SYSTEMS OF INTEGRABLE DERIVATIONS

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Let $A$ be a commutative $k$-algebra, with $k$ a subring of $A$. We give the definition of $n$-dimensional differentiation of $A$ over $k$ which formally extends the known one of unidimensional differentiation and we study the group of all $n$-dimensional differentiations of $A$ over $k$.

In the second part of the work we give some theorems of strong integrability for systems of derivations in terms of $n$-dimensional differentiation.

Introduction.

Let $A$ be a commutative $k$-algebra and let $\text{Der}_k(A)$ be the module of the $k$-derivations of $A$ in itself.

If $D$ is a derivation of $A$ over $k$ and $\text{char}(k) = 0$, it is known that $D$ is always strongly integrable over $k$, i.e. there exists an iterative unidimensional differentiation of $A$ over $k$ which lifts $D$ ([4]).

If $\text{char}(k) = p > 0$ and $A$ is a separable field over $k$, $D^p = 0$ is a necessary and sufficient condition for the strong integrability ([4]), while if $A$ is a ring separable over $k$, conditions for the integrability of $D$ such that $D^p = 0$ are given in [1].

Now, considered a system of derivations $D = \{D_1, \ldots, D_n\}$ in the sense of [7], [1], non necessary independent, we say that $D$ is integrable (strongly

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integrable) if there exists a n-dimensional (iterative) differentiation of A over k which lifts D.

If char(k) = 0, every commutative system of derivations is strongly integrable, while if char(k) = p > 0, systems of strongly integrable derivations are considered in [7].

In this work we study the lifting problem for a system of derivations to a differentiation of dimension n, where n is the number of independent derivations of the system of derivations.

Precisely in n. 1 we define a n-dimensional differentiation and we study its main properties. After having defined a system of integrable and strongly integrable derivations we study the group $HS^n_k(A)$ of the n-dimensional differentiations of A over k.

In n. 2 we show that by the existence of a n-dimensional differentiation in the ring we can deduce theoretic properties of the ring (existence of an A-sequence formed by n elements, analytic independence of the same n elements, etc.).

We obtain theorems of structure for I-adically complete rings, I ideal of the ring, module the strong integrability of a system of independent derivations of the ring expressed by the existence of a differentiation of dimension n.

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1.- In this work, all rings are assumed to be commutative, noetherian, with a unit element.

**Definition 1.** Let k be a ring and A be a k-algebra. A n-dimensional differentiation $D$ (an infinite n-higher derivation in A) of A is a set of linear maps $\{D_\alpha : A \to A; \alpha \in N^n\}$ such that $D_0 = \text{id}_A$ and

\[
D_\gamma(a \cdot b) = \sum_{\alpha + \beta = \gamma} D_\alpha(a) \cdot D_\beta(b)
\]

for $a, b \in A$ and $\alpha, \beta, \gamma \in N^n$.

Such a definition is a formal extension of the definition of 1-dimensional differentiation $D$ of A (in the sense of Hasse-Schmidt). The set of n-dimensional differentiations of A is denoted by $HS^n(A)$.

As for the 1-dimensional differentiations we can associate to a n-dimensionsal differentiation, a ring homomorphism $E$ from A into $A[[X_1, X_2, \ldots, X_n]]$ such that $E(a) \equiv a \mod (X_1, X_2, \ldots, X_n)$ and precisely

\[
E(a) = \sum_\alpha D_\alpha(a) X^\alpha
\]
with \( \alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \), \( X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n} \).

Then \( E \) is a ring homomorphism of \( A \) into \( A[[X_1, X_2, \ldots, X_n]] \), and conversely any such homomorphism \( E \) comes from a \( n \)-dimensional differentiation of \( A \). Sometimes we identify \( E \) with \( D \).

Let \( k \) be a ring and \( A \) be a \( k \)-algebra. Given a \( k \)-algebra homomorphism:

\[
\Phi_1 : A \to A[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2
\]

such that

\[
(2) \quad \Phi_1(a) \equiv a \mod (X_1, \ldots, X_n)
\]

we can associate \( n \) derivations of \( A \): \( D_1, \ldots, D_n \) and condition (2) can be expressed in this way:

\[
\Phi_1(a) = a + D_1(a)X_1 + \ldots + D_n(a)X_n.
\]

**Definition 2.** Let \( k \) a subring of \( A \). A \( n \)-dimensional differentiation \( D \) of \( A \) is called a \( n \)-dimensional differentiation (or \( n \)-differentiation) of \( A \) over \( k \) if \( D_{\alpha}(a) = 0 \) for all \( \alpha \neq 0 \) and for all \( \alpha \in k \).

The set of such \( n \)-differentiations is denoted by \( HS^n_k(A) \).

**Notation.** If \( \gamma = (\gamma_1, \ldots, \gamma_n) \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) then the symbol \( \binom{\gamma}{\alpha} \)

denotes the product of the binomial coefficients:

\[
\binom{\gamma}{\alpha} = \binom{\gamma_1}{\alpha_1} \ldots \binom{\gamma_n}{\alpha_n}.
\]

**Definition 3.** A \( n \)-differentiation \( D \) is said to be iterative if:

\[
D_\alpha \circ D_\beta = \binom{\alpha + \beta}{\alpha} D_{\alpha + \beta} \quad \text{for all } \alpha, \beta \in \mathbb{N}^n.
\]

**Remark 1.** We note that, in characteristic 0, for any commutative set of \( n \) derivations \( D_1, \ldots, D_n \), if we put \( D_{(h_1, \ldots, h_n)} = \frac{1}{h_1! \ldots h_n!} D_1^{h_1} \ldots D_n^{h_n} \), we get a \( n \)-dimensional differentiation and we have \( D_\alpha \circ D_\beta = \binom{\alpha + \beta}{\alpha} D_{\alpha + \beta} \).

