of B in $(\mathscr{B}(a), \leq)$. Hence the claim follows; and then, evidently, $x \leq z$ and $z \in \mathrm{fix}(T)$. The proof is complete. □

Remark. An alternative proof to this may be given under the lines below. Define a new ordering $\preceq$ on E by the convention

$$x \preceq y \quad \text{iff either} \quad x = y \quad \text{or} \quad T^p x \leq T^q y, \quad \text{for all} \quad p, q \in \mathbb{N}.$$

Let W be a well ordered (modulo $\preceq$) part of E, and put

$$Z = \cup\{T^n(W);\ n = 0, 1, \ldots\}(= W \cup T(W) \cup T^2(W) \cup \ldots).$$

Clearly, Z is T-invariant; we claim Z is well ordered (modulo $\preceq$) in E. Indeed, let X be any subset of Z. Each x in X is of the form $T^k y$, for some $k \in \mathbb{N}$, $y \in W$. Let Y be the subset (in W) of all such y, encountered in the representation of all $x \in X$. By the choice of W, Y admits a first element (modulo $\preceq$) say z. Now, by the very definition of Y, there exists $k \in \mathbb{N}$ with $x = T^k z$ belonging to X. We claim x is the first element (modulo $\preceq$) of X. In fact,

$$z \preceq y,\ y \in Y \Longrightarrow T^p z \leq T^q y,\ p, q \in \mathbb{N}.$$

This, in turn, gives

$$T^k z \preceq u, \quad \text{for all} \quad u \in X;$$

hence the claim. Now, by the admitted hypothesis, Z is bounded above (modulo $\preceq$); hence, by invariance, W is necessarily bounded above (modulo $\preceq$) in E. This, combined with the Zorn maximality principle shows $\max(E, \preceq)$ is non-empty and cofinal in $(E, \preceq)$. That is, for each $x \in E$ there exists $z \in E$ with

a) $x \preceq z$ (hence $x \leq z$)
b) $z \preceq y$ implies $z = y$.

Now, by the progressiveness assumption, $z \leq Tz \leq \ldots$; and, as $A = \{T^n z;\ n = 0, 1, \ldots\}$ is a T-invariant well ordered part of E, it admits, by hypothesis, an upper bound, y say. This shows that $z \preceq y$; and then, by b) above, $z = y$ (hence $z = Tz$). The proof is complete. □

We must underline here the basic role of C_1) in proving this statement. So, a natural question is to what extent can we retain (essentially) the conclusion above in case C_1) is to be substituted by its weaker counterpart $C_{1'}$). This requires a new convention. Precisely, call the selfmap T of E, *almost-increasing*, when

$$x, y \in E,\ x \leq Tx \leq T^2 x \leq \ldots \leq y \Longrightarrow x \leq Ty. \tag{2.2}$$

The motivation of this term is clear. Indeed, note that (2.2) may be also written as

$$\begin{cases} \text{whenever} \quad \{T^n x;\ n \in \mathbb{N}\} \quad \text{is ascending,} \\ \text{then} \quad \text{ubd}\{T^n x;\ n \in \mathbb{N}\} \quad \text{is } T\text{-invariant.} \end{cases} \tag{2.2'}$$

In other words each *increasing* self-map is necessarily almost-increasing. We are now in position to formulate an appropriate answer to the question above.

Theorem 3. *Let the quasi-progressive almost-increasing map $T : E \to$ be such that*

C$_4$) $G = \text{in}(T)$ *is* Y*-invariant*

C$_5$) $\{x, Tx\}$ *has an infimum, for all* $x \in E$

C$_6$) *each* T*-invariant well ordered part* X *of* $(E, \leq)$ *has an upper bound, minimal in* $(\text{ubd}(X), \leq)$.

Then,

(2.3) $H = \text{fix}(T)$ *is nonempty and cofinal in* $(G, \leq)$

(2.4) $\max(H, \leq)$ *is nonempty and cofinal in* $(H, \leq)$.

Hence (combining these facts)

(2.5) $\max(H, \leq)$ *is nonempty and cofinal in* $(G, \leq)$.

