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ON A CLASS OF ULTRAPARABOLIC OPERATORS OF KOLMOGOROV - FOKKER - PLANCK TYPE

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We consider the class of the ultraparabolic differential operators of the following type

$$Lu = \sum_{i,j=1}^{p_0} a_{i,j}(x,t)\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u,$$

where $B = \left(b_{i,j}\right)$ is a constant matrix and $0 < p_0 < N$. We give a definition of Hölder continuity related to suitable groups of translations and dilations. Then, assuming such a regularity on the coefficients $a_{i,j}(x,t)$, we construct the fundamental solution Γ of L by the Levi's parametrix method. Moreover we prove an accurate local estimate of Γ and an invariant Harnack inequality for non-negative solutions of the divergence form equation

$$Lu = \operatorname{div}(A(x,t)Du) + < x, BDu > -\partial_t u.$$

1. Introduction and main results.

We consider in $\mathbb{R}^{N+1}$ the second order differential operator

$$Lu = \sum_{i,j=1}^{p_0} a_{i,j}(z)\partial_{x_i x_j} u + < x, BDu > -\partial_t u, \tag{1.1}$$

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where $1 \le p_0 \le N, z = (x,t) \in \mathbb{R}^{N+1}, D = (\partial_{x_1}, ..., \partial_{x_N})$ and $< \cdot, \cdot >$ denote, respectively, the gradient and the inner product in $\mathbb{R}^N$. In (1.1) $B = (b_{i,j})$ is an $N \times N$ matrix with constant real entries, $A_0(z) = \left(a_{i,j}(z)\right)_{i,j=1,...,p_0}$ is a symmetric matrix, which is positive definite in $\mathbb{R}^{p_0}$ for any $z \in \mathbb{R}^{N+1}$. We shall make further hypotheses on matrices A_0 and B after having introduced suitable notation (Definitions 1.1 e 1.2).

Equations like (1.1) arise in the stochastic theory of diffusion processes. For example

$$L = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{x_{n+j}} - \partial_t \tag{1.2}$$

is the prototype of the Kolmogorov operator which, under suitable conditions, describes the probability density of a physical system with $2n$ degree of freedom (see [13], page 167). If we set $N = 2n$, (1.2) takes the form (1.1), with

$$A_0 = I_n, \qquad B = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$

Moreover, the equation $Lu = f$, is a linearized prototype for the Fokker - Planck equation which describes brownian motions of a particle in a fluid (Chandrasekhar [2]).

We would like to remark that, although the operator (1.2) is strongly degenerate, it is hypoelliptic. Indeed, letting $X_j = \partial_{x_j}$, $j = 1, ..., n$, and

$$Y =< x, BD > -\partial_t,$$

the operator L can be written as

$$L = \sum_{j=1}^{n} X_j^2 + Y$$

and satisfies the well known Hörmander's condition: $\text{rank}\big(\mathscr{L}(X_1, ..., X_n, Y)\big) = N + 1$ at each point of $\mathbb{R}^{N+1}$ (see [5]). Here $\mathscr{L}(X_1, ..., X_n, Y)$ denotes the Lie algebra generated by $X_1, ..., X_n, Y$.

When the coefficients $a_{i,j}$ of (1.1) are C^∞ functions, then there exist p_0 smooth vector fields $Y_1, ..., Y_{p_0}$ such that L can be written as $L = \sum_{j=1}^{p_0} Y_j^2 + Y$, and it belongs to the class introduced by Hörmander in [5] and later studied by Rotschild - Stein in [12]. Although Hörmander's operators of "parabolic" type

$$\sum_{j=1}^{p} X_j^2 - \partial_t$$

have been widely studied in recent years (see [17] and the references therein), only few significant results on the operators (1.1) have appeared in literature (see Weber [18], Il'in [6], Sonin [16], Genčev [4], Shatyro [14], [15]).

For A_0 a constant matrix it was shown in [8] that the operator (1.1) is hypoelliptic if, and only if, for some basis of $\mathbb{R}^N$ matrix B takes the form:

$$B = \begin{pmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{pmatrix}. \tag{1.3}$$

Each B_j is a $p_{j-1} \times p_j$ block matrix of rank p_j, $j = 1, 2, \dots, r$, with $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$ and $p_0 + p_1 + \dots + p_r = N$. In general, blocks "$*$" are arbitrary. If (and only if) these blocks are zero matrices, the operator L is invariant with respect to a certain dilation group $\mathcal{G} = (D(\lambda))_{\lambda>0}$, i.e.

$$L\left(u\left(D(\lambda)z\right)\right) = \lambda^2 Lu\left(D(\lambda)z\right)$$

for any $u \in C_0^\infty\left(\mathbb{R}^{N+1}\right)$ and for any $\lambda > 0$ and $z \in \mathbb{R}^{N+1}$. The dilation $D(\lambda)$ belonging to $\mathcal{G}$ is defined as

$$D(\lambda) = \operatorname{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2\right) \tag{1.4}$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix.

In the sequel we shall call *spatial homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $\left(D(\lambda)\right)_{\lambda>0}$ the positive integer

$$Q = p_0 + p_1 + \dots + (2r+1)p_r. \tag{1.5}$$

We also note that $\det D(\lambda) = \lambda^{Q+2}$ for every $\lambda > 0$.

In [8] it is also shown that to any "model" operator L (i.e. with the matrix A_0 constant) we can associate a particular "model" operator L_0 which is invariant with respect to some dilation groups, and such that the fundamental solution Γ_0 of L_0 is equivalent to the fundamental solution Γ of L, on the level set $\left\{z \in \mathbb{R}^{N+1} : \Gamma(z) > 1/r\right\}$.

Since the treatment of the dilation-invariant operators is simpler than the general case, in this work we study the operators (1.1) under the following

Hypothesis H.1. *B is an $N \times N$ constant matrix as in (1.3), where each block matrix B_j has rank p_j and every "$*$" block is a zero matrix.*

From Hypothesis H.1 it follows that any operator (1.1), with A_0 constant, is invariant with respect to the same dilation group $(D(\lambda))_{\lambda>0}$.

A remarkable property of "model" operators is the invariance with respect to left translations of the group $(\mathbb{R}^{N+1}, \circ)$, whose composition is defined as

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau), \quad E(t) = \exp(-tB^T). \tag{1.6}$$

It is easy to show that $(\mathbb{R}^{N+1}, \circ)$ is a *group* with identity element $(0,0)$, and

$$(\xi,\tau)^{-1} = (-E(-\tau)\xi, -\tau), \quad \forall (\xi,\tau) \in J\mathbb{R}^{N+1}. \tag{1.7}$$

Let $\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, $\ell_\zeta(z) = \zeta \circ z$. Then, for every operator (1.1) with A_0 constant, we have

$$\ell_\zeta \circ L = L \circ \ell_\zeta \qquad \forall \zeta \in \mathbb{R}^{N+1}. \tag{1.8}$$

Another interesting observation is that

$$D(\lambda)\,(\zeta\circ z) = (D(\lambda)\zeta)\circ(D(\lambda)z)\,, \quad D(\lambda)\left(z^{-1}\right) = (D(\lambda)z)^{-1}\,, \tag{1.9}$$

which relates $D(\lambda)$ to the group $(\mathbb{R}^{N+1}, \circ)$.

We would like to point out that both groups $(D(\lambda))_{\lambda>0}$ and $(\mathbb{R}^{N+1}, \circ)$ are only defined in terms of matrix B. In particular they *do not* depend on the point $z \in \mathbb{R}^{N+1}$ at which $A_0(z)$ is computed.

We next give the definition of norm and the definition of Hölder continuity associated with these groups.

Definition 1.1. *Let $(q_j)_{j=1,\dots,N}$ be such that $D(\lambda) = diag(\lambda^{q_1},\dots,\lambda^{q_N},\lambda^2)$. For every $z=(x,t)\in\mathbb{R}^{N+1}$, we put*

$$|x|_B = \sum_{j=1}^{N} |x_j|^{1/q_j}, \qquad \|z\|_B = |t|^{1/2} + |x|_B. \tag{1.10}$$

It is easy to see that $\|\cdot\|_B$ is a homogeneous function of degree 1 with respect the dilation $D(\lambda)$:

$$\|D(\lambda)z\|_B = \lambda\|z\|_B \qquad \forall \lambda > 0. \tag{1.11}$$

Definition 1.2. *Let α be a positive constant, $\alpha \le 1$. We say that a function f is Hölder continuous of exponent α with respect to the groups $(\mathbb{R}^{N+1}, \circ)$ and $(D(\lambda))_{\lambda>0}$ related to L (in short: B-Hölder continuous with exponent α) if there exists a positive constant k such that,*

$$|f(z) - f(\zeta)| \le k\|\zeta^{-1}\circ z\|_B^\alpha \qquad \forall z,\zeta \in \mathbb{R}^{N+1}. \tag{1.12}$$

We next give further hypotheses on the operator L in (1.1):

Hypothesis H.2. *There exists* $\Lambda > 0$ *such that*

$$\frac{1}{\Lambda}\sum_{j=1}^{p_0}\xi_j^2 \le \sum_{i,j=1}^{p_0} a_{i,j}(z)\xi_i\xi_j \le \Lambda\sum_{j=1}^{p_0}\xi_j^2$$

for every $(\xi_1, \dots\dots, \xi_{p_0}) \in \mathbb{R}^{p_0}$ *and for every* $z \in \mathbb{R}^{N+1}$.

Hypothesis H.3. *There exist* $\alpha \in]0, 1]$ *and* $M > 0$ *such that*

$$|a_{i,j}(z) - a_{i,j}(\zeta)| \le M\|\zeta^{-1} \circ z\|_B^\alpha, \qquad \forall z, \zeta \in \mathbb{R}^{N+1}$$

and for any $i, j = 1, \dots, p_0$.

With a suitable adaptation of the Levi's method of the parametrix, we shall construct in Section 2 the fundamental solution of operators (1.1) verifying Hypotheses H.1-H.3.

We note that the parametrix method was used by M. Weber [18] and by Il'In [6], assuming an "euclideous"-type regularity on the coefficients $a_{i,j}$, which is not related to the group law of $(\mathbb{R}^{N+1}, \circ)$. As a consequence, additional regularity hypotheses on $a_{i,j}$ were required, and the study was restricted to "Kolmogorov" type operators for which Lie algebra $\mathscr{L}(\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y)$ has degree 1. In 1967 Sonin [16] generalized the method to the class of the operators verifying Hypotheses H.1 and H.2, again requiring an unnecessary regularity on the coefficients of A_0.

The parametrix method also provides an upper bound for the fundamental solution Γ, which can considerably sharpened, as we shall see in Corollary 2.5. This new estimate allows us to determine an accurate *local estimate* of Γ (see Section 4) and to state an *invariant Harnack inequality* for non-negative solutions of $Lu = 0$ (see Section 5).

The main results of this work are the following Theorems 1.1 - 1.3.

Theorem 1.1. (Existence of the fundamental solution). *Let L be as in (1.1) verifying Hypotheses H.1, H.2, H.3. Then there exists the fundamental solution Γ of L.*

Theorem 1.2. (Local estimate of the fundamental solution). *Let L be as in (1.1) verifying Hypotheses H.1, H.2, H.3. Then, for every $\varepsilon > 0$ there exists $K > 0$ such that*

$$(1.13) \qquad (1-\varepsilon)Z(z,\zeta) \le \Gamma(z,\zeta) \le (1+\varepsilon)Z(z,\zeta)$$

for any $z, \zeta \in \mathbb{R}^{N+1}$ *such that* $Z(z, \zeta) \ge K$.

Here $\Gamma(z,\zeta)$ denotes the fundamental solution of L and $Z(z,\zeta)$ denotes the parametrix, i.e. the fundamental solution with pole at ζ of

$$L_\zeta = \sum_{i,j=1}^{p_0} a_{i,j}(\zeta)\partial_{x_i x_j} + < x, BD > -\partial_t. \tag{1.14}$$

We now introduce an additional condition.

Hypothesis H.4. *For every $i, j = 1, \ldots., p_0$ there exist the derivatives $\partial_{x_i} a_{i,j}(z)$ and they are bounded and B–Hölder continuous functions, with exponent α.*

If Hypothesis H.4 holds, Theorems 1.1 and 1.2 can be extended to the *divergence form* operator

$$M = \operatorname{div}\,(A(z)D) + < x, BD > -\partial_t, \tag{1.15}$$

where $A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}$.

The following Theorem states an invariant Harnack inequality for non-negative solutions of $Mu = 0$, where M is the divergence form operator (1.15). We first need to introduce some notation. For every $\rho > 0$ we put:

$$\begin{aligned} &H_\rho = \left\{(x,t) \in \mathbb{R}^{N+1} : -\rho^2 \le t \le 0;\ |D(\rho)x| \le 1\right\} \\ &H_\rho^- = \left\{(x,t) \in \mathbb{R}^{N+1} : t = -\rho^2;\ |D(\rho)x| \le 1\right\} \\ &H_\rho(z_0) = z_0 \circ H_\rho, \qquad H_\rho^-(z_0) = z_0 \circ H_\rho^-. \end{aligned} \tag{1.16}$$

Here and in the following we use the same notation for $D(\lambda)$ in $\mathbb{R}^{N+1}$ and for its restriction to $\mathbb{R}^N$. Next Theorem extends Theorem 5.1' in [8] to the variable $a_{i,j}$ coefficients.

Theorem 1.3. (Invariant Harnack inequality). *Let M be an operator (1.15) verifying Hypotheses H.1-H.4, and let Ω be an open subset of $\mathbb{R}^{N+1}$. Then there exist three constants $c, r_0 > 0$ and $\theta \in]0,1[$, only depending on the constants in Hypotheses H.1-H.4, such that*

$$\sup_{z \in H^-_{r\theta}(z_0)} u(z) \le c u(z_0) \tag{1.17}$$

for every non-negative solution u of $Mu = 0$ in Ω, for every $z_0 \in \Omega$ such that $H_r(z_0) \subset \Omega$ and for every $r \in]0, r_0[$.

We note that the proof of Theorem 1.1 yields an accurate upper bound for the fundamental solution Γ of L in terms of a "model" operator L^+. If Γ^+ denotes the fundamental solution of L^+, then

$$\Gamma(z;\zeta) \leq c^+\Gamma^+(z;\zeta) \tag{1.18}$$

for every $z = (x,t), \zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$ such that $0 < t-\tau < T$, for some positive constant c^+ depending on T.

By a suitable adaptation of a technique introduced by Aronson and Serrin in [1], Theorems 1.2 and 1.3 can be used for determining a lower bound analogous to (1.18) for divergence form operators (1.15). It is possible to construct a "model" operator L^- such that the corresponding fundamental solution Γ^- satisfies

$$\Gamma^-(z;\zeta)c^- \leq \Gamma(z;\zeta) \tag{1.19}$$

for every $z = (x,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$ such that $0 < t-\tau < T$, for some positive constant c^- depending on T (see [10], Teorema 4). This result will appear in a forthcoming paper [11].