(In substance, the Definition 3 is a generalization of that one given in [4]).
Definition 3'. We say that a $n$-differentiation $D$ is iterative if the following diagram:

\[
\begin{array}{c}
A \\ E_X \\ E_{X+Y} \\
\downarrow \\
A[[X]] \\ E_Y \\
\downarrow \\
A[[X+Y]] \\ i \\
\downarrow \\
A[[X, Y]]
\end{array}
\]

is commutative, where $X = (X_1, X_2, \ldots, X_n)$, $Y = (Y_1, Y_2, \ldots, Y_n)$, $i$ is the inclusion map and $E_Y(X) = X$.

Definitions 3 and 3' are equivalent because the usual check in the classic case holds this case too; more generally, the equivalence follows by the relation

\[
(X + Y)^\alpha = \sum \binom{\alpha}{\beta} X^\beta Y^{\alpha - \beta},
\]

even if $X, Y, \alpha, \beta$ are $n$-tuples.

Definition 4. Given $n$ independent derivations $D_1, \ldots, D_n \in \text{Der}(A)$ we say that they form a $n$-integrable set if there exists a ring homomorphism $E : A \to A[[X_1, X_2, \ldots, X_n]]$ such that

\[
E(a) = a + D_1(a)X_1 + \ldots + D_n(a)X_n + \ldots
\]

Such an homomorphism $E$ is called $n$-dimensional integral of $D_1, \ldots, D_n$.

Definition 5. Given $n$ independent derivations $D_1, \ldots, D_n \in \text{Der}(A)$ we say that they form a $n$-integrable set over $k$ if there exists a differentiation $D = \{D_\alpha : A \to A; \alpha \in N^n\} \in HS_k^n(A)$ with $D_{(0,\ldots,1,\ldots,0)} = D_i$.

Definition 6. We say that differentiations $D = \{D_\alpha : A \to A; \alpha \in N^n\}$ and $D' = \{D'_\beta : A \to A; \beta \in N^n\}$ commute if $D_\alpha$ and $D'_\beta$ commute for every pair $(\alpha, \beta)$.

Proposition 1. Let $k$ be a ring. If $A$ is a smooth $k$-algebra then every set of $n$ independent derivations of $A$ can be lifted to a $n$-differentiation.
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Proof. If \( D_1, \ldots, D_n \) is a set of independent derivations of \( A \) over \( k \) there exists a \( k \)-algebra homomorphism:

\[
\Phi_1 : A \rightarrow A[X]/(X)^2
\]

(where \( X = (X_1, X_2, \ldots, X_n) \)), such that \( \Phi_1(a) \equiv a \mod (X) \).

Since \( A \) is a smooth \( k \)-algebra, we have the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\Phi_1} & A[X]/(X)^2 = A[X]/(X)^2/(X)^3 \\
& \phi_2 \swarrow & \searrow & \phi_1 \\
k & & & A[X]/(X)^3
\end{array}
\]

For the formal smoothness there exists \( \Phi_2 : A \rightarrow A[X]/(X)^3 \) and so, for \( \Phi_n : A \rightarrow A[X]/(X)^{n+1} \) such that \( \Phi_{n-1}(a) \equiv \Phi_n(a) \mod (X)^n \); for all \( n \).

Since \( A[[X]] = \varprojlim A[X]/(X)^n \), \( \Phi_1 \) can be lifted to a \( k \)-algebra homomorphism \( E : A \rightarrow A[[X]] \) such that

\[
E(a) = a + \sum_{i=1}^{n} D_i(a) X_i + \ldots = \sum_{\alpha} D_\alpha(a) X^\alpha,
\]

and so the proof is complete. \( \square \)

Remark 2. In characteristic 0, every set commutative of \( n \) derivations \( D_1, D_2, \ldots, D_n \) is \( n \)-integrable and its integral is:

\[
E(a) = \sum_{\alpha} \frac{1}{\alpha_1! \ldots \alpha_n!} D_1^{\alpha_1} \ldots D_n^{\alpha_n}(a) X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n},
\]

with \( \alpha = (\alpha_1, \ldots, \alpha_n) \in N^n \), by Remark 1.

Remark 3. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in N^n \), then \( |\alpha| \) stands for \( \alpha_1 + \ldots + \alpha_n \).

Let \( E : A \rightarrow A[[X_1, \ldots, X_n]] \) be a \( k \)-algebra morphism so defined:

\[
E(a) = \sum_{\alpha} D_\alpha(a) X^\alpha,
\]

with \( D_0 = \text{id}_A \), then \( D_\alpha : A \rightarrow A \) is a \( k \)-derivations \( \forall \alpha \in N^n \) with \( |\alpha| = 1 \).
Remark 4. If $D = \{ D_\alpha : A \to A; \alpha \in N^n \}$ is a differentiation and $E : A \to A[[X_1, \ldots, X_n]] = A[X]$ is the corresponding ring homomorphism $(E(a) = \sum_{\alpha} D_\alpha(a) X^\alpha)$ then $E$ can be extended to a map $A[[X]] \to A[[X]]$, by:

$$E \left( \sum_{\alpha} a_\alpha X^\alpha \right) = \sum_{\alpha} E(a_\alpha) X^\alpha.$$  

(3)

It is easy to see that this is an endomorphism of the ring $A[[X]]$.

Any element $\xi$ of $A[[X]] = A[[X_1, \ldots, X_n]]$ can be written

$$\xi = \xi_r + \xi_{r+1} + \ldots,$$

where $\xi_i$ is a homogeneous polynomial of degree $i$ in $X_1, \ldots, X_n$ with coefficients in $A$. If $\xi_r \neq 0$ then $\xi_r$ is called the initial form and is denoted by $\text{In}(\xi)$. Its degree $r$ is called the initial degree and is denoted by $v(\xi)(\in N)$.