Proof. Let X be any T-invariant well ordered part of G. By C$_6$), it has an upper bound, z say, which is minimal in $(\text{ubd}(X), \leq)$. For each x in X we therefore have

$$x \leq Tx \leq T^2x \leq \ldots \leq z \quad (\text{because } X \subseteq G).$$

So, by the almost-increasing hypothesis C$_5$),

$$x \leq Tx \leq T^2x \leq \ldots \leq Tz.$$

In other words, Tz is an element of $\text{ubd}(X)$. But, in this case, $w = \inf(z, Tz)$ also belongs to $\text{ubd}(X)$; and moreover, $w \leq z$. This, combined with the minimal character of z, gives $w = z$; and then, z is necessarily in G. Summing up, each T-invariant well ordered part of G is bounded above in G. This, coupled with C$_4$), shows Theorem 2 applies (with G in place of E); and so, conclusion (2.3) follows. For the second part, let Y be any well ordered part of $(H, \leq)$. This subset is, by definition, T-invariant. As a consequence, the above reasoning may be reproduced to derive Y is bounded above, in G. On the other hand, by the conclusion (2.3), H is cofinal in $(G, \leq)$; hence Y is bounded above in H. Summing up, each well ordered part of H is bounded from above in H. This proves the desired conclusion, if we take into account the Zorn maximality principle. The last part is trivial. Hence the result. □

An immediate consequence of this statement is the following. Call the selfmap T of E, *half-increasing*, when

$$x \leq Tx \leq y \Longrightarrow Tx \leq Ty. \tag{2.6}$$

Clearly, "half-increasing" is more restrictive than "almost-increasing". In addition to this, any selfmap with such a property is necessarily endowed with the property C$_4$), as it can be directly seen.

Corollary 4. *Let the quasi-progressive half-increasing map $T : E \to E$ be as in* $C_5) + C_6)$. *Then, conclusion of Theorem 3 is retainable.*

In particular, when $C_6)$ is to be replaced by

$C_{6'})$ *each well ordered part X of $(E, \leq)$ has an upper bound, minimal in* $(\text{ubd}(X), \leq)$,

the statement above reduces to the one in Pasini [5], proved with the aid of a *strong transfinite induction*. So, the recourse to such a principle is not necessary in these developments.

Let us now return to Theorem 3. It is clear that, a sufficient condition for $C_6)$ is condition $C_{2'})$ (in the preceding section). Supposing this substitution were accepted, it is to be expected that $C_5)$ is not effectively necessary so as to conserve (2.3) - (2.5). It is our sam in the following to show this is indeed the case. Precisely, we have:

Theorem 4. *Let the quasi-progressive almost-increasing map $T : E \to E$ be such that* $C_4)+C_{2'})$ *are accepted. Then, conclusions of Theorem 3 are retainable.*

Proof. Let X be an arbitrary T-invariant well ordered part of $(G, \leq)$. By $C_{2'})$, $z = \sup(X)$ exists (in E). We have

$$x \leq Tx \leq T^2x \leq \ldots \leq z, \quad \text{for all } x \in X,$$

because X is T-invariant. This, in combination with the almost-increasing property, shows $Tz \in \text{ubd}(X)$; and so, by the definition of the supremum, $z \in G$. Hence, each T-invariant well ordered part of G admits a supremum in G. This, coupled with $C_4)$, shows, via Corollary 1, that conclusion (2.3) is true. Further, let Y be any well ordered part of $(H, \leq)$. This subset is, by definition, T-invariant. So, the above reasoning is also working in this context to get Y has a supremum in G; and this, combined with the obtained conclusion, shows Y is bounded above in H. Hence, each well ordered part of H is bounded from above in H. This, plus the Zorn maximality principle proves (2.4), and completes the argument. □

As already said, a sufficient condition for the almost-increasing property of the underlying map (which, in addition, fulfils $C_4)$) is the half-increasing one. We therefore have:

Corollary 5. *Let the quasi-progressive half-increasing map $T : E \to E$ be as in* $C_{2'})$. *Then, conclusion of Theorem 3 is holding.*

In particular, when $C_{2'}$) is substituted by C_2), this statement reduces to the one in Pasini [5] proved again by a strong transfinite induction. On the other hand, when "half-increasing" is replaced by "increasing", this result reduces to Corollary 3. Other aspects of the problem can be found in the paper by Manka [4].

3. Some extensions.

Let again $(E, \leq)$ be a partial order structure. It is our objective in what follows, to put the fixed point results above in a more general "coincidence" framework. Some other aspects will be discussed elsewhere.