We close this section by noting that Theorems 1.1, 1.2 and 1.3 are not sensitive to "small" perturbations of matrix A_0. On the other hand a C^∞ (or an analytic) perturbation of matrix B can destroy the regularity of L. For instance,

$$L_0 = \partial_{x_1}^2 + x_1\partial_{x_2} + x_2\partial_{x_3} - \partial_t \tag{1.20}$$

is hypoelliptic, whereas the "perturbated" operator

$$L = \partial_{x_1}^2 + x_1\partial_{x_2} + (x_2 - tx_1)\partial_{x_3} - \partial_t \tag{1.21}$$

is not hypoelliptic, since

$$\operatorname{rank}\left(\mathscr{L}(x_1, x_1\partial_{x_2} + x_2\partial_{x_3} - \partial_t)\right) = 4,$$

and

$$\operatorname{rank}\left(\mathscr{L}(x_1, x_1\partial_{x_2} + (x_2 - tx_1)\partial_{x_3} - \partial_t)\right) = 3,$$

(see [5], page 149; see also [8], Appendix).

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2. The parametrix method.

In this Section we shall apply the Levi's parametrix method to operators (1.1), verifying Hypotheses H.1-H.3, by making a systematic use of the definition of B-Hölder continuity given in Definition 1.2.

We start with a simple result.

Proposition 2.1. *A function f is locally B-Hölder continuous if, and only if, it is locally Hölder continuous.*

Proof. Let $f \in C(\Omega)$ be a locally Hölder continuous function, with exponent $\alpha \in]0, 1[$. We need to show that, for every $x \in \Omega$, there exist $\varepsilon \in]0, 1]$ and $M > 0$ such that

$$\frac{|f(z \circ \zeta) - f(z)|}{\|\zeta\|_B^\alpha} \leq M \tag{2.1}$$

for any $\zeta \in \mathbb{R}^N$ such that $\|\zeta\|_B \leq \varepsilon$. We note that, from the definition (1.6) of $E(t)$ there exists $c_0 > 0$ such that

$$\|E(\tau) - I\| \leq c_0|\tau| \qquad \forall \tau \in [-1, 1].$$

Then, the Hölder continuity of f on the compact set $\overline{B(z, \varepsilon)}$ gives

$$\begin{aligned} |f(z \circ \zeta) - f(z)| &= |f(\xi + E(\tau)x, t + \tau) - f(x, t)| \leq \\ &\leq M_0 \left[|\tau|^\alpha + |\xi + \big(E(\tau) - I\big)x|^\alpha\right] \leq \\ &\leq M_0 \left[|\tau|^\alpha + 2|\xi|^\alpha + 2c_0^\alpha |x|^\alpha |\tau|^\alpha\right] \leq \\ &\leq M\left[|\xi|^\alpha + |\tau|^\alpha\right] \leq M\|\zeta\|_B^\alpha \end{aligned} \tag{2.2}$$

for every $\zeta \in \mathbb{R}^{N+1}$ such that $\|\zeta\|_B \leq \varepsilon$. The last inequality holds since

$$|\xi|^\alpha + |\tau|^\alpha \leq |\xi|_B^\alpha + |\tau|^\alpha \leq \|\zeta\|_B^\alpha \tag{2.3}$$

for every $\zeta = (\xi, \tau)$ such that $\|\zeta\|_B \leq 1$. This proves (2.1). The proof of the converse is similar, since

$$\|\zeta\|_B \leq (N + 1)\|\zeta\|^{\frac{1}{q_N}}$$

for every ζ such that $\|\zeta\|_B \leq 1$, where q_N is as in Definition 1.1. □

We next construct the fundamental solution of equation (1.1). For every $\bar{z} \in R^{N+1}$, let

$$L_{\bar{z}} = \sum_{i,j=1}^{p_0} a_{i,j}(\bar{z})\partial_{x_i,x_j} + < x, BD > -\partial_t \tag{2.4}$$

and denote by $Z_{\bar{z}}(z,\zeta)$ the fundamental solution of $L_{\bar{z}}$. To simplify the notation, when $\bar{z}=\zeta$, we shall write

$$Z(z,\zeta)=Z_\zeta(z,\zeta). \tag{2.5}$$

According to Levi's method, we seek the fundamental solution Γ of L by using the *parametrix* $Z(z,\zeta)$. We put

$$\Gamma(z,\zeta)=Z(z,\zeta)+J(z,\zeta) \tag{2.6}$$

and we require that $\Gamma(\cdot\,;\zeta)$ is a solution of the differential equation (1.1), for $z\neq\zeta$

$$0=L\Gamma(z,\zeta)=LZ(z,\zeta)+LJ(z,\zeta). \tag{2.7}$$

Suppose that the function J can be written as

$$J(x,t;\xi,\tau)=\int_\tau^t\left(\int_{\mathbb{R}^N}Z(x,t;y,s)\Phi(y,s;\xi,\tau)\,dy\right)ds \tag{2.8}$$

for some unknown function Φ.

Assuming that $J(z;\zeta)$ can be differentiated under the integral sign, we obtain

$$LJ(z;\zeta)=\int_\tau^t\left(\int_{\mathbb{R}^N}LZ(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds-\Phi(z;\zeta), \tag{2.9}$$

then condition (2.7) can be written as

$$\Phi(z;\zeta)=(LZ)(z;\zeta)+\int_\tau^t\left(\int_{\mathbb{R}^N}LZ(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds. \tag{2.10}$$

It then follows that the differential equation $L\Gamma(z;\zeta)=0$ is transformed into the integral equation (2.10), where the unknown function is Φ.

The function Φ in (2.10) can be determined by means of the successive approximation method, which yields

$$\Phi(z;\zeta)=\sum_{k=1}^{\infty}(LZ)_k(z;\zeta), \tag{2.11}$$

where

$$\begin{aligned}(LZ)_1(z,\zeta)&=(LZ)(z;\zeta)\\(LZ)_{k+1}(z;\zeta)&=\int_\tau^t\left(\int_{\mathbb{R}^N}LZ(z;y,s)(LZ)_k(y,s;\zeta)\,dy\right)ds.\end{aligned} \tag{2.12}$$

We next analyse the convergence of the series (2.11).

Proposition 2.2. *There exists $k_0 \in \mathbb{N}$ such that, for every interval I of $\mathbb{R}$:*

i) *$(LZ)_k$ is a bounded function in $S_I = \mathbb{R}^N \times I$, for every $k \geq k_0$;*

ii) *the series*

$$\sum_{k=k_0}^{\infty} (LZ)_k(z;\zeta) \tag{2.13}$$

converges uniformly on $S_I = \mathbb{R}^N \times I$;

iii) *the function Φ defined in (2.11) satisfies the integral equation (2.10) for every $\zeta \in \mathbb{R}^{N+1}$ and for every $z \neq \zeta$.*

If we denote by Y the first order differential operator

$$Y = < x, BD > -\partial_t\,, \tag{2.14}$$

the following fundamental result can be proved

Proposition 2.3. *Let $J(z;\zeta)$ be the function defined in (2.8). Then, for $i, j = 1, \ldots, p_0$ the functions $\partial_{x_i} J$, $\partial^2_{x_i,x_j} J$, YJ exist and continuous. Moreover for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $z \neq \zeta$*

$$LJ(z;\zeta) = \int_{\tau}^{t} \left(\int_{\mathbb{R}^N} LZ(z; y, s)\Phi(y, s; \zeta)\, dy \right) ds - \Phi(z;\zeta)$$

as stated in (2.9).

Results (i) and (ii) of Proposition 2.2 give a precise meaning to definition (2.11) of $\Phi(z;\zeta)$ for every $z, \zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$. Moreover Propositions 2.2 and 2.3 ensure that the function Γ defined in (2.6) is a solution of $L\Gamma(z;\zeta) = 0$, for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $z \neq \zeta$.

The proof of Proposition 2.3 requires the study of some singular integrals, thus we postpone the proof to Section 3.

In order to prove Proposition 2.2, we first recall some results in [8].

For every $p_0 \times p_0$ constant symmetric and positive defined matrix A_0 let us define the $N \times N$ matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let B the matrix of L in (1.1) and put

$$C(t) = \int_0^t E(s) A E^T(s)\, ds, \qquad E(s) = \exp\left(-sB^T\right).$$

If L verify Hypothesis H.1, then

$$C(t) > 0 \qquad \text{for every } t > 0$$

and the operator

$$L = \operatorname{div}\left(AD\right) + < x, BD > -\partial_t \tag{2.15}$$

is hypoelliptic. If $D(\lambda)$ is the dilation matrix defined in (1.4), then the fundamental solution Γ of the operator (2.15), with pole at $(0, 0)$, is given by: $\Gamma(x, t) = 0$ if $t \leq 0$,

$$\Gamma(x, t) = \frac{c_N}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{4} < C^{-1}(1) D\left(\frac{1}{\sqrt{t}}\right) x, D\left(\frac{1}{\sqrt{t}}\right) x >\right), \tag{2.16}$$

if $t > 0$, where $c_N = (4\pi)^{-N/2} \left(\det C(1)\right)^{-1/2}$ and Q is the homogeneous spatial dimension of $\mathbb{R}^{N+1}$ with respect to $D(\lambda)$ defined in (1.4). The fundamental solution $\Gamma(\cdot\,; \zeta)$ of (2.15) with pole at ζ, can be obtained from Γ as

$$\Gamma(z; \zeta) = \Gamma(x, t; \xi, \tau) = \Gamma(x - E(t - \tau)\xi, t - \tau).$$

Let A_0^-, A_0^+ be two constant, symmetric positive defined $p_0 \times p_0$ matrices, and let

$$A^- = \begin{pmatrix} A_0^- & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^+ = \begin{pmatrix} A_0^+ & 0 \\ 0 & 0 \end{pmatrix} \tag{2.17}$$

be the corresponding $N \times N$ matrices. Denote by Γ^-, Γ^+ the fundamental solutions of

$$\begin{aligned} L^- &= \operatorname{div}\left(A^- D\right) + < x, BD > -\partial_t, \\ L^+ &= \operatorname{div}\left(A^+ D\right) + < x, BD > -\partial_t, \end{aligned} \tag{2.18}$$

respectively. Then

Lemma 2.1. *If $A_0^- \leq A_0^+$, then there exists a constant $c > 0$ such that*

$$\Gamma^-(z;\zeta) \leq c\,\Gamma^+(z;\zeta) \qquad \forall z, \zeta \in \mathbb{R}^{N+1}.$$

Proof. Let

$$\begin{aligned} C^- &= \int_0^1 E(s)A^-E^T(s)\,ds, \quad c_N^- = (4\pi)^{-N/2}\big(\det C^-\big)^{-1/2}, \\ C^+ &= \int_0^1 E(s)A^+E^T(s)\,ds, \quad c_N^+ = (4\pi)^{-N/2}\big(\det C^+\big)^{-1/2}. \end{aligned} \tag{2.19}$$

Then we have

$$\begin{aligned} \Gamma^-(z;\zeta) &= c_N^-(t-\tau)^{-\frac{Q}{2}} \exp\left(-\frac{1}{4} < \big(C^-\big)^{-1}\eta, \eta >\right), \\ \Gamma^+(z;\zeta) &= c_N^+(t-\tau)^{-\frac{Q}{2}} \exp\left(-\frac{1}{4} < \big(C^+\big)^{-1}\eta, \eta >\right), \end{aligned} \tag{2.20}$$

where $\eta = D\left(\frac{1}{\sqrt{t-\tau}}\right)\big(x - E(t-\tau)\xi\big)$. Here $z = (x,t)$ and $\zeta = (\xi, \tau)$.

Hypothesis $A_0^- \leq A_0^+$ yields $C^- \leq C^+$, then, for every $\eta \in \mathbb{R}^N$,

$$\exp\left(-\frac{1}{4} < \big(C^-\big)^{-1}\eta, \eta >\right) \leq \exp\left(-\frac{1}{4} < \big(C^+\big)^{-1}\eta, \eta >\right). \tag{2.21}$$

The result follows from (2.20) and (2.21), with $c = c_N^+/c_N^-$. □

Remark 2.1. The validity of (2.21) relies on the following result: if C_1 and C_2 are symmetric and positive definite matrices, and $C_1 \leq C_2$, then $C_2^{-1} \leq C_1^{-1}$. For sake of completeness, we give a short proof of this elementary proposition.

We first note that, for any symmetric matrix G,

$$C_1 \leq C_2 \ \Rightarrow\ GC_1G \leq GC_2G.$$

Then, if we set $G = C_1^{-1/2}$, from $C_1 \leq C_2$ it follows $I \leq C_1^{-1/2}C_2C_1^{-1/2}$. Choosing $G = \big(C_1^{-1/2}C_2C_1^{-1/2}\big)^{-1/2}$, we have $C_1^{1/2}C_2^{-1}C_1^{1/2} \leq I$, and finally, setting again $G = C_1^{-1/2}$, we get $C_2^{-1} \leq C_1^{-1}$.

The following result is an immediate consequence of Lemma 2.1.

Proposition 2.4. *There exist two operators L^- and L^+ as in (2.15) and two constants $c^-, c^+ > 0$ such that, if Γ^- and Γ^+ are the corresponding fundamental solutions, then*

$$c^-\Gamma^-(z;\zeta) \le Z_{\bar{z}}(z;\zeta) \le c^+\Gamma^+(z;\zeta) \quad \forall z,\zeta,\bar{z} \in \mathbb{R}^{N+1}.$$

Proof. It is sufficient to use Lemma 2.1 since, by Hypothesis H.1,

$$\frac{1}{\Lambda} I_{p_0} \le A(\bar{z}) \le \Lambda I_{p_0}, \quad \forall \bar{z} \in \mathbb{R}^{N+1}. \qquad \square$$

Corollary 2.1. *There exists a constant $c > 0$ such that, for every $i, j = 1, \dots, p_0$, and for every $z = (x,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$,*

$$(2.22) \qquad \left|\partial_{x_i} Z(z;\zeta)\right| \le \frac{c}{\sqrt{t-\tau}} \left| D\left((t-\tau)^{-\frac{1}{2}}\right)(x - E(t-\tau)\xi)\right| \Gamma^+(z;\zeta)$$

and

$$(2.23) \qquad \left|\partial^2_{x_i,x_j} Z(z;\zeta)\right| \le \le \frac{c}{t-\tau}\left(1 + \left|D\left((t-\tau)^{-\frac{1}{2}}\right)(x - E(t-\tau)\xi)\right|^2\right)\Gamma^+(z;\zeta).$$

Proof. Since the function Z_ζ can be written as in (2.16), with $C(1) = C_\zeta(1) = \int_0^1 E(s)A(\zeta)E^T(s)ds$, we have

$$(2.24) \qquad \partial_{x_i} Z_\zeta(z;\zeta) = = \frac{-1}{2\sqrt{t-\tau}} < l_i(\zeta), D\left((t-\tau)^{-\frac{1}{2}}\right)(x - E(t-\tau)\xi) > Z_\zeta(z;\zeta),$$

where $l_i(\zeta)$ is the i-th row of the matrix $C_\zeta^{-1}(1)$. From Hypothesis H.1, the coefficients of the matrix C_ζ^{-1} are bounded functions of ζ, then (2.24) yields (2.22).