Then, for $\xi \in A[[X]]$ we have

$$\text{In}(E(\xi)) = \text{In}(\xi)$$

or equivalently

$$v(E(\xi) - \xi) > v(\xi).$$

(4)

(5)

Formula (4) shows that $E(\xi) \neq 0$ if $\xi \neq 0$, i.e. that $E$ is injective. We claim that $E$ is also surjective (hence an automorphism of $A[[X]]$). To prove this, take $\eta = \eta_0 + \eta_1 + \ldots \in A[[X]]$, and construct $\xi = \xi_0 + \xi_1 + \xi_2 + \ldots$ (where $\xi_i$ is a form of degree $i$) such that $E(\xi) = \eta$ inductively by

$$\xi_0 = \eta_0,$$

$$\xi_i = \text{the homogeneous part of degree } i \text{ of } \eta - \sum_{j=0}^{i-1} E(\xi_j).$$

Then $v \left( \eta - \sum_{j=0}^{i} E(\xi_j) \right) > i$, and so $E \left( \sum_{i=0}^{\infty} \xi_i \right) = \eta$.

The set $G$ of all such automorphisms of $A[[X]]$ is characterised by formula (4). Therefore it is easy to see that

1) $E, E' \in G \Rightarrow E \cdot E' \in G$
2) \( E \in G \Rightarrow E^{-1} \in G \).

Therefore \( G \) is a subgroup of \( \text{Aut} (A[[X]]) \).

Explicitly, if \( E(a) = \sum_{\alpha} D_\alpha(a)X^\alpha \), \( E'(a) = \sum_{\beta} D'_\beta(a)X^\beta \), \( (EE')(a) = \sum_{\gamma} D''_\gamma(a)X^\gamma \), then

\[
(EE')(a) = E(E'(a)) = \sum_{\beta} X^\beta E(D'_\beta(a)) = \sum_{\alpha, \beta} X^{\alpha+\beta} (D_\alpha(D'_\beta(a)))
\]

hence

\[
D''_\gamma = \sum_{\alpha + \beta = \gamma} D_\alpha D'_\beta = \sum_{\alpha \leq \gamma} D_\alpha D'_{\gamma - \alpha}
\]

(where if \( \alpha, \beta \in N^n, \alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \), then we write \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \forall i \), and \( \alpha < \beta \) if \( \alpha \leq \beta \) and \( \alpha \neq \beta \).

Let us write \( D_i \) for \( D_{(0, \ldots, 1, \ldots, 0)} \). Then we have \( D''_i = D_i + D'_i \).

The explicit form of \( E^{-1} \) can be obtained as follows. Set \( E^{-1} = E^*, \) \( E^*(a) = \sum_{\alpha} D^*_\alpha(a)X^\alpha \) \( (a \in A) \). Then \( D^* \)'s are obtained by solving the equations

\[
\sum_{\alpha \leq \gamma} D_\alpha D^*_{\gamma - \alpha} = 0 \quad (\gamma \neq 0).
\]

Hence inductively

\[
D^*_0 = \text{id}, \quad D^*_\gamma = -\sum_{0 < \alpha \leq \gamma} D_\alpha D^*_{\gamma - \alpha}.
\]

For instance

\[
D^*_i = -D_i \quad (i = 1, 2, \ldots, n),
\]

\[
D^*_{(2,0,\ldots,0)} = -D_{(2,0,\ldots,0)} - D_1 D^*_1 = D^2_1 - D_{(2,0,\ldots,0)},
\]

\[
D^*_{(1,1,0,\ldots,0)} = -D_1 D^*_2 - D^2_2 D_1 - D_{(1,1,0,\ldots,0)} = D_1 D_2 + D_2 D_1 - D_{(1,1,0,\ldots,0)}.
\]

**Definition 7.** Let \( \mathcal{D} = \{D_\alpha : A \rightarrow A; \alpha \in N^n\} \) and \( \mathcal{D}' = \{D'_\alpha : A \rightarrow A; \alpha \in N^n\} \) be n-dimensional differentiations. If \( E \) and \( E' \) are their corresponding homomorphisms, we define \( \mathcal{D} \cdot \mathcal{D}' \) as the n-dimensional differentiations corresponding to \( E \circ E' \). As a consequence of Remark 4, we have: \( (D \circ D')_\gamma = \sum_{0 < \alpha \leq \gamma} D_\alpha D'_{\gamma - \alpha} \).

Moreover, the set of all n-dimensional differentiations is a group.
Example. Let $A = k[T_1, \ldots, T_r] = k[[T]]$ and let $\Psi_1(X), \Psi_2(X), \ldots, \Psi_r(X) \in A[[X]] = A[[X_1, \ldots, X_n]]$ be such that $\Psi_i(0) = T_i, (1 \leq i \leq r)$. Then there exists one and only one homomorphism of $k$-algebras $E : A \to A[[X]]$ such that:

$$E(T_i) = \Psi_i = T_i + \sum_{j=1}^{n} f_{ij} X_j + \sum_{j,k=1}^{n} f_{ijk} X_j X_k + \ldots$$

This is the $n$-dimensional differentiation which lifts $D_1, D_2, \ldots, D_n \in \text{Der}_k(A)$, where $D_j = \sum_{i=1}^{r} f_{ij} \frac{\partial}{\partial T_i}$.

This differentiation is not iterative because it doesn’t check the diagram of Definition 3′.

2.- We extend, some results relative to unidimensional differentiations and contained in [7] to the $n$-dimensional differentiations.

From now on, if $E(a) = \sum_{\alpha} D_{\alpha}(a) X^\alpha$ is a $n$-dimensional differentiation, we always denote $D_{(0,\ldots,1,\ldots,0)}$ by $D_i$.