Let S, T be two selfmaps of E. A point x in E will be called a *coincidence point* for S and T when $Sx = Tx$. The set of all such points will be denoted $\mathrm{co}(S, T)$. (Note that, when $S = I$ (the identity selfmap of E) then $\mathrm{co}(S, T)$ is just $\mathrm{fix}(T)$). Denote in the following

$$\mathrm{in}(S, T) = \{x \in E;\ Sx \leq Tx\}.$$

We term the couple (S, T), *progressive* in case $\mathrm{in}(S, T) = E$; and *quasi-progressive*, provided $\mathrm{in}(S, T) \neq \emptyset$. Further, call the selfmap S of E, *order-continuous*, in case

$$\text{(3.1)}\quad \begin{cases} S\sup(X)) = \sup(S(X)), & \text{for each well ordered part } X \text{ of } (E, \leq), \\ \text{which admits a supremum (in } E). \end{cases}$$

Under these notations, one has

Theorem 5. *Let the partial order structure $(E, \leq)$ be as in* C_2). *And, let (S, T) be a couple of selfmaps (of E) with*

C_7) *for each $x \in E$ with $Sx < Tx$, there exists $y \in E$ with $x \leq y$ and $Sy = Tx$.*

In this case,

- *a) if, in addition, (S, T) is progressive then, necessarily,* $\mathrm{co}(S, T)$ *is nonempty and cofinal in $(E, \leq)$*
- *b) if S is order continuous, T is increasing and (S, T) is quasi-progressive then,* $\mathrm{co}(S, T)$ *is nonempty and cofinal in* $(G = \mathrm{in}(S, T), \leq)$.

Proof. By C_2) plus the Zorn maximality principle, $\max(E, \leq)$ is nonempty and cofinal in $(E, \leq)$. Let u be such a maximal element. By the progressiveness assumption, $Su \leq Tu$. Assume $Su < Tu$. By C_7), there exists $v \in E$ with

$$u \leq v, \quad Sv = Tu\,. \tag{3.2}$$

The former of these gives, by maximality, $u = v$; and then, $Su = Sv$. This, however, contradicts the latter of these. Hence, $Su = Tu$ (that is, $u \in \mathrm{co}(S, T)$) and this proves a). Passing to b), let X be a well ordered part of $(G, \leq)$. By C_2) again, $z = \sup(X)$ exists, as an element of E. We thus have

$$x \leq z, \quad \text{for all} \quad x \text{ in } X.$$

This, combined with X being a part of G and T increasing, gives directly

$$Sx \leq Tx \leq Tz, \quad x \in X.$$

So, by the order continuity assumption about S, one clearly has $Sz \leq Tz$; that is, $z \in G$. That is, each well ordered part of G has a supremum in G. The Zorn maximality principle tells us $\max(G, \leq)$ is nonempty and cofinal in $(G, \leq)$. We now claim that each point u of $\max(G, \leq)$ is a coincidence point for S and T. Suppose not; i.e., $Su < Tu$. Again by C_7), there exists v in E with the properties (3.2). The former of these gives (in combination with T being increasing) $Tu \leq Tv$; and then, adding the latter, $Sv \leq Tv$; i.e., v belongs to G. This, plus the maximal character of u (in G) gives $u = v$; and then, the contradiction follows as in the part a). Hence, $u \in \mathrm{co}(S, T)$, and the proof is complete. □

In particular, if S is the identity selfmap of E, the result above reduces to Corollary 1 (in the progressive case) and, respectively, Corollary 3 (in the quasi-progressive case). The reciprocal is also valid, in some special cases. For instance, assume that, under the acceptation of C_2), there may be found a selfmapping V of E, with

$$S(Vx) = Tx, \quad \text{for all } x \text{ in } E\,. \tag{3.3}$$

Then, conclusion a) of the above result is holding whenever V is progressive; likewise, conclusion b) of the same is to be retained, with $G = \mathrm{in}(V)$, in case V is quasi-progressive and increasing. This is essentially the result in Abian and Brown [2].

Finally, a simple inspection of the argument developed for the conclusion b) above shows the increasing property for the selfmap T may be removed if, in compensation to this, we require C_2) be valid in $(G, \leq)$; that is

C_{2*}) *each well ordered part of* $(G, \leq)$ *has a supremum in* G

and condition C_7) be replaced by

C_8) *no* $x \in G$ *with* $Sx < Tx$ *can be maximal in* $(G, \leq)$.

This, in the particular case of S being the identity selfmap of E is essentially, the result in Manka [4]. A further enlargement of these facts is that related to the possibility of developing a coincidence point theory for commuting families of selfmaps (of E), under the lines in Smithson [7]. These will be discussed elsewhere.

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