The bound (2.23) can be obtained in a similar manner, by differentiating (2.24). $\square$

Lemma 2.2. *Let A_0^-, A_0^+ be two constant matrices such that $0 < A_0^- < A_0^+$. Then, for every polynomial p there exists a positive constant c_p such that, if we set $\eta = D\left(\frac{1}{\sqrt{t-\tau}}\right)\left(x - E(t-\tau)\xi\right)$, we have*

$$\left|p(|\eta|)\right|\Gamma^-(z;\zeta) \le c_p\,\Gamma^+(z;\zeta) \quad \forall z,\zeta \in \mathbb{R}^{N+1}.$$

Proof. From (2.20) it follows

$$\frac{\Gamma^-(z;\zeta)}{\Gamma^+(z;\zeta)} = \frac{c_N^-}{c_N^+}\exp\left(\frac{1}{4} < \left[\left(C^+\right)^{-1} - \left(C^-\right)^{-1}\right]\eta, \eta >\right). \tag{2.25}$$

The Lemma follows by noting that the quadratic form in (2.25) is negative defined, since from (2.19)

$$C^- - C^+ = \int_0^1 E(s)\left[A^- - A^+\right]E^T(s)ds < 0$$

and (see Remark 2.1)

$$\left(C^+\right)^{-1} - \left(C^-\right)^{-1} < 0. \qquad \square$$

Corollary 2.2. *Fix $\varepsilon > 0$, put $\widetilde{A}_0 = (\Lambda + \varepsilon)I_{p_0}$ and denote by $\widetilde{\Gamma}$ the fundamental solution of the corresponding operator $\widetilde{L}$. Then there exists a constant $\widetilde{c} > 0$ such that, for every $i, j = 1, \ldots, p_0$, and for every $z, \zeta \in \mathbb{R}^{N+1}$, we have*

$$\left|\partial_{x_i} Z_\zeta(z;\zeta)\right| \le \frac{\widetilde{c}}{\sqrt{t-\tau}}\widetilde{\Gamma}(z;\zeta) \tag{2.26}$$

and

$$\left|\partial^2_{x_i,x_j} Z_\zeta(z;\zeta)\right| \le \frac{\widetilde{c}}{t-\tau}\widetilde{\Gamma}(z;\zeta). \tag{2.27}$$

Lemma 2.3. *There exist an operator $\widetilde{L}$ and a constant $\widetilde{c} > 0$ such that, if $\widetilde{\Gamma}$ is the fundamental solution of $\widetilde{L}$, then*

$$|LZ(z;\zeta)| \le \frac{\widetilde{c}}{(t-\tau)^{1-\frac{\alpha}{2}}}\widetilde{\Gamma}(z;\zeta) \quad \forall z \ne \zeta. \tag{2.28}$$

Proof. Since Z_ζ is the fundamental solution of L_ζ, for every $z \ne \zeta$ we have

$$LZ_\zeta(z;\zeta) = \sum_{i,j=1}^{p_0} \left[a_{i,j}(z) - a_{i,j}(\zeta)\right]\partial^2_{x_i,x_j} Z_\zeta(z;\zeta). \tag{2.29}$$

From Hypothesis H.3, for $i, j = 1, \ldots, p_0$,

$$\left|a_{i,j}(z) - a_{i,j}(\zeta)\right| \le M|\zeta^{-1}\circ z|_B^\alpha = M(t-\tau)^{\frac{\alpha}{2}}\,|(\eta, 1)|_B^\alpha, \tag{2.30}$$

where $\eta = D\left(\frac{1}{\sqrt{t-\tau}}\right)(x - E(t-\tau)\xi)$. Then (2.28) follows from Corollary 2.1 and Lemma 2.2. $\square$

Estimate (2.28) plays a crucial role in the proof of Proposition 2.2. We next prove that Lemma 2.3 holds under Hypothesis H.4 for divergence form operators.

Lemma 2.3'. *Let M be a divergence form operator (1.15), verifying Hypotheses H.1-H.4. Then there exists an operator $\widetilde{L}$ such that, for every bounded interval $I \subset \mathbb{R}$, there exists a constant $\widetilde{c}_I > 0$ such that, if $\widetilde{\Gamma}$ is the fundamental solution of $\widetilde{L}$, then*

$$|MZ(z;\zeta)| \leq \frac{\widetilde{c}_I}{(t-\tau)^{1-\frac{\alpha}{2}}} \widetilde{\Gamma}(z;\zeta). \tag{2.31}$$

Proof. If we put, for $j = 1, \ldots, N$

$$b_j(z) = \sum_{i=1}^{p_0} \partial_{x_i} a_{i,j}(z), \tag{2.32}$$

then

$$MZ(z;\zeta) = \sum_{i,j=1}^{p_0} \left[a_{i,j}(z) - a_{i,j}(\zeta)\right] \partial^2_{x_i,x_j} Z(z;\zeta) + \tag{2.33}$$
$$+ \sum_{j=1}^{p_0} b_j(z) \partial_{x_j} Z(z;\zeta) = LZ(z;\zeta) + < b(z), DZ(z;\zeta) > .$$

By Hypothesis H.3 the coefficients of L are B-Hölder continuous. Moreover, by Hypothesis H.4, vector $b(z)$ is bounded and B-Hölder continuous, and its j-th component is zero for $j > p_0$. Using Lemma 2.3 and Corollary 2.1, we obtain (2.31). □

Corollary 2.3. *For every $k \in \mathbb{N}$ we have*

$$\left|(LZ)_k(z;\zeta)\right| \leq \frac{c_k}{(t-\tau)^{1-\frac{\alpha k}{2}}} \widetilde{\Gamma}(z;\zeta) \qquad \forall z, \zeta \in \mathbb{R}^{N+1}; \tag{2.34}$$

where

$$c_k = \frac{\widetilde{c}^k \Gamma(\alpha/2)^k}{\Gamma(\alpha k/2)}. \tag{2.35}$$

Here $\widetilde{c}$ and $\widetilde{\Gamma}$ are the constant and the function of Lemma 2.3 and, only in (2.35), Γ denotes the Euler's Gamma function.

Proof. Let us proceed by induction. If $k = 1$, (2.34) is nothing but (2.28). Suppose that (2.34) holds for k and compute

$$\left|(LZ)_{k+1}(z;\zeta)\right| = \left|\int_\tau^t \left(\int_{\mathbb{R}^N} LZ(z;y,s)(LZ)_k(y,s;\zeta)\,dy\right)ds\right| \tag{2.36}$$

(for induction hypothesis and (2.28))

$$\leq \int_\tau^t \frac{c_1}{(t-s)^{1-\frac{\alpha}{2}}}\frac{c_k}{(s-\tau)^{1-\frac{\alpha k}{2}}}\left(\int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\widetilde{\Gamma}(y,s;\zeta)\,dy\right)ds.$$

The *reproduction property* of the function $\widetilde{\Gamma}$ gives, for every $s \in]\tau, t[$,

$$\int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\,\widetilde{\Gamma}(y,s;\zeta)\,dy = \widetilde{\Gamma}(z;\zeta). \tag{2.37}$$

Using the basic properties of Euler's Gamma function, we obtain directly

$$\int_\tau^t \frac{c_1}{(t-s)^{1-\frac{\alpha}{2}}}\frac{c_k}{(s-\tau)^{1-\frac{\alpha k}{2}}}\,ds = c_{k+1}(t-\tau)^{-1+\frac{\alpha(k+1)}{2}},$$

which proves (2.34) for $k+1$. □

Proof of Proposition 2.2. (i) follows from Corollary 2.3 and from the explicit expression of $\widetilde{\Gamma}(z;y,s)$, for $k_0 \in \mathbb{N}$ and $k_0 \geq \frac{Q+2}{\alpha}$.
(ii) follows from (2.34), noting that the power series

$$\sum_{j=1}^{\infty} c_{k_0+j}t^j,$$

where c_k is defined in (2.35), has radius of convergence equal to infinity.
(iii) using (2.34), (2.35) and (2.37), for every $z \neq \zeta$ we obtain:

$$\int_\tau^t \left(\int_{\mathbb{R}^N} (LZ)(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds$$

$$= \sum_{k=1}^{\infty}\int_\tau^t \left(\int_{\mathbb{R}^N} (LZ)(z;y,s)(LZ)_k(y,s;\zeta)\,dy\right)ds$$

$$= \sum_{k=1}^{\infty}(LZ)_{k+1}(z;\zeta) = \Phi(z;\zeta) - (LZ)(z;\zeta).$$

This proves (2.10). □

The following result is a straightforward consequence of Lemma 2.3

Corollary 2.4. *For every bounded interval $I \subset \mathbb{R}$ there exists a constant $\widetilde{k}_I > 0$ such that*

$$(2.38) \qquad |\Phi(z;\zeta)| \le \frac{\widetilde{k}_I}{(t-\tau)^{1-\frac{\alpha}{2}}}\widetilde{\Gamma}(z;\zeta) \qquad \forall z,\zeta \in S_I, z \neq \zeta.$$

Corollary 2.5. *For every bounded interval $I \subset \mathbb{R}$ there exists a constant $c_I > 0$ such that*

$$(2.39) \qquad \Gamma(z;\zeta) \le c_I\,\widetilde{\Gamma}(z;\zeta) \qquad \forall z,\zeta \in S_I, z \neq \zeta.$$

Proof. Substitute in (2.6) the bounds of Corollary 2.4 and of Proposition 2.4. Then, for every $z, \zeta \in S_I$,

$$(2.40) \quad \Gamma(z;\zeta) \le Z(z;\zeta) + \int_\tau^t \left(\int_{\mathbb{R}^N} Z(z;y,s)\,|\Phi(y,s;\zeta)|\,dy \right) ds \le$$

$$\le \widetilde{c}\,\widetilde{\Gamma}(z;\zeta) + \int_\tau^t \frac{\widetilde{c}\,\widetilde{k}_I}{(s-\tau)^{1-\frac{\alpha}{2}}} \left(\int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\,\widetilde{\Gamma}(y,s;\zeta)\,dy \right) ds =$$

$$= \widetilde{c}\left(1 + (t-\tau)^{\frac{\alpha}{2}}\widetilde{k}_I c_\alpha\right)\widetilde{\Gamma}(z;\zeta).$$

Since the interval I is bounded, (2.40) yields the estimate (2.39). □

Propositions 2.2 and 2.3 show that Γ is a solution of the equation $L\Gamma = 0$ in $\mathbb{R}^{N+1}\backslash\{\zeta\}$, for any $\zeta \in \mathbb{R}^{N+1}$. In order to show that Γ is *the fundamental solution* of L, we only need to prove the following

Proposition 2.5. *For every function $f \in C_0\left(\mathbb{R}^N\right)$ we have*

$$(2.41) \qquad \lim_{t\to\tau^+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau) f(\xi)\,d\xi = f(x),$$

for every $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$.

Proof. Recall that Γ was defined as

$$\Gamma(z,\zeta) = Z(z,\zeta) + J(z,\zeta).$$

Since $Z_{\bar{z}}$ is the fundamental solution of $L_{\bar{z}}$, for every $x \in \mathbb{R}^N$,

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} Z_{(x,\tau)}(x,t;\xi,\tau) f(\xi)\, d\xi = f(x). \tag{2.42}$$

Moreover, since, from Hypothesis H.3, it follows

$$\lim_{w \to w'} \int_{\mathbb{R}^N} |Z_w(z;\xi,\tau) - Z_{w'}(z;\xi,\tau)|\, d\xi =$$
$$= \lim_{w \to w'} \int_{\mathbb{R}^N} |Z_w(0,0;\xi,\tau - t) - Z_{w'}(0,0;\xi,\tau - t)|\, d\xi = 0,$$

and f is a bounded function, we can easily derive

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \left[Z_{(\xi,\tau)}(z;\xi,\tau) - Z_{(x,\tau)}(z;\xi,\tau) \right] f(\xi)\, d\xi = 0. \tag{2.43}$$

Finally, applying Corollary 2.4 and Proposition 2.4 to the definition of J in (2.8), we get

$$|J(z;\zeta)| \le \int_{\tau}^{t} \frac{\tilde{c}\tilde{k}_I}{(s-\tau)^{1-\frac{\alpha}{2}}} \left(\int_{\mathbb{R}^N} \tilde{\Gamma}(z;y,s)\tilde{\Gamma}(y,s;\zeta)\, dy \right) ds$$
$$= c'(t-\tau)^{\frac{\alpha}{2}} \tilde{\Gamma}(z;\zeta),$$

from which

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} J(z;\xi,\tau) f(\xi)\, d\xi = 0. \tag{2.44}$$

The result (2.41) follows from (2.42), (2.43) and (2.44). $\square$

3. Singular integrals.

In this Section we shall give the proof of Proposition 2.3. To this purpose, we first prove some results on the function

$$J(z;\zeta)=\int_{\mathbb{R}^N\times]\tau,t[} Z(z;w)\Phi(w;\zeta)\,dw. \tag{3.1}$$

Proposition 3.1. *For every $\zeta\in\mathbb{R}^{N+1}$, for $i=1,\ldots,p_0$, function $J(z;\zeta)$ has continuous derivatives $\partial_{x_i}J(z;\zeta)$ and*

$$\partial_{x_i}J(z;\zeta)=\int_{\mathbb{R}^N\times]\tau,t[} \partial_{x_i}Z(z;w)\Phi(w;\zeta)\,dw. \tag{3.2}$$

Remark 3.1. The integral (3.2) converges since, by Corollary 2.2 and Corollary 2.4 there exists a positive constant c such that

$$\begin{aligned}\int_{\mathbb{R}^N\times]\tau,t[} &\left|\partial_{x_i}Z(z;w)\Phi(w;\zeta)\right|dw\leq\\ &\leq\int_{\mathbb{R}^N\times]\tau,t[} \widetilde{\Gamma}(z;w)\widetilde{\Gamma}(w;\zeta)(t-s)^{-1/2}(s-\tau)^{-1+\alpha/2}\,dw=\\ &=\widetilde{\Gamma}(z;\zeta)\int_\tau^t (t-s)^{-1/2}(s-\tau)^{-1+\alpha/2}\,ds<\infty.\end{aligned}$$

Proof of Proposition 3.1. Let φ be a function of class $C^2(\mathbb{R})$, such that $0\leq\varphi(t)\leq1$ for every $t\geq0$, $\varphi(t)=1$ for every $t\leq\frac{1}{2}$, and $\varphi(t)=0$ for every $t\geq1$. For any fixed $\varepsilon>0$, put

$$\eta_\varepsilon(z;w)=1-\varphi\left(\left\|D\left(\frac{1}{\varepsilon}\right)\left(w^{-1}\circ z\right)\right\|_B\right), \tag{3.3}$$

and

$$J_\varepsilon(z;\zeta)=\int_{\mathbb{R}^N\times]\tau,t[} Z(z;w)\eta_\varepsilon(z;w)\Phi(w;\zeta)\,dw. \tag{3.4}$$

We first show that for every $z = (x, t)$, $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, with $t > \tau$, it holds

$$\partial_{x_i} J_\varepsilon(z; \zeta) = \int\limits_{\mathbb{R}^N \times]\tau, t[} \partial_{x_i} (Z\eta_\varepsilon)(z; w)\Phi(w; \zeta)\, dw. \tag{3.5}$$

Note that, for every $\varepsilon > 0$, functions η_ε, $\partial_{x_i}\eta_\varepsilon$ are bounded. From Proposition 2.4 and Lemma 2.3, it thus follows that there exists $c_\varepsilon > 0$ such that

$$\left|\partial_{x_i} (Z\eta_\varepsilon)(z; w)\right| \leq \frac{c_\varepsilon}{\sqrt{t-s}} \widetilde{\Gamma}(z; w). \tag{3.6}$$

On the other hand, if we set

$$B(z, \rho) = \left\{\zeta \in \mathbb{R}^{N+1} : \|\zeta^{-1} \circ z\|_B < \rho\right\}, \tag{3.7}$$

then $(Z\eta_\varepsilon)(z; w) = 0$ for every $w \in B\left(z, \frac{\varepsilon}{2}\right)$, from which

$$\left|\partial_{x_i} (Z\eta_\varepsilon)(z; w)\right| \leq c_\varepsilon \sup_{w \in \mathbb{R}^{N+1} \setminus B\left(z, \frac{\varepsilon}{2}\right)} \frac{\widetilde{\Gamma}(z; w)}{\sqrt{t-s}} \stackrel{\text{def}}{=} \widetilde{c}_\varepsilon < \infty. \tag{3.8}$$

Hence, from Corollary 2.4, the bound

$$\left|\partial_{x_i} (Z\eta_\varepsilon)(z; w)\Phi(w; \zeta)\right| \leq \frac{\widetilde{c}_\varepsilon \widetilde{k}_I}{(s-\tau)^{1-\frac{\alpha}{2}}} \widetilde{\Gamma}(w; \zeta) \tag{3.9}$$

holds *uniformly with respect to the z variable*, where the constant $\widetilde{k}_I$ in Corollary 2.4 corresponds to the interval $I =]\tau, t[$. Since the function in (3.9) is absolutely integrable, (3.5) follows by Lebesgue's Theorem, for any fixed $\varepsilon > 0$.