Proposition 2. Let $A$ be a commutative ring and $E : A \to A[[X_1, \ldots, X_n]]$ a $n$-dimensional differentiation. Put $A_0 = \{a \in A \mid E(a) = a\}$. Let $x \in A$ such that

$$E(x) = x + \sum_{i=1}^{n} a_i X_i$$

with some $a_i$ invertible and $\bigcap_{n=1}^{\infty} x^n A = (0)$.

Then, if $z \in A$ and $xz \in A_0$, we have $z = 0$. Consequently $x$ is not a zero-divisor of $A$ and $A_0 \cap x A = (0)$.

Proof.

$$xz = E(xz) = E(x) E(z) = \left( x + \sum_{i=1}^{n} a_i X_i \right) \left( z + \sum_{i=1}^{n} D_i(z) X_i + \ldots \right) =$$

$$= xz + \sum_{i=1}^{n} X_i (a_i z + x D_i(z)) + \sum_{i=1}^{n} X_i^2 \left( a_i D_i(z) + x D_{(0,\ldots,2,\ldots,0)} \right) + \ldots$$
Therefore, we have:

\[ a_i z + x D_i(z) = 0 , \]
\[ a_i D_i(z) + x D_{(0, \ldots, 2, \ldots, 0)}(z) = 0 , \]
\[ \ldots \ldots \ldots \ldots \]
\[ a_i D_{(0, \ldots, n, \ldots, 0)}(z) + x D_{(0, \ldots, n+1, \ldots, 0)}(z) = 0 \]

and so on. It follows that, if \( a_i \) is invertible, \( z \in x^n A \) for all \( n > 0 \). Therefore \( z = 0 \). \( \square \)

**Proposition 3.** Let \( A \) be a noetherian ring and let \( x_1, x_2, \ldots, x_r \) be elements of the Jacobson radical \( \text{Rad}(A) \) of \( A \). We put \( I = A x_1 + \ldots + A x_r \). Suppose either:

(a) \( A \) contains \( Q \), and there exist \( r \) derivations \( D^{(1)}, \ldots, D^{(r)} \in \text{Der}(A) \) such that \( \det(D^{(i)}(x_j)) \in U(A) = \{ \text{units of } A \} \) and \( [D^{(i)}, D^{(j)}] = 0 \); or

(b) there exists a \( r \)-dimensional differentiation \( E : A \to A[[Y_1, \ldots, Y_r]] \) such that

\[ E(x_j) = x_j + \sum_{i=1}^{r} D^{(i)}(x_j) Y_i \]

\( j = 1, \ldots, r, \) and \( \det(D^{(i)}(x_j)) \in U(A). \) Put \( F = \{ a \in A \mid D^{(1)} a = \ldots = D^{(r)} a = 0 \} \) in case (a), and \( F = \{ a \in A \mid E(a) = a \} \) in case (b). Then \( F \) is a subring of \( A \) and \( F \cap I = (0) \).

**Proof.** In case (a), replacing \( D^{(i)} \) by suitable linear combinations we may assume that \( D^{(i)}(x_i) = \delta_{ij} \). Then the iterative differentiation:

\[ E(a) = \sum_{\alpha} \frac{1}{\alpha_1! \ldots \alpha_r!} D^{(1)^{\alpha_1}} D^{(2)^{\alpha_2}} \ldots D^{(r)^{\alpha_r}} (a) Y_1^{\alpha_1} Y_2^{\alpha_2} \ldots Y_r^{\alpha_r} \]

satisfies condition (b). Since the condition (a) implies (b), it suffices to prove the proposition by hypothesis (b). We proceed by induction on \( r \). When \( r = 1 \) the assertion is true by [7] Theorem 2.

We consider the differentiation \( E^* \) constructed by \( D^{(2)}, \ldots, D^{(r)} \). Since \( D^{(2)}(x_1) = \ldots = D^{(r)}(x_1) = 0, D^{(2)}, \ldots, D^{(r)} \) are derivations of the quotient. Thus \( E^* \) passes to the quotient \( \text{mod} x_1 A \). \( E^*(x_1) = x_1 \in x_1 A \) and we have

\[ \overline{E}^*(x_j) = x_j + \overline{D}^{(2)} x_j Y_2^{+} \ldots + \overline{D}^{(r)} x_j Y_r . \]

Now let \( y \in F \cap I, \ y \in F \). It results then \( \overline{E}^*(\overline{y}) = \overline{y} \) and \( \overline{y} \in (\overline{x}_2, \ldots, \overline{x}_r) \). Therefore \( \overline{y} = 0 \) by the induction hypothesis. Thus \( y = x_1 z \) with some \( z \in A \). Since

\[ E(x_1) = x_1 + D^{(1)}(x_1) Y_1 + \ldots + D^{(r)}(x_1) Y_r \]

and since \( x_1 \in \text{Rad}(A) \), we can apply Proposition 2 to conclude \( z = 0, y = 0 \).

\[ \square \]

**Proposition 4.** Let \( A \) be a noetherian ring, let \( x_1, x_2, \ldots, x_n \in \text{Rad}(A) \) and let \( E : A \to A[[X_1, \ldots, X_n]] \) be a \( n \)-dimensional differentiation such that \( \det(D_i(x_j)) \in U(A) \), where \( D_i \) \( i = 1, \ldots, n \) are the derivations associated to \( E \). Then \((x_1, x_2, \ldots, x_n)\) is a regular sequence.

**Proof.** Replacing \( D_i \) with \( \sum c_{ij} D_j \), where \( c_{ij} \) are elements of the inverse matrix of \( (D_i(x_j)) \), we may assume that \( D_j(x_j) = \delta_{ij} \). In particular we have \( D_1(x_1) = 1, D_2(x_1) = \ldots = D_n(x_1) = 0 \). We consider the homomorphism \( E : A \to A[[X_1, \ldots, X_n]] \) such that

\[
E(x_j) = x_j + \sum_{i=1}^{n} D_i(x_j)X_i;
\]

it results \( E(x_1) = x_1 + X_1 \), then by Proposition 2, \( x_1 \) is regular. In the ring \( A/(x_1) \) we can consider the derivations of the quotient \( \overline{D}_2, \ldots, \overline{D}_n \) and the differentiation

\[
\overline{E}(\overline{x}_j) = \overline{x}_j + \overline{D}_2\overline{x}_jX_2 + \ldots + \overline{D}_n\overline{x}_jX_n.
\]

Since \( \overline{D}_2\overline{x}_2 = 1 \), by the same reasoning, we can find that \( \overline{x}_2 \) is regular in \( A/(x_1) \) and \( \overline{D}_3\overline{x}_2 = \ldots = \overline{D}_n\overline{x}_2 = 0 \). Proceeding by induction, we deduce that \( \forall i, x_i \) is regular in \( A/(x_1, \ldots, x_{i-1}) \), i.e. \( (x_1, \ldots, x_n) \) is an \( A \)-regular sequence. \( \square \)

**Proposition 5.** Given \( n \) 1-dimensional differentiations

\[
E_i : A \to A[[X_i]], \quad i = 1, \ldots, n,
\]

there exists a \( n \)-dimensional differentiation

\[
E : A \to A[[X_1, \ldots, X_n]]
\]

such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{E} & A[[X_1, \ldots, X_n]] \\
\downarrow{E_i} & & \downarrow{\pi_i} \\
A[[X_i]] & & 
\end{array}
\]

(where \( \pi_i \) is the projection of \( A[[X_1, \ldots, X_n]] \) into the residue class ring

\[
A[[X_1, \ldots, X_n]]/(X_1, \ldots, \widehat{X}_1, \ldots, X_n) \cong A[[X_i]]
\]

is commutative for all \( i, i = 1, \ldots, n \).

Moreover, if the homomorphisms \( E_i \) are iterative and commute then \( E \) is iterative too.
Proof. Using the \( n \) unidimensional differentiations \( E_i \) defined by \( E_i (a) = \sum_{n=0}^{\infty} X_i^n D_n (a) \), we set \( E (a) = \sum_{\alpha} D_{\alpha} (a) X^{\alpha} \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), where \( D_{\alpha} = D_{\alpha_1}^{(1)} \circ D_{\alpha_2}^{(2)} \circ \cdots \circ D_{\alpha_n}^{(n)} \). Obviously, \( E \equiv E_i \mod (X_1, \ldots, \hat{X}_i, \ldots, X_n) \). Let us prove that \( E \) is an \( n \)-dimensional differentiation. The only non-trivial fact is the equality \( E(ab) = E(a)E(b) \). Let us use induction on \( n \). For \( n = 1 \), there is nothing to prove; so, let us suppose the equality true for \( n - 1 \). This means that \( E^* (a) = \sum D_{\alpha_2}^{(2)} \cdots D_{\alpha_n}^{(n)} (a) X^{(\alpha_2, \ldots, \alpha_n)} \) satisfies \( E^*(ab) = E^*(a)E^*(b) \).

But, put \( X_i^{\alpha_i} = X^{\alpha_i} \),

\[
E(a) = \sum_{\alpha_1} D'_{\alpha_1} \left( \sum_{(\alpha_2, \ldots, \alpha_n)} D_{\alpha_2}^{(2)} \cdots D_{\alpha_n}^{(n)} (a) X^{\alpha_2} \cdots X^{\alpha_n} \right) X^{\alpha_1} = \sum_{\alpha_1} D'_{\alpha_1} (E^*(a)) X^{\alpha_1}
\]

so that

\[
E(ab) = \sum_{\alpha_1} D'_{\alpha_1} (E^*(ab)) X_1^{\alpha_1} = \sum_{\alpha_1} D'_{\alpha_1} (E^*(a)E^*(b)) X_1^{\alpha_1} = \sum_{\alpha_1} \sum_{\gamma_1 + \delta_1 = \alpha_1} D'_{\gamma_1} (E^*(a)) D'_{\delta_1} (E^*(b)) X^{\alpha_1}.
\]

\[
E(a)E(b) = \sum_{\gamma_1} D'_{\gamma_1} E^*(a) X^{\gamma_1} \sum_{\delta_1} D'_{\delta_1} E^*(b) X^{\delta_1} = \sum_{\alpha_1} \sum_{\gamma_1 + \delta_1 = \alpha_1} D'_{\gamma_1} (E^*(a)) D'_{\delta_1} (E^*(b)) X^{\alpha_1}.
\]

If the homomorphisms \( E_i \) are iterative and commute then the differentiation constructed above is iterative too. In fact:

\[
D_{\alpha} \circ D_{\beta} (a) = D_{\alpha} (D_{\beta} (a)) = D_{\alpha} (D_{\beta_1}^{(1)} D_{\beta_2}^{(2)} \cdots D_{\beta_n}^{(n)} (a)) = D_{\alpha_1}^{(1)} D_{\alpha_2}^{(2)} \cdots D_{\alpha_n}^{(n)} (D_{\beta_1}^{(1)} D_{\beta_2}^{(2)} \cdots D_{\beta_n}^{(n)} (a)) = D_{\alpha_1}^{(1)} \circ D_{\alpha_2}^{(2)} \circ \cdots \circ D_{\alpha_n}^{(n)} \circ D_{\beta_n}^{(n)} (a) = \frac{(\alpha_1 + \beta_1)!}{\alpha_1! \beta_1!} \cdots \frac{(\alpha_n + \beta_n)!}{\alpha_n! \beta_n!} D^{(1)}_{(\alpha_1 + \beta_1)} \cdots D^{(n)}_{(\alpha_n + \beta_n)} (a).
\]

\[
D_{\alpha + \beta} (a) = D_{(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)} (a) = D_{(\alpha_1 + \beta_1)}^{(1)} D_{(\alpha_2 + \beta_2)}^{(2)} \cdots D_{(\alpha_n + \beta_n)}^{(n)} (a).
\]

Remark 5. We can reformulate Lipman's theorem using differentiations instead of derivations.
Let \((A, m)\) be a noetherian local ring containing the rational number field \(Q\) and let \(x_1, \ldots, x_r \in m, I = (x_1, \ldots, x_r)\).