Fix $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ and for any $T_0, T_1 \in \mathbb{R}$, $\tau < T_0 < T_1$, let $S_I = \mathbb{R}^N \times I = \mathbb{R}^N \times]T_0, T_1[$. We claim that

$$J_\varepsilon(z; \zeta) \underset{\varepsilon \to 0}{\longrightarrow} J(z; \zeta) \qquad \forall z \in S_I \tag{3.10}$$

and, for every $i = 1, \ldots, p_0$,

$$\partial_{x_i} J_\varepsilon(z; \zeta) \underset{\varepsilon \to 0}{\overset{z \in S_I}{\rightrightarrows}} \int\limits_{\mathbb{R}^N \times]\tau, t[} \partial_{x_i} Z(z; w)\Phi(w; \zeta)\, dw \tag{3.11}$$

(we denote by $\overset{z\in S_I}{\underset{\varepsilon\to 0}{\rightrightarrows}}$ the uniform convergence on S_I). If (3.10) and (3.11) hold, function $J(\cdot\,;\zeta)$ has derivatives $\partial_{x_i} J(\cdot\,;\zeta)$ on S_I. Moreover these derivatives are continuous functions and, for every $z \in S_I$

$$\partial_{x_i} J(z;\zeta) = \int_{\mathbb{R}^N\times]\tau,t[} \partial_{x_i} Z(z;w)\Phi(w;\zeta)\,dw.$$

The arbitrariness on the choice of $I\ =]T_0, T_1[$, with $\tau < T_0 < T_1$, yields the results of Proposition 3.1.

We are thus left with the proof of (3.10) and (3.11).

Without loss of generality, we can suppose $0 < \varepsilon \le \varepsilon_0$, with $\varepsilon_0 = \sqrt{\frac{T_0-\tau}{2}}$. Then

$$J_\varepsilon(z;\zeta) - J(z;\zeta) = \int_{\mathbb{R}^N\times]\tau,t[} Z(z;w)\,[\eta_\varepsilon(z;w)-1]\,\Phi(w;\zeta)\,dw. \tag{3.12}$$

From the definition of the function η_ε, we have

$$\begin{aligned} |\eta_\varepsilon(z;w)-1| &\le 1 \qquad \forall w\in\mathbb{R}^{N+1},\\ \eta_\varepsilon(z;w)-1 &= 0 \qquad \forall w\in\mathbb{R}^{N+1}\setminus B(z,\varepsilon), \end{aligned} \tag{3.13}$$

and from Proposition 2.4, there exists a constant $\widetilde{c} > 0$ such that

$$Z(z;w) \le \widetilde{c}\,\widetilde{\Gamma}(z;w).$$

Therefore from (3.13) and Corollary 2.4 it follows

$$|J_\varepsilon(z;\zeta) - J(z;\zeta)| \le \int_{\substack{B(z,\varepsilon)\\ s<t}} \widetilde{c}\,\widetilde{k}_I\,\widetilde{\Gamma}(z;w)\frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}}\,dw. \tag{3.14}$$

Note that, since $z\in S_I$, $w\in B(z,\varepsilon)$ and $\varepsilon \le \varepsilon_0 = \sqrt{\frac{T_0-\tau}{2}}$, then $s-\tau \ge \varepsilon_0^2$, and

$$\frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}} \le \widetilde{c}_N\,\varepsilon_0^{\alpha-Q-2}. \tag{3.15}$$

Hence there exists a constant $c_S > 0$, depending only on the set S_I, such that

$$|J_\varepsilon(z;\zeta) - J(z;\zeta)| \le c_S \int_{\substack{B(z,\varepsilon)\\ s<t}} \widetilde{\Gamma}(z;w)\,dw. \tag{3.16}$$

Recall that $\widetilde{\Gamma}(z;w)=\widetilde{\Gamma}(w^{-1}\circ z;0)$ and that

$$\widetilde{\Gamma}(D(\lambda)z;0)=\lambda^{-Q}\widetilde{\Gamma}(z;0);\qquad \det(D(\lambda))=\lambda^{Q+2}. \tag{3.17}$$

Hence, by setting $w'=D\left(\frac{1}{\varepsilon}\right)\left(w^{-1}\circ z\right)$ in (3.16), we obtain

$$|J_\varepsilon(z;\zeta)-J(z;\zeta)|\le\varepsilon^2 c_S\int\limits_{\substack{B(0,1)\\ s<0}}\widetilde{\Gamma}(0;w)\,dw=\varepsilon^2 k_S \tag{3.18}$$

from which (3.10) follows.

In order to prove (3.11) we again assume $0<\varepsilon\le\varepsilon_0=\sqrt{\frac{T_0-\tau}{2}}$. From (3.5)

$$\partial_{x_i}J_\varepsilon(z;\zeta)-\int\limits_{\mathbb{R}^N\times]\tau,t[}\partial_{x_i}Z(z;w)\Phi(w;\zeta)\,dw= \tag{3.19}$$

$$=\int\limits_{\mathbb{R}^N\times]\tau,t[}\partial_{x_i}Z(z;w)\,[\eta_\varepsilon(z;w)-1]\,\Phi(w;\zeta)\,dw+$$

$$+\int\limits_{\mathbb{R}^N\times]\tau,t[}Z(z;w)\partial_{x_i}\eta_\varepsilon(z;w)\Phi(w;\zeta)\,dw=I_\varepsilon(z;\zeta)+II_\varepsilon(z;\zeta).$$

We next evaluate the quantities I_ε and II_ε. Using (3.13), Corollary 2.2 and Corollary 2.4 we get

$$|I_\varepsilon(z;\zeta)|\le\int\limits_{\substack{B(z,\varepsilon)\\ s<t}}\frac{\widetilde{c}\widetilde{k}_I}{\sqrt{t-s}}\widetilde{\Gamma}(z;w)\frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}}\,dw\le \tag{3.20}$$

$$\le c'_S\int\limits_{\substack{B(z,\varepsilon)\\ s<t}}\frac{\widetilde{\Gamma}(z;w)}{\sqrt{t-s}}\,dw.$$

Last inequality follows from (3.15). Substituting again $w'=D\left(\frac{1}{\varepsilon}\right)\left(w^{-1}\circ z\right)$ we finally obtain

$$|I_\varepsilon(z;\zeta)|\le\varepsilon c'_S\int\limits_{\substack{B(0,1)\\ s<0}}\frac{\widetilde{\Gamma}(0;w)}{\sqrt{-s}}\,dw=\varepsilon k'_S. \tag{3.21}$$

To evaluate $II_\varepsilon(z;\zeta)$, we note at first that

$$\partial_{x_i}\eta_\varepsilon(z;w) = \frac{1}{\varepsilon}\varphi'\left(\left\| D\left(\frac{1}{\varepsilon}\right)(w^{-1}\circ z)\right\|_B\right). \tag{3.22}$$

Then, letting $m = \sup_{\mathbb{R}}|\varphi'|$,

$$\begin{aligned} \left|\partial_{x_i}\eta_\varepsilon(z;w)\right| &\le \frac{m}{\varepsilon} && \forall w\in\mathbb{R}^{N+1},\\ \partial_{x_i}\eta_\varepsilon(z;w) &= 0 && \forall w\in\mathbb{R}^{N+1}\setminus B(z,\varepsilon). \end{aligned} \tag{3.23}$$

By using (3.23) II_ε can be treated as I_ε in the previous case. We have

$$|II_\varepsilon(z;\zeta)| \le \frac{m\tilde{c}\tilde{k}_I}{\varepsilon}\int\limits_{B(z,\varepsilon),s<t}\widetilde{\Gamma}(z;w)\frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}}\,dw \le \tag{3.24}$$

$$\le \frac{m\,c'_S}{\varepsilon}\int\limits_{B(z,\varepsilon),s<t}\widetilde{\Gamma}(z;w)dw = m\,c'_S\varepsilon\int\limits_{B(0,1),s<0}\widetilde{\Gamma}(0;w)\,dw = m\,k_S\varepsilon.$$

Then (3.11) follows from combining (3.19), (3.21) and (3.24). □

Lemma 3.1. *For every bounded interval I there exist three constants $c>0$, $\gamma,\gamma'\in]0,1[$ such that*

$$\left|\Phi(x,t;\zeta)-\Phi(x',t;\zeta)\right| \le c\frac{|x-x'|_B^{\gamma'}}{(t-\tau)^{1-\frac{\gamma}{2}}}\left[\widetilde{\Gamma}(x,t;\zeta)+\widetilde{\Gamma}(x',t;\zeta)\right], \tag{3.25}$$

for every $\zeta=(\xi,\tau)\in\mathbb{R}^{N+1}$, for every $t,\tau\in I$, $t>\tau$ and for every $x,x'\in\mathbb{R}^N$.

Lemma 3.2. *There exists a constant $k>0$ such that*

$$\left|\partial^2_{x_i,x_j}Z_\zeta(z;w)-\partial^2_{x_i,x_j}Z_{\zeta'}(z;w)\right| \le \frac{k}{t-s}\|\zeta^{-1}\circ\zeta'\|_B^\alpha\,\widetilde{\Gamma}(z;w) \tag{3.26}$$

for every $z,w,\zeta,\zeta'\in\mathbb{R}^{N+1}$ and for every $i,j=1,\ldots,p_0$.

The proof of these lemmas is postponed to the end of this Section.

Proposition 3.2. *For every fixed $\zeta\in\mathbb{R}^{N+1}$ and for every $i,j=1,\ldots,p_0$ the derivatives $\partial^2_{x_i,x_j}J(\cdot\,;\zeta)$ exist and are continuous functions. Moreover*

$$\partial^2_{x_i,x_j}J(z;\zeta) = \int_\tau^t\left(\int_{\mathbb{R}^N}\partial^2_{x_i,x_j}Z(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds. \tag{3.27}$$

Remark 3.2. We cannot assert that the integrating function in (3.27) is absolutely integrable on $\mathbb{R}^N \times]\tau, t[$. However we can consider the integral in (3.27) as a "repeated integral". Indeed, since the function Φ is absolutely integrable, it is sufficient to prove that, for every fixed $t_0 \in]\tau, t[$, the integral

$$\int_{t_0}^{t} \left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s) \Phi(y, s; \zeta)\, dy \right) ds \tag{3.28}$$

converges. For every fixed $s \in]\tau, t[$ and for every $y' \in \mathbb{R}^N$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s) \Phi(y, s; \zeta)\, dy = \\ &= \int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s) \left[\Phi(y, s; \zeta) - \Phi(y', s; \zeta) \right] dy + \\ &+ \Phi(y', s; \zeta) \int_{\mathbb{R}^N} \partial^2_{x_i,x_j} \left[Z_{(y,s)} - Z_{(y',s)} \right] (z; y, s)\, dy + \\ &+ \Phi(y', s; \zeta) \int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z_{(y',s)}(z; y, s)\, dy = \\ &= I'(z; \zeta; y', s) + I''(z; \zeta; y', s) + I'''(z; \zeta; y', s). \end{aligned} \tag{3.29}$$

We next give some estimates for the three addends above. Let $y' = E(s-t)x$. From Corollary 2.1 and Lemma 3.1

$$\begin{aligned} \left| I'(z; \zeta; y', s) \right| \le \int_{\mathbb{R}^N} \frac{c^+}{t-s} \left[1 + \left| D\left((t-s)^{\frac{1}{2}} \right) (x - E(t-s)y) \right|^2 \right] \cdot \\ \cdot \Gamma^+(z; y, s) c \frac{|y - y'|_B^{\gamma'}}{(s-\tau)^{1-\frac{\gamma}{2}}} \left[\widetilde{\Gamma}(y, s; \zeta) + \widetilde{\Gamma}(y', s; \zeta) \right] dy. \end{aligned} \tag{3.30}$$

If $s \in]t_0, t[$, the explicit expression of $\widetilde{\Gamma}$ allows one to derive the existence of $k = k(t_0) > 0$ such that

$$\frac{\widetilde{\Gamma}(y, s; \zeta)}{(s-\tau)^{1-\frac{\gamma}{2}}} \le k \quad \forall (y, s) \in \mathbb{R}^{N+1} : s \ge t_0. \tag{3.31}$$

Moreover, using the identity (see [8], (2.20))

$$D(\lambda) E(t) D\left(\frac{1}{\lambda} J \right) = E(\lambda^2 t) \quad \forall \lambda > 0, \forall t \in \mathbb{R}, \tag{3.32}$$

we can prove that there exists a constant $M = M(T_0, T_1) > 0$ such that

$$(3.33)\qquad \begin{aligned} &\left|y - y'\right|_B^{\gamma'} = \left|y - E(s-t)x\right|_B^{\gamma'} \leq \\ &\leq M(t-s)^{\frac{\gamma'}{2}} \left|D\left((t-s)^{-\frac{1}{2}}\right)(x - E(t-s)y)\right|_B . \end{aligned}$$

Therefore, from (3.30), (3.31), (3.33) it follows that there exists a constant $c'(t_0) > 0$ such that

$$(3.34)\quad \left|I'(z;\zeta;E(s-t)x,s)\right| \leq \frac{c'(t_0)}{(t-s)^{1-\frac{\gamma'}{2}}} \int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\,dy = \frac{c'(t_0)}{(t-s)^{1-\frac{\gamma'}{2}}}.$$

Arguing as above, using Lemma 3.2 instead of Lemma 3.1, we can show that there exists $c''(t_0) > 0$ such that

$$(3.35)\quad \left|I''(z;\zeta;E(s-t)x,s)\right| \leq \frac{c''(t_0)}{(t-s)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\,dy = \frac{c''(t_0)}{(t-s)^{1-\frac{\alpha}{2}}}.$$

To evaluate I''', note that, for every $\bar{z} \in \mathbb{R}^{N+1}$ we have $\int_{\mathbb{R}^N} Z_{\bar{z}}(z;y,s)\,dy = 1$. Then, for every $i, j = 1, \dots, p_0$,

$$(3.36)\qquad \partial^2_{x_i,x_j} \int_{\mathbb{R}^N} Z_{\bar{z}}(z;y,s)dy = 0.$$

On the other hand, using the Lebesgue's Theorem, for every $s \in]\tau, t[$ it holds

$$(3.37)\qquad \partial^2_{x_i,x_j} \int_{\mathbb{R}^N} Z_{\bar{z}}(z;y,s)\,dy = \int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z_{\bar{z}}(z;y,s)\,dy.$$

Setting $\bar{z} = (E(t-s)x, s)$, we obtain $I''' = 0$. This proves the existence of the integral (3.27).