Suppose that

1. \(A\) is \(I\)-adically complete,
2. there exists a \(r\)-dimensional differentiation

\[ E : A \to A[[Y_1, \ldots, Y_r]] = A[[Y]], \]

associated to a commutative set of independent derivations, such that

\[ E(x_j) = x_j + \sum_{i=1}^{r} D_i(x_j) Y_i \]

and \(\det(D_i(x_j)) \notin m\).

Then \(A_0 = \{a \in A \mid E(a) = a\}\) is a subring of \(A\) such that \(A = A_0[[x_1, \ldots, x_r]]\), with \(x_1, \ldots, x_r\) analytically independent over \(A_0\).

Since we are in characteristic 0, every commutative set of \(r\) derivations \(D_1, \ldots, D_r\), associated to a \(r\)-dimensional \(E\), is \(r\)-integrable and its integral is

\[ E(a) = \sum_{\alpha} \frac{1}{\alpha_1! \ldots \alpha_r!} D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_r^{\alpha_r}(a) Y_1^{\alpha_1} Y_2^{\alpha_2} \ldots Y_r^{\alpha_r}. \]

**Proposition 6.** Let \(K\) be a separable extension field of a field \(k\) of characteristic \(p > 0\). Let \(D_1, \ldots, D_n \in \text{Der}_k(K)\) and \(x_1, \ldots, x_n \in K\) such that \(\det(D_i(x_j)) \neq 0\).

Then there exists an iterative \(n\)-dimensional \(E : K \to K[[X_1, \ldots, X_n]]\) associated to the system of derivations \(\{D_1, D_2, \ldots, D_n\} \Rightarrow \)

\[ [D_i, D_j] \in \sum_{\alpha=1}^{n} K D_{\alpha}, \quad D_i^p \in \sum_{\alpha=1}^{n} K D_{\alpha} \]

\[ 1 \leq i, j \leq n \]

\[ 1 \leq i \leq n \]

**Proof.** It follows by Theorem 1 of [1] and by Proposition 4.

**Definition 8.** Let \(S\) be a ring of characteristic \(p\) and \(S^p\) denote the subring \(\{x^p \mid x \in S\}\). Let \(S'\) be a subring of \(S\). A subset \(\Gamma\) of \(S\) is said to be \(p\)-independent over \(S'\), if the monomials \(b_1^{e_1} \ldots b_n^{e_n}\), where \(b_1, \ldots, b_n\) are distinct elements of \(\Gamma\) and \(0 \leq e_i \leq p - 1\), are linearly independent over \(S'^p[S']\). \(\Gamma\) is called a \(p\)-basis of \(S\) over \(S'\) if it is \(p\)-independent over \(S'\) and \(S'^p[S', \Gamma] = S\).
Let $B$ be a subring of $A$. We denote the differential module of $A$ over $B$ by $\Omega_B(A)$ and the canonical $B$-derivation by $d : A \to \Omega_B(A)$. For the definition and elementary properties we refer to [5], par. 25. It is not difficult to verify that if $\Gamma$ is a $p$-basis of $A$ over $B$ then $\Omega_B(A)$ is a free $A$-module with $(dy, y \in \Gamma)$ as a basis. Not every rings have a $p$-basis and there are sets $p$-independent which cannot be extended to a $p$-basis.

**Theorem 1.** Let $(A, m)$ be a local ring containing a field $k$ of characteristic $p > 0$. Let $A$ be separable over $k$. Let $D_1, \ldots, D_n \in \text{Der}_k(A)$. Set $A_0 = \{ a \in A \mid D_i(a) = 0, i = 1, \ldots, n \}$. Suppose that

1) there exist $x_1, \ldots, x_n \in A / \text{det}(D_i(x_j)) \notin m$,
2) $\Omega_k(A_0)$ is a free $A_0$-module and $A_0$ is finite over $k[A_0^p]$,
3) there exists a $p$-basis $B_0$ of $A_0$ over $k$ containing $x_1^p, \ldots, x_n^p$.

4) \[ [D_i, D_j] \in \sum_{\alpha=1}^{n} AD_\alpha, \quad D_i^p \in \sum_{\alpha=1}^{n} AD_\alpha \quad 1 \leq i, j \leq n \quad 1 \leq i \leq n. \]

Then the system of derivations \{ $D_1, D_2, \ldots, D_n$ \} can be prolonged to an iterative $n$-dimensional differentiation.

**Proof.** Let $(c_{ij})$ be the inverse matrix of $(D_i x_j)$ and put $\partial_i = \sum_j c_{ij} D_j$.

Then $\partial_i x_j = \delta_{ij}$, and $\sum_{i=1}^{n} AD_i = \sum_{i=1}^{n} A \partial_i$, $[\partial_i, \partial_j] \in \sum A \partial_i$, $\partial_i^p \in \sum A \partial_j$

(this last follows from the Hochschild formula $(aD)^p = a^p D^p + (aD)^{p-1}(a)D$, [5], p. 197). But if $[\partial_i, \partial_j] = \sum b_{ijk} \partial_k$ then $b_{ijk} = [\partial_i, \partial_j] x_k = 0$ for all $k$, therefore $[\partial_i, \partial_j] = 0$. Similarly $\partial_i^p = 0$.