Proof of Proposition 3.2. Arguing exactly as in the proof of Proposition 3.1, it can be shown that

$$(3.38)\qquad \partial^2_{x_i,x_j} J_\varepsilon(z;\zeta) = \int_{\mathbb{R}^N \times]\tau,t[} \partial^2_{x_i,x_j}(Z\eta_\varepsilon)(z;w)\Phi(w;\zeta)\,dw.$$

For every fixed $\zeta = (\xi, \tau)$, for every $T_0, T_1 \in \mathbb{R}$, with $\tau < T_0 < T_1$, let $S_I = \mathbb{R}^N \times]T_0, T_1[$. If

$$\partial^2_{x_i,x_j} J_\varepsilon(z;\zeta) \underset{\varepsilon\to 0}{\overset{z\in S_I}{\rightrightarrows}} \int_\tau^t \left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s)\Phi(y, s; \zeta)\, dy \right) ds, \tag{3.39}$$

then Proposition 3.2 follows from (3.10) and (3.11).

Without loss of generality, we can suppose that $0 < \varepsilon \le \varepsilon_0 = \sqrt{\frac{T_0-\tau}{2}}$. From (3.38) we have

$$\begin{aligned} \partial^2_{x_i,x_j} J_\varepsilon(z;\zeta) - \int_\tau^t \left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s)\Phi(y, s; \zeta)\, dy \right) ds = \\ = \int_\tau^t \left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} \left[(\eta_\varepsilon - 1)\, Z\right](z; y, s)\Phi(y, s; \zeta)\, dy \right) ds = \\ = \int_\tau^t \left(\widetilde{I}'_\varepsilon(z;\zeta;y',s) + \widetilde{I}''_\varepsilon(z;\zeta;y',s) + \widetilde{I}'''_\varepsilon(z;\zeta;y',s) \right) ds. \end{aligned} \tag{3.40}$$

Functions $\widetilde{I}'_\varepsilon$, $\widetilde{I}''_\varepsilon$ and $\widetilde{I}'''_\varepsilon$ are obtained as in (3.29) with Z replaced by $(\eta_\varepsilon - 1)\, Z$.

In order to prove (3.39) it is sufficient to find bounds independent of ε, for $\widetilde{I}'_\varepsilon$ and $\widetilde{I}''_\varepsilon$, as in (3.34) and(3.35). This requirement can be met by showing that

$$\left| \partial^2_{x_i,x_j} \left[(\eta_\varepsilon - 1)\, Z\right](x, t; y, s) \right| \le \frac{k(\varepsilon_0)}{t-s} \widetilde{\Gamma}(x, t; y, s), \tag{3.41}$$

for every $x, y \in \mathbb{R}^N$, $s \in]\tau, t[$, $\varepsilon \in]0, \varepsilon_0[$. As mentioned above

$$\begin{aligned} \left|\partial_{x_i} \eta_\varepsilon(z; w)\right| &\le \frac{m}{\varepsilon} \qquad \forall w \in \mathbb{R}^{N+1}, \\ \partial_{x_i} \eta_\varepsilon(z; w) &= 0 \qquad \forall w \in B(z, \varepsilon/2). \end{aligned}$$

Moreover

$$\|w^{-1} \circ z\|_B = |x - E(t-s)y| + \sqrt{t-s} \le \varepsilon/2 \quad \Rightarrow \quad \frac{1}{\varepsilon} \le \frac{1}{2\sqrt{t-s}},$$

then

$$\left|\partial_{x_i} \eta_\varepsilon(z; w)\right| \le \frac{m}{2\sqrt{t-s}} \qquad \forall w, z \in \mathbb{R}^{N+1}.$$

Analogously

$$\left|\partial^2_{x_i,x_j}\eta_\varepsilon(z;w)\right| \le \frac{m'}{4(t-s)} \qquad \forall w, z \in \mathbb{R}^{N+1}.$$

Using Corollary 2.1, the estimate (3.41) follows.

Since the integrating function on the right hand side of (3.40) is zero for every $(y,s) \in \mathbb{R}^{N+1}$ satisfying $s \ge t - \varepsilon^2$, estimates analogous to (3.34) and (3.35) guarantee that there exist two constants $\beta \in]0,1[$ and $c(\varepsilon_0) > 0$ such that

$$\left|\partial^2_{x_i,x_j} J_\varepsilon(z;\zeta) - \int_\tau^t \left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z;y,s)\Phi(y,s;\zeta)\,dy\right) ds\right| \le$$

$$\le \int_{t-\varepsilon^2}^t \left|\int_{\mathbb{R}^N} \partial^2_{x_i,x_j}\left[(\eta_\varepsilon - 1)\,Z\right](z;y,s)\Phi(y,s;\zeta)\,dy\right| ds \le$$

$$\le \int_{t-\varepsilon^2}^t \frac{c(\varepsilon_0)}{(t-s)^{1-\beta}}\,ds = c'(\varepsilon_0)\varepsilon^\beta.$$

This proves (3.29) and thus, Proposition 3.2. □

Proposition 3.3. *For every $\zeta \in \mathbb{R}^{N+1}$ the derivative*

$$YJ(z;\zeta) = < x, BD_x J(z;\zeta) > -\partial_t J(z;\zeta)$$

exists and is a continuous function with respect to z. Moreover

$$YJ(z;\zeta) = \int_\tau^t \left(\int_{\mathbb{R}^N} YZ(z;y,s)\Phi(y,s;\zeta)\,dy\right) ds - \Phi(z;\zeta). \tag{3.42}$$

Proof. Here, for every $\varepsilon > 0$ we set

$$J_\varepsilon(z;\zeta) = \int_{\mathbb{R}^N \times]\tau, t-\varepsilon[} Z(z;w)\Phi(w;\zeta)\,dw. \tag{3.43}$$

Similarly to (3.10), it can be shown that, for $z, \zeta \in \mathbb{R}^{N+1}$, $z \ne \zeta$,

$$J_\varepsilon(z;\zeta) \underset{\varepsilon\to 0}{\longrightarrow} J(z;\zeta). \tag{3.44}$$

Let $z = (x, t) \in \mathbb{R}^{N+1}$, $\delta > 0$ and define the path

$$\gamma :]-\delta, \delta[\to \mathbb{R}^{N+1}, \quad \gamma(s) = \big(x(s), t(s)\big) = \big(E(s)x, t + s\big). \tag{3.45}$$

From the definition of $E(s)$, it follows

$$\gamma(0) = \big(x, t\big), \quad \dot\gamma(s) = \big(-B^T x(s), 1\big). \tag{3.46}$$

Let $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ be given, with $t > \tau$ and set $\varepsilon > 0$ such that $\varepsilon \le \varepsilon_0 = \frac{t-\tau}{2}$. We next show that

$$YJ_\varepsilon(z;\zeta) = \int_{\mathbb{R}^N \times]\tau, t-\varepsilon[} YZ(z;w)\Phi(w;\zeta)\,dw - \tag{3.47}$$
$$- \int_{\mathbb{R}^N} Z(z; y, t-\varepsilon)\Phi(y, t-\varepsilon;\zeta)\,dy.$$

Consider the path γ defined in (3.45) with $\delta = \frac{\varepsilon}{2}$. We have

$$\frac{J_\varepsilon(\gamma(\sigma);\zeta) - J_\varepsilon(\gamma(0);\zeta)}{\sigma} = \tag{3.48}$$
$$= \int_{\mathbb{R}^N \times]\tau, t-\varepsilon[} \frac{Z(\gamma(\sigma);w) - Z(\gamma(0);w)}{\sigma}\Phi(w;\zeta)\,dw +$$
$$+ \frac{1}{\sigma} \int_{\mathbb{R}^N \times]t-\varepsilon, t+\sigma-\varepsilon[} Z(\gamma(\sigma);w)\Phi(w;\zeta)dw\,.$$

Being $Z(z;w)$ the fundamental solution of L_w, it follows from (3.46) that there exists $\sigma^* \in]-|\sigma|, |\sigma|[$ satisfying

$$\frac{Z(\gamma(\sigma);w) - Z(\gamma(0);w)}{\sigma} = \frac{d}{d\sigma} Z(\gamma(\sigma);w)_{|\sigma=\sigma^*} \tag{3.49}$$
$$= -YZ(\gamma(\sigma^*);w) = \sum_{i,j=1}^{p_0} a_{i,j}(w)\partial^2_{x_i,x_j} Z(\gamma(\sigma^*);w).$$

Using the fact that $a_{i,j}$ are bounded functions and that $t + \sigma^* \ge t - \frac{\varepsilon}{2}$, Corollary 2.1 gives that the function $\sum_{i,j=1}^{p_0} a_{i,j}(w)\partial^2_{x_i,x_j} Z(\gamma(\sigma^*);w)$ is bounded on $\mathbb{R}^N \times]\tau, t-\varepsilon[$. The summability of Φ yields

$$\lim_{\sigma\to 0} \int_{\mathbb{R}^N \times]\tau, t-\varepsilon[} \frac{Z(\gamma(\sigma);w) - Z(\gamma(0);w)}{\sigma}\Phi(w;\zeta)\,dw = \tag{3.50}$$
$$= - \int_{\mathbb{R}^N \times]\tau, t-\varepsilon[} YZ(z;w)\Phi(w;\zeta)\,dw,$$

hence

$$(3.51)\qquad \int_{\mathbb{R}^N} Z(z; y, t-\varepsilon)\Phi(y, t-\varepsilon; \zeta)\, dy - \\ -\frac{1}{\sigma}\int_{t-\varepsilon}^{t+\sigma-\varepsilon}\left(\int_{\mathbb{R}^N} Z(\gamma(\sigma); y, s)\Phi(y, s; \zeta)\, dy\right) ds =$$

$$\left(\text{setting } r = \tfrac{s-t+\varepsilon}{\sigma}\right)$$

$$= \int_0^1\left(\int_{\mathbb{R}^N} [Z(z; y, t-\varepsilon) - Z(\gamma(\sigma); y, t-\varepsilon+r\sigma)]\,\Phi(y, t-\varepsilon; \zeta)\, dy\right) dr + \\ + \int_0^1\left(\int_{\mathbb{R}^N} Z(\gamma(\sigma); y, t-\varepsilon+r\sigma)\big[\Phi(y, t-\varepsilon; \zeta) - \\ - \Phi(y, t-\varepsilon+r\sigma; \zeta)\big]\, dy\right) dr = \widehat{I}'_\sigma(z;\zeta) + \widehat{I}''_\sigma(z;\zeta).$$

Since $\delta = \frac{\varepsilon}{2}$, it follows that $Z(z; y, t-\varepsilon) - Z(\gamma(\sigma); y, t-\varepsilon+r\sigma)$ is a bounded function, while $(y, r) \mapsto \Phi(y, t-\varepsilon; \zeta)$ is an absolutely integrable function on $\mathbb{R}^N \times]0, 1[$. Hence, from (3.45) and the Lebesgue's Theorem,

$$(3.52)\qquad \lim_{\sigma\to 0} \widehat{I}'_\sigma(z;\zeta) = 0.$$

Applying on $\widehat{I}''_\sigma$ the change of variable

$$\eta = D\left(\frac{1}{\sqrt{\varepsilon+(1-r)\sigma}}\right)(E(\sigma)x - E(\varepsilon+(1-r)\sigma)y),$$

with an obvious meaning of the notations we obtain

$$|\widehat{I}''_\sigma(z,\zeta)| \le \widetilde{c}\int_0^1\left(\int_{\mathbb{R}^N} \exp\left(< \widetilde{C}^{-1}(1)\eta, \eta >\right)\cdot \\ \cdot\, |\Phi(y(\eta), t-\varepsilon; \zeta) - \Phi(y(\eta), t-\varepsilon+r\sigma; \zeta)|\, d\eta\right) dr.$$

Since $\Phi(y, t-\varepsilon, \zeta)-\Phi(y, t-\varepsilon+r\sigma, \zeta)$ is a bounded function, from Lebesgue's Theorem it holds

$$\lim_{\sigma\to 0} \widehat{I''_\sigma}(z;\zeta) = 0. \tag{3.53}$$

Then, from (3.51), (3.52) and (3.53) it follows

$$\begin{aligned}\lim_{\sigma\to 0}\frac{1}{\sigma}\int_{t-\varepsilon}^{t+\sigma-\varepsilon}\left(\int_{\mathbb{R}^N} Z(\gamma(\sigma); y, s)\Phi(y, s; \zeta)\,dy\right)ds \\ = \int_{\mathbb{R}^N} Z(z; y, t-\varepsilon)\Phi(y, t-\varepsilon; \zeta)\,dy.\end{aligned} \tag{3.54}$$

Hence (3.48), (3.50) and (3.54) yield (3.47).

Fix now $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, and, for every $T_0, T_1 \in \mathbb{R}$, with $\tau < T_0 < T_1$, put $S_I = \mathbb{R}^N \times]T_0, T_1[$. Let us prove that

$$YJ_\varepsilon(z;\zeta) \underset{\varepsilon\to 0}{\overset{z\in S_I}{\rightrightarrows}} \int_\tau^t\left(\int_{\mathbb{R}^N} YZ(z; y, s)\Phi(y, s; \zeta)\,dy\right)ds - \Phi(z;\zeta). \tag{3.55}$$

Without loss of generality, we can assume $0 < \varepsilon \le \varepsilon_0 = \frac{T_0-\tau}{2}$. As above,

$$\begin{aligned}\int_{t-\varepsilon}^{t}\left(\int_{\mathbb{R}^N} YZ(z; y, s)\Phi(y, s; \zeta)\,dy\right)ds = \\ = -\sum_{i,j=1}^{p_0}\int_{t-\varepsilon}^{t}\left(\int_{\mathbb{R}^N} a_{i,j}(y, s)\partial^2_{x_i,x_j} Z(z; y, s)\Phi(y, s; \zeta)\,dy\right)ds.\end{aligned} \tag{3.56}$$

Using Hypothesis H.3 and Lemma 3.1, we have that $a_{i,j}(y, s)\Phi(y, s; \zeta)$ is uniformly Hölder continuous with respect to y. Following the lines of Proposition 3.2 (see (3.29)), we can show that there exist two constants $c(\varepsilon_0) > 0$ and $\beta \in]0, 1[$ such that

$$\left|\int_{t-\varepsilon}^{t}\left(\int_{\mathbb{R}^N} \partial^2_{x_i,x_j} Z(z; y, s)a_{i,j}(y, s)\Phi(y, s; \zeta)\,dy\right)ds\right| \le \frac{c(\varepsilon_0)}{(t-s)^{1-\frac{\beta}{2}}}, \tag{3.57}$$

for every $z \in S_I$, $s \in]t-\varepsilon, t[$. Hence, from (3.56),

$$\int_{\tau}^{t-\varepsilon}\left(\int_{\mathbb{R}^N} YZ(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds \underset{\varepsilon\to 0}{\overset{z\in S_I}{\rightrightarrows}} \int_{\tau}^{t}\left(\int_{\mathbb{R}^N} YZ(z;y,s)\Phi(y,s;\zeta)\,dy\right)ds. \tag{3.58}$$

Since $\Phi(\cdot\,;\zeta)$ is a continuous an bounded function on S_I, we obtain (see the proof of Proposition 2.2)

$$\int_{\mathbb{R}^N} Z(z;y,t-\varepsilon)\Phi(y,t-\varepsilon;\zeta)\,dy \underset{\varepsilon\to 0}{\overset{z\in S_I}{\rightrightarrows}} \Phi(z;\zeta). \tag{3.59}$$

From (3.47), (3.58) and (3.59) relation (3.55) follows, and therefore, the proof of Proposition 3.3. □

Proof of Proposition 2.3. It is a straightforward consequence of Proposition 3.1, Proposition 3.2 and Proposition 3.3. □

We next prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. We first consider operators of the type (1.1), not in divergence form.

We start by proving that there exists a constant $k > 0$ such that, for every $i, j = 1, \ldots, p_0$

$$\begin{aligned}&\left|\partial^2_{x_i,x_j} Z(x,t;w) - \partial^2_{x_i,x_j} Z(x',t;w)\right| \\ &\qquad \le \frac{k}{(t-s)^{\frac{3}{2}}}\sum_{h=1}^{N}\left|(x-x')_h\right|^{\frac{1}{q_h}}\widetilde{\Gamma}(x,t;w)\end{aligned} \tag{3.60}$$

for every $w \in \mathbb{R}^{N+1}$, for every $t, s \in I$, $t > s$ and for every $x, x' \in \mathbb{R}^N$, satisfying $\left|x-x'\right|_B \le \sqrt{t-s}$. Let $v = x - x'$, $W(x) = \partial^2_{x_i,x_j} Z(x,t;w)$. Since $W \in C^1(\mathbb{R}^N)$, there exists $r \in]0,1[$ such that

$$\begin{aligned}&\partial^2_{x_i,x_j} Z(x,t;w) - \partial^2_{x_i,x_j} Z(x',t;w) = \\ &\qquad = W(x) - W(x') = \frac{d}{ds}W(x+sv)_{|s=r}\,.\end{aligned} \tag{3.61}$$

From the explicit expression of W:

$$W(x) = (t-s)^{-\frac{Q}{2}-1}\left[c_{i,j}(w) + < l_i(w), \eta >< l_j(w), \eta >\right] \cdot \tag{3.62}$$
$$\cdot \exp\left(-\frac{1}{4} < C_w^{-1}\eta, \eta >\right),$$
$$\eta = D\left((t-s)^{-\frac{1}{2}}\right)(x - E(t-s)y),$$

we can compute

$$\partial_{x_h} W(x) = (t-s)^{-\frac{Q}{2}-1-\frac{q_h}{2}} p_h(\eta) \exp\left(-\frac{1}{4} < C_w^{-1}\eta, \eta >\right), \tag{3.63}$$

where p_h is a polynomial whose coefficients only depend on w and are *uniformly bounded* in $\mathbb{R}^{N+1}$. Using (3.63), (3.61) and Lemma 2.2, there exists a function Γ', namely the fundamental solution of some operator L' of type (1.1), and a positive constant c_0 such that

$$\left|W(x) - W(x')\right| = \left|\sum_{h=1}^{N} v_h \partial_{x_h} W(x + rv)\right| \leq \tag{3.64}$$
$$\leq \frac{c_0}{t-s} \sum_{h=1}^{N} \frac{|v_h|}{(t-s)^{\frac{q_h}{2}}} \Gamma'(x + rv, t; w).$$

From the assumption $\left|x - x'\right|_B \leq \sqrt{t-s}$ it holds

$$\frac{|v_h|}{(t-s)^{\frac{q_h}{2}}} = \left(\frac{|x_h - x_h'|^{\frac{1}{q_h}}}{\sqrt{t-s}}\right)^{q_h} \leq \frac{|x_h - x_h'|^{\frac{1}{q_h}}}{\sqrt{t-s}},$$

then

$$\left|W(x) - W(x')\right| \leq \frac{c_0}{(t-s)^{\frac{3}{2}}} \left|x - x'\right|_B \Gamma'(x + rv, t; w). \tag{3.65}$$

To complete the proof of (3.60) it is sufficient th show that there exists a positive constant c_1 such that

$$\Gamma'(x + rv, t; w) \leq c_1 \widetilde{\Gamma}(x, t; w), \tag{3.66}$$

for every $z = (x, t), w = (y, s) \in \mathbb{R}^{N+1}$, with $t > s$, for any $r \in]0, 1[$ and for every $v \in \mathbb{R}^N$ such that $\left|v\right|_B \leq \sqrt{t-s}$. From the explicit expression of the functions Γ' and $\widetilde{\Gamma}$ it follows that (3.66) will hold if, for any given constant symmetric positive definite matrix C and for any given constant $\delta > 0$, there exists a constant $c_2 = c_2(C, \delta) > 0$ such that

$$(3.67) \qquad < C\eta, \eta > \leq (1+\delta) < C(\eta + \eta'), \eta + \eta' > + c_2,$$

for every $\eta, \eta' \in \mathbb{R}^N$, $|\eta'| \leq 1$. The inequality (3.60) will then a consequence of (3.66).

We thus need to show that there exist three constants $k_1 > 0$, $\gamma, \gamma' \in]0, 1[$ such that

$$(3.68) \qquad \begin{aligned} &\left|LZ(x,t;w) - LZ(x',t;w)\right| \leq \\ &\leq \frac{k_1}{(t-s)^{1-\frac{\gamma}{2}}} \left|x - x'\right|_B^{\gamma'} \left[\widetilde{\Gamma}(x,t;w) + \widetilde{\Gamma}(x',t;w)\right] \end{aligned}$$

for every $w \in \mathbb{R}^{N+1}$, for every $x, x' \in \mathbb{R}^N$ and for every $t, s \in I$, $t > s$.

We first consider the case $\left|x - x'\right|_B \leq \sqrt{t-s}$. Then

$$(3.69) \qquad \begin{aligned} &LZ(x,t;w) - LZ(x',t;w) = \\ &= \sum_{i,j=1}^{p_0} \left[a_{i,j}(x,t) - a_{i,j}(x',t)\right] \partial^2_{x_i,x_j} Z(x',t;w) + \\ &+ \sum_{i,j=1}^{p_0} \left[a_{i,j}(x,t) - a_{i,j}(w)\right] \left[\partial^2_{x_i,x_j} Z(x,t;w) - \partial^2_{x_i,x_j} Z(x',t;w)\right]. \end{aligned}$$

From (3.60) and Hypothesis H.3, for every $\beta \in]1-\alpha, 1[$, we obtain

$$(3.70) \qquad \begin{aligned} &\left|\sum_{i,j=1}^{p_0} \left[a_{i,j}(x',t) - a_{i,j}(w)\right] \left[\partial^2_{x_i,x_j} Z(x,t;w) - \partial^2_{x_i,x_j} Z(x',t;w)\right]\right| \leq \\ &\leq M(t-s)^{\frac{\alpha}{2}} \left[1 + \left|D\left((t-s)^{-\frac{1}{2}}\right)(x - E(t-s)y)\right|_B^{\alpha}\right] \cdot \\ &\cdot k \frac{\left|x - x'\right|_B}{(t-s)^{\frac{3}{2}}} \Gamma'(x,t;w) \leq k_1 \frac{\left|x - x'\right|_B^{1-\beta}}{(t-s)^{\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}}} (1 + |\eta'|^{\alpha}) \Gamma'(x,t;w), \end{aligned}$$

where $\eta' = x' - E(t-s)y$. Now, (3.66) and Lemma 2.2 give

$$|\eta'|^{\alpha} \Gamma'(x,t;w) \leq c_1 \widetilde{\Gamma}(x',t;w).$$

Then the last expression in (3.70) is bounded by

$$k_2 \frac{|x - x'|_B^{1-\beta}}{(t-s)^{\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}}} \left[\widetilde{\Gamma}(x, t; w) + \widetilde{\Gamma}(x', t; w)\right].$$

On the other hand, using once more Hypothesis H.3,

$$|a_{i,j}(x, t) - a_{i,j}(x', t)| \le M|x - x'|_B^{\alpha}. \tag{3.71}$$

From Corollary 2.2,

$$\left|\sum_{i,j=1}^{p_0} [a_{i,j}(x, t) - a_{i,j}(x', t)]\, \partial^2_{x_i,x_j} Z(x', t; w)\right| \le \tag{3.72}$$

$$\le k_3 \frac{|x - x'|_B^{\alpha}}{(t-s)} \widetilde{\Gamma}(x', t; w) \le k_4 \frac{|x - x'|_B^{\alpha-\beta'}}{(t-s)^{1-\frac{\beta'}{2}}} \widetilde{\Gamma}(x, t; w)$$

for every $\beta' \in]0, \alpha[$.

If we choose $\gamma = \max\left\{\frac{\beta+\alpha-1}{2}, \frac{\beta'}{2}\right\}$, $\gamma' = \min\left\{\alpha - \beta', 1 - \beta\right\}$, then (3.70) and (3.72) yield (3.68), when $|x - x'|_B \le \sqrt{t-s}$.

If instead $|x - x'|_B > \sqrt{t-s}$, (3.68) is a direct consequence of Corollary 2.2. Substituting (3.68) in the definition of Φ, and applying Lemma 2.3, we immediately obtain Lemma 3.1, for operators of type (1.1), not in divergence form.

Now let M be an operator in divergence form (1.15). Proceeding as in the proof of (3.60), it is easy to show that there exists a constant $k > 0$ such that, for every $j = 1, ..., p_0$

$$|\partial_{x_j} Z(x, t; w) - \partial_{x_j} Z(x', t; w)| \le k \frac{|x - x'|_B}{(t-s)} \widetilde{\Gamma}(x, t; w) \tag{3.73}$$

for every $w \in \mathbb{R}^{N+1}$, for every $t, s \in I$, $t > s$ and for every $x, x' \in \mathbb{R}^N$, such that $|x - x'|_B \le \sqrt{t-s}$. From Hypothesis H.4, functions

$$b_j(z) = \sum_{i=1}^{p_0} \partial_{x_i} a_{i,j}(z),$$

are bounded and B-Hölder continuous, thus from (3.73) and Corollary 2.2 there exists a positive constant k_1 such that, for every $j = 1, ..., p_0$,

$$|b_j(x, t)\partial_{x_j} Z(x, t; w) - b_j(x', t)\partial_{x_j} Z(x', t; w)| \le \tag{3.74}$$

$$\le k_1 \frac{|x - x'|^{1-\alpha}}{(t-s)^{1-\alpha/2}} \left[\widetilde{\Gamma}(x, t; w) + \widetilde{\Gamma}(x', t; w)\right]$$

for every $z = (x, t)$, $w = (y, s) \in \mathbb{R}^{N+1}$, with $t > s$. Note that

$$MZ(z;\zeta) = \sum_{i,j=1}^{p_0} \left[a_{i,j}(z) - a_{i,j}(\zeta)\right] \partial^2_{x_i,x_j} Z(z;\zeta) + \sum_{j=1}^{p_0} b_j(z)\partial_{x_j} Z(z;\zeta)$$
$$= LZ(z;\zeta) + < b(z), DZ(z;\zeta) >$$

and that $b_j(z) = 0$ for $j > p_0$. Then from (3.68) and (3.74), Lemma 3.1 also holds for operators in divergence form. □

Proof of Lemma 3.2. Recall the explicit expression of $\partial^2_{x_i,x_j} Z_\zeta(z;w)$

$$\partial^2_{x_i,x_j} Z_\zeta(z;w) = \frac{c_N}{(t-\tau)^{\frac{Q}{2}+1}\sqrt{\det C_\zeta(1)}} \cdot \tag{3.75}$$
$$\cdot \left[c_{i,j}(\zeta) + < l_i(\zeta), \eta >< l_j(\zeta), \eta >\right] \exp\left(-\frac{1}{4} < C_\zeta^{-1}\eta, \eta >\right),$$

where $c_{i,j}(\zeta)$ is the element of $C_\zeta^{-1}(1)$ at location i, j, $l_i(\zeta)$ is the $i-$th row of $C_\zeta^{-1}(1)$ and $\eta = D\left((t-\tau)^{-\frac{1}{2}}\right)(x - E(t-\tau)\xi)$. Since

$$C_\zeta = C_\zeta(1) = \int_0^1 E^T(s)A(\zeta)E(s)\,ds, \tag{3.76}$$

from Hypothesis H.2 there exists a constant symmetric positive defined matrix C^+, satisfying $C_\zeta^{-1} \le \left(C^+\right)^{-1}$ for every $\zeta \in \mathbb{R}^{N+1}$. Moreover, from Hypothesis H.3, there exists a constant $c > 0$ such that

$$\left\|C_\zeta^{-1} - C_{\zeta'}^{-1}\right\| \le \left\|C_{\zeta'}^{-1}\right\| \cdot \left\|C_{\zeta'} - C_\zeta\right\| \cdot \left\|C_\zeta^{-1}\right\| \le c\left\|\zeta'^{-1} \circ \zeta\right\|_B^\alpha. \tag{3.77}$$

for every $\zeta, \zeta' \in \mathbb{R}^{N+1}$. Then

$$\left|\exp\left(-\frac{1}{4} < C_\zeta^{-1}\eta, \eta >\right) - \exp\left(-\frac{1}{4} < C_{\zeta'}^{-1}\eta, \eta >\right)\right| \le \tag{3.78}$$
$$\le \frac{1}{4}\left|< [C_\zeta^{-1} - C_{\zeta'}^{-1}]\eta, \eta >\right| \exp\left(-\frac{1}{4} < \left(C^+\right)^{-1}\eta, \eta >\right) \le$$
$$\le c'\left\|\zeta'^{-1} \circ \zeta\right\|_B^\alpha |\eta|^2 \exp\left(-\frac{1}{4} < \left(C^+\right)^{-1}\eta, \eta >\right).$$

Moreover, from (3.76) and Hypothesis H.3, the functions $c_{i,j}(\zeta)$, $l_i(\zeta)$, $\frac{1}{\sqrt{\det C_\zeta(1)}}$ are B-Hölder continuous in $\mathbb{R}^{N+1}$.
Then (3.78) and (3.76) prove the Lemma. □

4. Local estimates.

In this Section we shall prove some local estimates for the fundamental solution of the operator L as (1.1), as well as in (1.15), by using its expression given in Section 2 in terms of the parametric $Z(z, \zeta)$.