By Theorem 1 of [2], $x_1, x_2, \ldots, x_n$ is a $p$-basis of $A$ over $A_0$. Thus $x_1, x_2, \ldots, x_n$ are $p$-independent over $A_0$.

Moreover, with the same process that is found in theorem 7 of [4], for the separability of $A$ over $k$, we can obtain that $x_i^p \notin A_0^p k$, and $x_i^p$ can belong to a $p$-basis $B_0$ of $A_0$ over $k$.

Let $B_0^* = B_0 \setminus \{ x_1^p, \ldots, x_n^p \}$.

Put $B = B_0^* \cup \{ x_1, x_2, \ldots, x_n \}$.

It results $A = A_0[x_1, x_2, \ldots, x_n]$ (since $x_1, x_2, \ldots, x_n$ is a $p$-basis of $A$ over $A_0$).

Set $y_i = x_i^p$, $1 \leq i \leq n$, we have

$A = A_0[X_1, \ldots, X_n]/(X_1^p - y_1, \ldots, X_n^p - y_n)$.

Set $I = (X_1^p - y_1, \ldots, X_n^p - y_n)$ and $(X) = (X_1, X_2, \ldots, X_n)$.

Since

$\Omega_k(A_0[X]) \cong (\Omega_k(A_0) \otimes_{A_0} A_0[X]) \oplus A_0[X]d X_1 \oplus \ldots \oplus A_0[X]d X_n$, 

This completes the proof.
\( \Omega_k(A_0) \) is a free module with basis \( dB_0 \) and the following sequence

\[
I/I^2 \to \Omega_k(A_0[X]) \otimes A \to \Omega_k(A) \to 0,
\]
is exact, we have

\[
\Omega_k(A) \cong (\Omega_k(A_0) \otimes_{A_0} A)/(Ady_1 \oplus Ady_2 \oplus \ldots \oplus Ady_n) \oplus Adx_1 \oplus \ldots \oplus Adx_n,
\]
i.e. \( \Omega_k(A) \) is free with differential basis \( B = B_0^* \cup \{x_1, x_2, \ldots, x_n\} \), i.e. \( B \) is a \( p \)-basis of \( A \) over \( k \).

Since \( A \) is separable over \( k \), \( A \otimes_k k^{p^{-1}} \) is reduced and we can settle on the grounds of the theorem of [8].

The homomorphism \( E : A \to A[[X_1, X_2, \ldots, X_n]] \) defined by \( E(x_j) = x_j + \sum_{i=1}^n D_i(x_j)X_i, 1 \leq j \leq n \) is the iterative differentiation sought.

The following results concern the case of a set of derivations \( \mathcal{D} = \{D_1, D_2, \ldots, D_n\} \) which aren’t necessarily independent.

In such ambit we have the following result:

**Corollary 1.** ([8] Corollary 3.2). Let \( A \) be a \( k \)-algebra such that the ring \( A \otimes_k k^{p^{-1}} \) is reduced and let \( \Gamma \) be a \( p \)-basis of \( A \) over \( k \). For any set of \( k \)-derivations \( d_1, \ldots, d_m : A \to A \) there exists a \( m \)-dimensional differentiation \( D : A \to A[[X_1, \ldots, X_m]] \)

\[
D(a) = \sum_{\alpha} D_\alpha(a)X^\alpha
\]

with \( \alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_m) \in N^m, X^\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\ldots X_m^{\alpha_m} \), such that \( D_0 = \text{id}_A \) and \( D_{(0, \ldots, 0, 1, 0, \ldots, 0)} = d_i, i = 1, \ldots, m, \) with 1 on the \( i \)-th position.

**Remark 5.** We notice that the \( d_1, \ldots, d_m \) of Corollary 1 are ordinary derivations and they can be dependent or independent. If \( A \) is a field \( K \), \( k \) doesn’t necessarily coincide with the subfield of the constants \( K_0 \). In general \( k \subseteq K_0 \). The hypothesis \( K \) has a \( p \)-basis (also infinite over \( k \)) is too strong. In fact we can deduce that \( K \) has a finite \( p \)-basis over \( K_0 \). (Theorem 7 [4]).

If \( d_1, \ldots, d_m \) are dependent derivations of which are independent \( s < m \), by renaming and reordering we can suppose that the first \( s \) derivations are independent. If \( \Gamma = \{x_1, \ldots, x_s\} \) is a \( p \)-basis of \( K \) over \( K_0 \), we can define \( d_i(x_j) = \delta_{ij}1 \leq i, j \leq s \). If \( d_1, \ldots, d_m \) are not linearly independent we can’t suppose that \( d_i(x_j) = \delta_{ij}1 \leq i, j \leq m \).

We reformulate Proposition 6 and Theorem 1 in the case of a set of derivations not necessarily independent.
Theorem 2. Let $K$ be a separable extension field of a field $k$ of characteristic $p > 0$. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ be a set of derivations of $K$ over $k$ such that $D_1, D_2, \ldots, D_r$ with $r < n$ are independent. Let $x_1, x_2, \ldots, x_r \in K$ such that

$$\det(D_i(x_j)) \neq 0, \quad 1 \leq i, j \leq r.$$ 

We suppose that $\det(D_i, D_j) = 0$ and $D_i^p = 0$, $1 \leq i, j \leq r$, $1 \leq i \leq r$.

Then we have the following facts:

1) $D_1, D_2, \ldots, D_r$ are strongly integrable.

2) We can find integrals of $E^{(1)}, \ldots, E^{(r)}$ of $D_1, D_2, \ldots, D_r$ such that $E^{(i)}$ and $E^{(j)}$ commute, for any $i, j$.