Let A_0^+, A_0^- be two $p_0 \times p_0$ constant symmetric matrices, such that

$$\lambda I_{p_0} \leq A_0^-, A_0^+ \leq \Lambda I_{p_0}; \quad 0 \leq A_0^+ - A_0^- \leq \mu I_{p_0}. \tag{4.1}$$

As in Lemma 2.1, define the $N \times N$ matrices

$$A^- = \begin{pmatrix} A_0^- & 0 \\ 0 & 0 \end{pmatrix}; \qquad A^+ = \begin{pmatrix} A_0^+ & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, put

$$C = \int_0^1 E(s) \begin{pmatrix} I_{p_0} & 0 \\ 0 & 0 \end{pmatrix} E^T(s)\, ds, \tag{4.2}$$

and denote by c^- and c^+ the minimum and the maximum eigenvalue of C, respectively. The following lemma will be needed

Lemma 4.1. *Let Γ^- and Γ^+ be the fundamental solutions of*

$$L^- = div\left(A^- D\right) + < x, BD > -\partial_t,$$

and

$$L^+ = div\left(A^+ D\right) + < x, BD > -\partial_t,$$

respectively. Then, for every $K > 0$ there exists a constant $c_K > 0$, depending only on K, on matrix B and on the constants λ, Λ, μ in (4.1), such that

$$\Gamma^+(z, \zeta) \leq c_K (t - \tau)^{-\frac{Q\delta}{2}} \Gamma^-(z, \zeta) \tag{4.3}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$ satisfying $\Gamma^+(z, \zeta) \geq K$, where

$$\delta = \mu \Lambda \left(\frac{c^+}{\lambda c^-} \right)^2. \tag{4.4}$$

Proof. Set

$$C^- = \int_0^1 E(s)A^-E^T(s)\,ds; \quad C^+ = \int_0^1 E(s)A^+E^T(s)\,ds. \tag{4.5}$$

From (4.1)

$$\lambda C \le C^-, C^+ \le \Lambda C; \quad 0 \le C^+ - C^- \le \mu C. \tag{4.6}$$

Moreover, from

$$\left(C^-\right)^{-1} - \left(C^+\right)^{-1} = \left(C^-\right)^{-1}\left[C^+ - C^-\right]\left(C^+\right)^{-1}, \tag{4.7}$$

it follows

$$\left\|\left(C^-\right)^{-1} - \left(C^+\right)^{-1}\right\| \le \left\|\left(C^-\right)^{-1}\right\| \left\|C^+ - C^-\right\| \left\|\left(C^+\right)^{-1}\right\| \le \frac{\mu c^+}{(\lambda c^-)^2}. \tag{4.8}$$

Hence, for every $\eta \in \mathbb{R}^N$,

$$\begin{aligned} 0 \le &< \left[\left(C^-\right)^{-1} - \left(C^+\right)^{-1}\right]\eta, \eta > \le \frac{\mu c^+}{(\lambda c^-)^2}|\eta|^2 \le \\ &\le \mu\Lambda\left(\frac{c^+}{\lambda c^-}\right)^2 < \left(C^+\right)^{-1}\eta, \eta > . \end{aligned} \tag{4.9}$$

Letting $\eta = D\left(\frac{1}{\sqrt{t-\tau}}\right)\left(x - E(t-\tau)\xi\right)$, $c_N^+ = (4\pi)^{-N/2}\left(\det C^+\right)^{-1/2}$ and $c_N^- = (4\pi)^{-N/2}\left(\det C^-\right)^{-1/2}$, we have

$$\begin{aligned} &\exp\left(-\frac{1}{4} < \left[\left(C^+\right)^{-1} - \left(C^-\right)^{-1}\right]\eta, \eta >\right) \le \\ &\le \exp\left(\frac{\delta}{4} < \left(C^+\right)^{-1}\eta, \eta >\right)\left(\frac{c_N^+(t-\tau)^{Q/2}}{c_N^+(t-\tau)^{Q/2}}\right)^\delta = \\ &= (c_N^+)^\delta (t-\tau)^{-\frac{Q\delta}{2}}\left(\Gamma^+(z,\zeta)\right)^{-\delta} \le \left(\frac{c_N^+}{K}J\right)^\delta (t-\tau)^{-\frac{Q\delta}{2}}, \end{aligned} \tag{4.10}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $\Gamma^+(z,\zeta) \ge K$.

Relation (4.3) is a direct consequence of (4.10) and of the explicit expression of $\Gamma^-$and Γ^+, after setting

$$c_K = (c_N^+)^{\delta-1}(c_N^-)K^{-\delta}. \qquad \square \tag{4.11}$$

Proof of Theorem 1.2. We first consider operators of the type (1.1), not in divergence form.

Suppose $z_0 = 0$. Note that this is not a restrictive assumption, since the estimate we shall prove only depends on the constants defined in Hypotheses H.1-H.3. These Hypotheses are invariant with respect to the left translation of the group $(\mathbb{R}^{N+1}, \circ)$; this implies that the result holds for every $z_0 \in \mathbb{R}^{N+1}$.

Since $z_0 = 0$, we shall use the simpler notation $Z(z) \equiv Z(z, 0)$.

Let ψ be a decreasing function of class $C^1(\mathbb{R})$, such that $\psi(t) = 0$ for every $t \geq 1$, $\psi(t) = 1$ for every $t \leq 1/2$. For any fixed $\rho > 0$ and for every $i, j = 1, ..., p_0$, let

$$a^{\rho}_{i,j}(z) = a_{i,j}(0) + \psi\left(\frac{\|z\|_B}{\rho}\right)\left[a_{i,j}(z) - a_{i,j}(0)\right], \tag{4.12}$$

$$L^{\rho} = \sum_{i,j=1}^{p_0} a^{\rho}_{i,j}(z)\partial_{x_i x_j} + < x, BD > -\partial_t. \tag{4.13}$$

Function ψ is bounded and belongs to C^1, thus Proposition 2.1 ensures that functions $a^{\rho}_{i,j}$ are B-Hölder continuous.The fundamental solution Γ^{ρ} of L^{ρ} can be determined using the function $Z^{\rho}(z, \zeta)$ as parametrix, and then setting

$$\Phi^{\rho}(z; \zeta) = \sum_{k=1}^{\infty}(L^{\rho}Z^{\rho})_k(z; \zeta).$$

For any fixed $\rho > 0$, which will be chosen in the sequel, we have

$$\begin{aligned}\Gamma(z) &= Z(z) + \int_0^t \left(\int_{\mathbb{R}^n} Z^{\rho}(z; y, s)\Phi^{\rho}(y, s)\, dy\right) ds + \\ &+ \int_0^t \left(\int_{\mathbb{R}^n} \left[Z(z; y, s) - Z^{\rho}(z; y, s)\right]\Phi(y, s)\, dy\right) ds + \\ &+ \int_0^t \left(\int_{\mathbb{R}^n} Z^{\rho}(z; y, s)\left[\Phi(y, s) - \Phi^{\rho}(y, s)\right] dy\right) ds = \\ &= Z(z) + I(z) + II(z) + III(z).\end{aligned} \tag{4.14}$$

We next evaluate $I(z)/Z(z)$, $II(z)/Z(z)$ and $III(z)/Z(z)$ on the set

$$O_K = \left\{ z \in \mathbb{R}^{N+1} : Z(z) \geq K \right\}. \tag{4.15}$$

Let Λ be the constant of Hypothesis H.2; let c^- and c^+ be the minimum and the maximum eigenvalue of the matrix C defined in (4.2). Put

$$\mu = \frac{\alpha (c^-)^2}{\Lambda^3 Q (c^+)^2} \tag{4.16}$$

and apply Lemma 4.1 to matrices

$$A_0^- = A_0^\rho(0), \qquad A_0^+ = A_0^\rho(0) + \mu I_{p_0}.$$

For any $z \in O_K^+ = \{\zeta : \Gamma^+(\zeta) \geq K\}$, it holds

$$\Gamma^+(z) \leq c_K t^{-\frac{Q\delta}{2}} \Gamma^-(z) = \frac{c_0}{K^{\frac{\alpha}{Q}}} t^{-\frac{\alpha}{2}} Z^\rho(z). \tag{4.17}$$

Using the B-Hölder continuity of the coefficients of A_0, we can choose $\rho > 0$, only depending on the constants of Hypotheses H.1-H.3, so that

$$A_0^\rho(z) \leq A_0(0) + \frac{\mu}{2} I_{p_0} < A_0^+ \qquad \forall z \in \mathbb{R}^{N+1}. \tag{4.18}$$

From Proposition 2.4 and Lemma 2.3, using Γ^+ instead of $\widetilde{\Gamma}$, we have that, for every $z \in \mathbb{R}^{N+1}$,

$$|I(z)| \leq \int_0^t \frac{\widetilde{c}\widetilde{k}}{s^{1-\alpha/2}} \Gamma^+(z)\, ds = c_\rho t^{\alpha/2} \Gamma^+(z). \tag{4.19}$$

Hence (4.19) and (4.17) yield

$$|I(z)| \leq \frac{c_0 c_\rho}{K^{\frac{\alpha}{Q}}} Z^\rho(z) \qquad \forall z \in O_K^+. \tag{4.20}$$

Moreover there exists $r_0 = r_0(\rho) > 0$ such that, for every $r > r_0$,

$$O_r^+ \subset B(0, \rho/2) = \left\{ z \in \mathbb{R}^{N+1} : \|z\|_B \leq \rho/2 \right\}, \tag{4.21}$$

and, from Proposition 2.4,

$$O_r \subset O_{r/c^+}^+ \qquad \forall r > 0. \tag{4.22}$$

Since $Z(z) = Z^\rho(z)$, from (4.20), (4.21), (4.22) it follows that, for every $\varepsilon > 0$ there exists a constant $K = K(\varepsilon, \rho)$ such that

$$|I(z)| \leq \frac{\varepsilon}{3} Z(z) \qquad \forall z \in O_K. \tag{4.23}$$

In order to evaluate II and II in (4.14), we need to show that there exists a positive constant c_ρ such that

$$|II(z)| \leq c_\rho, \quad |III(z)| \leq c_\rho \quad \forall z \in \mathbb{R}^{N+1}. \tag{4.24}$$

Since L and L^ρ verify Hypothesis H.2 (with the same constant Λ, independent of ρ), from Proposition 2.4 there exists $\widetilde{c} > 0$ such that

$$Z^\rho(z;\zeta), Z(z;\zeta) \leq \widetilde{c}\,\widetilde{\Gamma}(z;\zeta). \tag{4.25}$$

Moreover

$$Z(z;\zeta) = Z^\rho(z;\zeta) \qquad \forall z, \zeta \in \mathbb{R}^{N+1} : \|\zeta\|_B \leq \rho/2, \tag{4.26}$$

therefore

$$|Z(z;\zeta) - Z^\rho(z;\zeta)| \leq \widetilde{c} \sup_{\|\zeta\|_B \geq \rho/2} \widetilde{\Gamma}(\zeta) = m_\rho < \infty. \tag{4.27}$$

On the other hand Corollary 2.4 yields

$$|\Phi(\zeta)| \leq \frac{\widetilde{k}}{\tau^{1-\frac{\alpha}{2}}} \widetilde{\Gamma}(\zeta),$$

hence

$$|II(z)| \leq m_\rho \widetilde{k} \int_0^t \tau^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} \widetilde{\Gamma}(z; y, s)\, dy \right) d\tau = c_\rho t^{\alpha/2}. \tag{4.28}$$

The first bound in (4.24) follows from (4.28), since from (4.21) and (4.22) we have $O_K \subset B(0, \rho/2)$.

The second bound of (4.24) can be obtained in a similar manner, by replacing (4.27) with $\square$

Lemma 4.2. *There exists a positive constant b_ρ such that*

$$\left|(\Phi - \Phi_\rho)(\zeta)\right| \leq b_\rho. \tag{4.29}$$

Proof. In proving statements (i) and (ii) of Proposition 2.2, we showed that there exists a positive integer k_0 such that $(LZ)_k$, $(L^\rho Z^\rho)_k$ are bounded functions for every $k > k_0$, and the series

$$\sum_{k=k_0+1}^{\infty} (LZ)_k(z;\zeta), \qquad \sum_{k=k_0+1}^{\infty} (L^\rho Z^\rho)_k(z;\zeta)$$

converge uniformly for $z, \zeta \in S_I$.

To prove the lemma it is sufficient to show that there exist some positive constants $M_1, M_2, \ldots, M_{k_0}$ such that

$$|(LZ - L^\rho Z^\rho)_k(\zeta;0)| \le M_k \quad \forall k = 1, \ldots, k_0, \forall \zeta \in \mathbb{R}^{N+1}. \tag{4.30}$$

Using Lemma 2.3, for $k = 1, \ldots, k_0$

$$|(LZ)_k(z)|,\ |(L^\rho Z^\rho)_k(z)| \le \frac{c_k}{t^{1-\alpha/2}} \widetilde{\Gamma}(z) \le \widetilde{c}_k \tag{4.31}$$

for every $z \in \mathbb{R}^{N+1}$ such that $\|Jz\|_B \ge \rho/2$, thus it is sufficient to prove (4.30) for $z \in B(0, \rho/2)$.

We proceed by induction: if $k = 1$, relation (4.30) follows from noting that $(LZ)(z) = (L^\rho Z^\rho)(z)$, $\forall z \in B(0, \rho/2)$.

Suppose now that (4.30) holds for k and evaluate

$$\begin{aligned}
&(LZ)_{k+1}(z) - (L^\rho Z^\rho)_{k+1}(z) = \\
&\quad = \int_0^\tau \left(\int_{\mathbb{R}^n} LZ(z;y,s) \left[(LZ)_k - (L^\rho Z^\rho)_k \right](y,s)\, dy \right) ds + \\
&\quad + \int_0^t \left(\int_{\mathbb{R}^n} \left[LZ - (L^\rho Z^\rho) \right](z;y,s) (L^\rho Z^\rho)_k(y,s)\, dy \right) ds = \\
&\quad = J_1(\zeta) + J_2(\zeta).
\end{aligned}$$

From induction hypothesis and from Lemma 2.3 we have

$$|J_1(z)| \le \widetilde{c} M_k \int_0^t \frac{1}{(t-s)^{1-\alpha/2}} \left(\int_{\mathbb{R}^n} \Gamma^+(z;y,s)\, dy \right) ds = c' M_k t^{\alpha/2}. \tag{4.32}$$

To evaluate J_2, note that

$$LZ(z;y,s) = (L^\rho Z^\rho)(z;y,s) \quad \forall (y,s) \in B(0, \rho/2),$$

and, from (4.31),

$$|(L^\rho Z^\rho)_k(y,s)| \leq \widetilde{c}_k \quad \forall (y,s) \in \mathbb{R}^{N+1} \backslash B(0,\rho/2).$$

Hence, from Lemma 2.3

$$|J_2(z)| \leq 2\widetilde{c}\,\widetilde{c}_k \int_0^t \frac{1}{s^{1-\alpha/2}} \left(\int_{\mathbb{R}^n} \widetilde{\Gamma}(z; y,s)\, dy \right) ds = c_k' t^{\alpha/2}. \tag{4.33}$$

Then (4.30) holds for $k+1$, as a consequence of inequalities (4.32) and (4.33).

This completes the proof of Lemma, and then the proof of Theorem 1.2, for operators of type (1.1). □

In order to extend this result to operators M in divergence form (1.15), we only need to observe that if ψ belongs to $C^2(\mathbb{R})$, then all corresponding operators M^ρ defined in terms of the functions $a_{i,j}^\rho$ in (4.12) verify Hypothesis H.4 for some constant which may depend on ρ. On the other hand, since the selection of ρ only depends on the constants of Hypotheses H.2 and H.3, and these ones can be chosen for M^ρ independently of ρ, Theorem 1.2 holds for operators (1.15).