3) There exist integrals of $D_{r+1}, \ldots, D_n$ of the form $E^{(1)}_{\lambda_1} \circ \cdots \circ E^{(r)}_{\lambda_r}$ where $\lambda_1, \ldots, \lambda_r \in K$ and $E^{(i)}_{\lambda_i}$ is the homomorphism associated to the differentiation $D_{\lambda_i} = \{1, \lambda_i D_j^{(1)}, \lambda_i^2 D_j^{(2)}, \ldots, \lambda_i^n D_j^{(n)} \}$, $1 \leq i, j \leq r$.

Proof. To integrate $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ we find $n$ ring homomorphisms $E^{(1)}, \ldots, E^{(n)}$ that lift $D_1, D_2, \ldots, D_n$ respectively. We observe that $E^{(1)}, \ldots, E^{(r)}$ don’t have ties because $D_1, D_2, \ldots, D_r$ are independent. Given two differentiations $D_i = \{1, D_{i_1}, D_{i_2}, \ldots\}$ with $D_{i_1} = D_i$ and $D_j = \{1, D_{j_1}, D_{j_2}, \ldots\}$ with $D_{j_1} = D_j$, we can define the product $D_i \cdot D_j$ by the automorphisms $E^{(i)}$ and $E^{(j)}$ associated to $D_i$ and $D_j$ respectively. The differentiation obtained

$$D_i \cdot D_j = \{1, D_{i_1} + D_{j_1}, \ldots\}$$

is an unidimensional differentiation that lifts the sum $D_{i_1} + D_{j_1}$.

If $D_j = \{1, D_{j_1}, D_{j_2}, \ldots\}$ is an unidimensional differentiation of $A$ and $\lambda_i \in A$ then the differentiation $D_{\lambda_i} = \{1, \lambda_i D_{i_1}, \lambda_i^2 D_{i_2}, \ldots\}$ is a differentiation that lifts $\lambda_i D_{i_1}$. Then there exists an unidimensional differentiation that lifts $\lambda_1 D_1 + \lambda_2 D_2 + \ldots + \lambda_r D_r$, that one which we obtain putting in order $E^{(1)}_{\lambda_1}, \ldots, E^{(r)}_{\lambda_r}$ and thus there exist integrals of $D_{r+1}, \ldots, D_n$ of the form $E^{(1)}_{\lambda_1} \circ \cdots \circ E^{(r)}_{\lambda_r}$. \hfill $\square$

Definition 9. We say that a set $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ of derivations of a field $K$ is integrable if there are $s$ unidimensional differentiations $E^{(1)}, \ldots, E^{(s)}$ that commute and integrate $D_1, D_2, \ldots, D_s$ if $s < n$ are independent derivations and $D_{s+1}, \ldots, D_n$ have each one an integral unidimensional that we can obtain from $E^{(1)}, \ldots, E^{(s)}$. 
Theorem 3. Let \((A, m)\) be a local ring containing a field \(k\) of characteristic \(p > 0\). Let \(A\) be separable over \(k\). Let \(D_1, \ldots, D_n\) be a set of derivations of \(A\) over \(k\), such that \(D_1, \ldots, D_r\) with \(r < n\) are independent. Set \(A_0 = \{a \in A \mid D_i(a) = 0, \ i = 1, \ldots, n\}\). We suppose that

1) there exist \(x_1, \ldots, x_r \in A / \det(D_j(x_i)) \notin m 1 \leq i, j \leq r\),
2) \(\Omega_k(A_0)\) is a free \(A_0\)-module and \(A_0\) is finite over \(k[A_0^p]\),
3) there exists a \(p\)-basis \(B_0\) of \(A_0\) over \(k\) containing \(x_1^p, \ldots, x_r^p\),
4) \([D_i, D_j] = 0 \quad D_i^p = 0 \quad 1 \leq i, j \leq r \quad 1 \leq i \leq r\).

Then

1) There exists an iterative \(r\)-dimensional integral \(E^{(r)}\) for \(D_1, D_2, \ldots, D_r\).
2) For any \(D_{r+1}, \ldots, D_n\) there exists an iterative unidimensional integral of the form \(E_{\lambda_1}^{(1)} \cdots \cdots E_{\lambda_r}^{(r)}\) where \(E^{(i)} 1 \leq i \leq r\) is obtained by \(E^{(r)}\) putting \(X_1 = \ldots = \widehat{X}_i = \ldots = X_r = 0\).

Proof. Let \(E^{(r)}\) be the integral relative to \(D_1, D_2, \ldots, D_r\) (see Theorem 2). Let \(E^{(1)}, \ldots, E^{(r)}\) be unidimensional integrals obtained by \(E^{(r)}\) putting \(X_1 = \ldots = \widehat{X}_i = \ldots = X_r = 0\).

\[E^{(i)}\] integrates \(D_1, \ldots, E^{(r)}\) integrates \(D_r\).

Let \(D_i, r + 1 \leq i \leq n, D_i = \lambda_1 D_1 + \lambda_2 D_2 + \cdots + \lambda_r D_r\).

We consider \(E_{\lambda_1}^{(1)}, \ldots, E_{\lambda_r}^{(r)}\), it is clear that we can get \(E_{\lambda_1}^{(1)} \cdots \cdots E_{\lambda_r}^{(r)}\) integral of \(D_i, r + 1 \leq i \leq n\). \(\square\)

It is allowed to give the following:

Definition 10. Let \(\mathcal{D} = \{D_1, D_2, \ldots, D_n\}\) a set of derivations of a ring \(A\) with \(s < n\) independent derivations. To integrate \(\mathcal{D}\) means to give an integral \(E\) of dimension \(r\) for the independent derivations and \(n - r\) integrals of dimension 1 that we can always construct by \(E\).

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