Next result will be used in Section 5:

Proposition 4.1. *There exist two positive constants c, K_0 such that, for every $z, \zeta \in \mathbb{R}^{N+1}$ satisfying $\Gamma(z,\zeta) \geq K_0$, and for any $i = 1, \ldots, p_0$,*

$$\left|\partial_{x_i} \Gamma(z,\zeta)\right| \leq c \left[\frac{\left| D\big((t-\tau)^{-1/2}\big)\big(x - E(t-\tau)\xi\big) \right|}{\sqrt{t-\tau}} + 1 \right] \Gamma(z,\zeta)\,. \tag{4.34}$$

Proof. Assume $z_0 = 0$ and, as in the proof of Theorem 1.2, choose $\rho > 0$ such that (4.18) holds. For any $\varepsilon \in]0,1[$, let $K = K(\varepsilon)$ be a positive constant such that (1.13) holds on the set $O_K = \{z \in \mathbb{R}^{N+1} : Z(z) \geq K\}$ (see Theorem 1.2). Then, for $i = 1, \ldots, p_0$,

$$\begin{aligned} \partial_{x_i}\Gamma(z) &= \partial_{x_i} Z(z) + \int_0^t \left(\int_{\mathbb{R}^n} \partial_{x_i} Z^\rho(z; y,s) \Phi^\rho(y,s)\, dy \right) ds + \\ &+ \int_0^t \left(\int_{\mathbb{R}^n} \partial_{x_i} \left[Z(z;y,s) - Z^\rho(z;y,s) \right] \Phi(y,s)\, dy \right) ds + \\ &+ \int_0^t \left(\int_{\mathbb{R}^n} \partial_{x_i} Z^\rho(z;y,s) \left[\Phi(y,s) - \Phi^\rho(y,s) \right] dy \right) ds = \\ &= \partial_{x_i} Z(z) + I'(z) + II'(z) + III'(z). \end{aligned} \tag{4.35}$$

Note that there exists $c_0 > 0$, depending only on the constant Λ in Hypothesis H.2, such that

$$|\partial_{x_i} Z(z;\zeta)| \le \frac{c_0}{\sqrt{t-\tau}} \left| D\left(\frac{1}{\sqrt{t-\tau}}\right)\left(x - E(t-\tau)\xi\right)\right| Z(z;\zeta) \tag{4.36}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$. From (4.36), and following the lines of the proof of Theorem 1.2, it is possible to determine two positive constants c_1, c_2, depending only on ρ, such that

$$\begin{aligned} &|I'(z)| \le c_1 t^{\alpha/2}\widetilde{\Gamma}(z), \\ &|II'(z)|;\ |III'(z)| \le c_2 t^{\alpha/2}. \end{aligned} \tag{4.37}$$

Finally, from Lemma 2.2 and (4.17), the inequality

$$|I'(z)| \le c_1 t^{\alpha/2}\widetilde{\Gamma}(z) \le \frac{c_0 c_1}{K^{\frac{\alpha}{Q}}} Z(z) \tag{4.38}$$

holds for every $z \in O_K$.

Using (4.35), (4.36), (4.37), (4.38) and Theorem 1.2,

$$\begin{aligned} |\partial_{x_i}\Gamma(z)| &\le \left[\frac{c_0}{\sqrt{t}}\left|D(t^{-1/2})x\right| + \frac{c_0 c_1}{K^{\alpha/Q}} + \frac{2c_2}{K}\right] Z(z) \le \\ &\le c\left[\frac{\left|D(t^{-1/2})x\right|}{\sqrt{t}} + 1\right]\Gamma(z) \end{aligned} \tag{4.39}$$

for every $z \in O_K$. □

5. Harnack inequality.

In order to prove the Harnack inequality stated in Theorem 1.3, we need to introduce some mean value formulas for solutions of $Lu = 0$, where L is an operator in divergence form (1.15). Such mean value formulas can be obtained from the Green's identity, as for classical parabolic operators (see [8], Section 4).

For every $z_0 \in \mathbb{R}^{N+1}$ and for any $r > 0$ let

$$\Omega_r(z_0) = \left\{z \in \mathbb{R}^{N+1} : \Gamma(z_0; z) > \frac{1}{r}\right\}, \tag{5.1}$$

$$M(z_0; z) = \frac{< A(z) D_x\Gamma(z_0; z), D_x\Gamma(z_0; z) >}{\Gamma^2(z_0; z)}. \tag{5.2}$$

Then the following mean value formula holds

Proposition 5.1. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let u be a solution of $Lu = 0$. Then, for every $z_0 \in \Omega$ such that $\overline{\Omega_r(z_0)} \subset \Omega$, we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r(z_0)} M(z_0; z) u(z)\, dz. \tag{5.3}$$

By means of the Hadamard's *descent method*, relation (5.3) is a starting point for the derivation of mean value formulas with more regular kernel (see [8], Section 4 and [7]). The descent method relies on the following remark.

If $u = u(x, t)$ is a solution of (1.15), then, for every $m \in \mathbb{N}$, the function

$$\widetilde{u}(y, x, t) = u(x, t), \qquad y \in \mathbb{R}^m \tag{5.4}$$

is a solution of

$$\widetilde{L}\widetilde{u} = \left(\Delta_y + L\right)\widetilde{u} = 0. \tag{5.5}$$

Operators $\widetilde{L}$ and L are of the same type, thus a mean value formula analogous to (5.3) holds

$$\widetilde{u}(y_0, z_0) = \frac{1}{r} \int_{(\widetilde{\Omega})_r(y_0, z_0)} \widetilde{M}\left(y_0, z_0; y, z\right) \widetilde{u}(y, z)\, dy dz, \tag{5.6}$$

where

$$\widetilde{\Omega}_r(y_0, z_0) = \left\{ (y, z) \in \mathbb{R}^{m+N+1} : \widetilde{\Gamma}\left(y_0, z_0; y, z\right) > \frac{1}{r} \right\}, \tag{5.7}$$

$$\widetilde{M}\left(y_0, z_0; y, z\right) = M\left(z_0; z\right) + \frac{|y_0 - y|^2}{4(t_0 - t)^2}. \tag{5.8}$$

Moreover, if u is a solution of $Lu = 0$, we can apply the mean value formula (5.6) on $\widetilde{u}$ as defined in (5.4)

$$u(z_0) = \frac{1}{r} \int_{(\widetilde{\Omega})_r(y_0, z_0)} \widetilde{M}\left(y_0, z_0; y, z\right) u(z)\, dy dz, \tag{5.9}$$

and since u only depends on z, we can integrate (5.9) with respect to y. Letting, for every $m \in \mathbb{N}$, $r > 0$,

$$\Gamma^{(m)}(z_0; z) = \left(4\pi(t_0 - t)\right)^{-\frac{m}{2}} \Gamma(z_0; z), \tag{5.10}$$

$$\Omega_r^{(m)}(z_0) = \left\{ z \in \mathbb{R}^{N+1} : \Gamma^{(m)}(z_0; z) > \frac{1}{r} \right\}, \tag{5.11}$$

$$N_r(z_0; z) = 2\sqrt{t_0 - t}\sqrt{\log\left(r\Gamma^{(m)}(z_0; z)\right)}, \tag{5.12}$$

$$M_r^{(m)}(z_0; z) = \omega_m N_r^m(z_0; z)\left[M(z_0; z) + \frac{m}{m+2}\frac{N_r^2(z_0; z)}{4(t_0 - t)^2} \right], \tag{5.13}$$

where ω_m denotes the measure of the unit ball in $\mathbb{R}^m$, we have

Proposition 5.2. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let u be a solution of $Lu = 0$. Then, for any $m \in \mathbb{N}$, for every $z_0 \in \Omega$ such that $\overline{\Omega_r^{(m)}(z_0)} \subset \Omega$, we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r^{(m)}(z_0)} M_r^{(m)}(z_0; z) u(z)\, dz. \tag{5.14}$$

Before proving Theorem 1.3, we would like to point out that Theorem 1.2 and Proposition 4.1 also hold relatively to the sets $\Omega_r^{(m)}$ as defined in (5.11).

Proposition 5.3. *For any $m \in \mathbb{N}$ and for every $\varepsilon > 0$ there exist two positive constants c, δ, depending only on the quantities in Hypotheses H.1-H.4 and on matrix B, such that*

$$(1 - \varepsilon) Z(z_0; z) \leq \Gamma(z_0; z) \leq (1 + \varepsilon) Z(z_0; z), \tag{5.15}$$

$$\left|\partial_{x_i} \Gamma(z_0; z)\right| \leq \leq c \left[\frac{\left| D\left((t_0 - t)^{-1/2}\right)\left(x_0 - E(t_0 - t)x\right) \right|}{\sqrt{t_0 - t}} + 1 \right] \Gamma(z_0; z), \tag{5.16}$$

for any $i, j = 1, 2, \ldots, p_0$, for every $z_0 \in \mathbb{R}^{N+1}$ and for every $z \in \Omega_\delta^{(m)}(z_0)$.

Proof. Denote by $\widetilde{\Gamma}$ and $\widetilde{Z}$ the fundamental solution and the parametric of operator $\widetilde{L}$ in (5.5), respectively. From Theorem 1.2 there exists $k > 0$ such that

$$(1-\varepsilon)\widetilde{Z}(0, z_0; y, z) \leq \widetilde{\Gamma}(0, z_0; y, z) \leq (1+\varepsilon)\widetilde{Z}(0, z_0; y, z) \tag{5.17}$$

for every $(y, z) \in \{w \in \mathbb{R}^{m+N+1} : \widetilde{\Gamma}(0, z_0; w) > k\} = \widetilde{\Omega}_{\frac{1}{k}}(0, z_0)$. Moreover

$$\widetilde{Z}(0, z_0; 0, z) = \left(4\pi(t_0 - t)\right)^{-\frac{m}{2}} Z(z_0; z), \tag{5.18}$$

and, if $H(y; t)$ is the fundamental solution of the heat equation $\Delta_y - \partial_t = 0$,

$$\widetilde{\Gamma}(0, z_0; 0, z) = H(0; t_0 - t)\Gamma(z_0; z) = \left(4\pi(t_0 - t)\right)^{-\frac{m}{2}} \Gamma(z_0; z). \tag{5.19}$$

Putting $\delta = \frac{1}{k}$, then (5.10) and (5.11) yield $\Omega_\delta^{(m)}(z_0) = \widetilde{\Omega}_\delta(0, z_0) \cap \mathbb{R}^{N+1}$, and thus

$$(1-\varepsilon)\widetilde{Z}(0, z_0; 0, z) \leq \widetilde{\Gamma}(0, z_0; 0, z) \leq (1+\varepsilon)\widetilde{Z}(0, z_0; 0, z)$$

for every $y \in \Omega_\delta^{(m)}(z_0)$. Hence, by using (5.17) and (5.18) the estimate (5.15) follows.

In order to prove (5.16), we first observe that differentiation in (5.16) is carried out with respect to the "adjoint" variables of Γ, whereas in Proposition 4.1 differentiation was done with respect to its "principal" variables.

This problem can be overcome by working with the adjoint operator L^* of L, defined as

$$L^* = \operatorname{div}\left(A(z)D\right) - Y. \tag{5.20}$$

Note that L^* is a *backward* operator in the t variable and that it satisfies Hypotheses H.1-H.4. By means of an obvious adaptation of the parametrix method (see Theorem 1.1), we are able to construct the fundamental solution Γ^* of L^*. We then proceed as in the classical parabolic case (see [3], Chap. I, Theorem 15). Using the Green's identity and the basic properties of fundamental solutions we can prove that

$$\Gamma^*(z_0; z) = \Gamma(z; z_0) \qquad \forall z, z_0 \in \mathbb{R}^{N+1}, z \neq z_0. \tag{5.21}$$

We can now prove relation (5.16). Let

$$\widetilde{L}^* = \left(\Delta_y + L^*\right). \tag{5.22}$$

For every $z \in \Omega_\delta^{(m)}(z_0)$ we have (see Proposition 4.1)

$$H^*\big(0; t-t_0\big)\big|\partial_{x_i}\Gamma^*\big(z; z_0\big)\big| = \big|\partial_{x_i}\widetilde{\Gamma}^*\big(0, z; 0, z_0\big)\big| \leq \tag{5.23}$$
$$\leq c\left[\frac{\big|\widetilde{D}\big((t-t_0)^{-1/2}\big)\big((0,x)-\widetilde{E}(t-t_0)(0,x_0)\big)\big|}{\sqrt{t-t_0}}+1\right]\widetilde{\Gamma}\big(0, z; 0, z_0\big) =$$
$$= c\,H^*\big(0; t-t_0\big)\left[\frac{\big|D\big((t-t_0)^{-1/2}\big)\big(x-E(t-t_0)x_0\big)\big|}{\sqrt{t-t_0}}+1\right]\Gamma^*\big(z; z_0\big).$$

which, together with (5.21), prove the result. □

Fix $m \in \mathbb{N}$ and, for every $r > 0$, $z_0 \in \mathbb{R}^{N+1}$, denote by $\Omega_r^{(m)-}(z_0)$ and $\Omega_r^{(m)+}(z_0)$ the level sets (5.11) corresponding to operators L^- and L^+ as defined in Proposition 2.4.

Lemma 5.1. *For every $m \in \mathbb{N}$ there exist three constants $c^-, c^+, s_0 > 0$ such that*

$$\Omega_{sc^-}^{(m)-}(z_0) \subset \Omega_s^{(m)}(z_0) \subset \Omega_{sc^+}^{(m)+}(z_0) \tag{5.24}$$

for every $z_0 \in \mathbb{R}^{N+1}$ and for every $s \in]0, s_0]$.

Proof. As in (5.5), let

$$\widetilde{L} = \big(\Delta_y + L\big), \quad \widetilde{L^+} = \big(\Delta_y + L^+\big), \quad \widetilde{L^-} = \big(\Delta_y + L^-\big). \tag{5.25}$$

Apply Theorem 1.2 to the corresponding fundamental solutions $\widetilde{\Gamma}$, $\widetilde{\Gamma^+}$, $\widetilde{\Gamma^-}$, with $\varepsilon = \frac{1}{2}$, and subsequently apply Proposition 2.4. Then there exist three positive constants c^-, c^+, k_0 such that

$$c^-\widetilde{\Gamma^-}\big(y_0, z_0; y, z\big) \leq \frac{1}{2}\widetilde{Z}\big(y_0, z_0; y, z\big) \leq \widetilde{\Gamma}\big(y_0, z_0; y, z\big) \leq \tag{5.26}$$
$$\leq \frac{3}{2}\widetilde{Z}\big(y_0, z_0; y, z\big) \leq c^+\widetilde{\Gamma^+}\big(y_0, z_0; y, z\big)$$

for every $(y_0, z_0), (y, z) \in \mathbb{R}^{m+N+1}$ satisfying $\widetilde{\Gamma}\big(y_0, z_0; y, z\big) \geq k_0$.

From (5.26), with an obvious meaning of notation, it follows

$$c^-\Gamma^{(m)-}(z_0; z) \leq \frac{1}{2}Z^{(m)}(z_0; z) \leq \Gamma^{(m)}(z_0; z)$$
$$\leq \frac{3}{2}Z^{(m)}(z_0; z) \leq c^+\Gamma^{(m)+}(z_0; z)$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma^{(m)}(z_0; z) \geq k_0$, from which (5.24) follows. $\square$

Proof of Theorem 1.3. Fix $m \in \mathbb{N}, m \geq 3$ and let $c^-, c^+, s_0 > 0$ be the constants determined in Lemma 5.1, and put

$$r_0 = \inf\left\{r > 0 : \Omega^{(m)+}_{s_0 c^+} \subset H_r\right\}, \tag{5.27}$$

where H_r was defined in (1.16). Since $\Omega^{(m)+}_{s_0 c^+}$ is a bounded set, the definition of r_0 is well posed and, by continuity, we have $\Omega^{(m)+}_{s_0 c^+} \subset H_{r_0}$. Suppose that u is a non-negative solution of $Lu = 0$ in Ω and that, for a fixed $z_0 \in \Omega$ and for some $r \in]0, r_0]$, we have $H_r \subset \Omega$. We remark that it is not restrictive to assume $z_0 = 0$, since it will be shown that c and θ do not depend on z_0.

Since Γ^- and Γ^+ are invariant with respect the dilation group $\left(D(\lambda)\right)_{\lambda>0}$,

$$\begin{aligned} \Omega^{(m)-}_{r\lambda}(0) &= D\left(\lambda^{\frac{1}{Q+m}}\right)\Omega^{(m)-}_{r}(0), \\ \Omega^{(m)+}_{r\lambda}(0) &= D\left(\lambda^{\frac{1}{Q+m}}\right)\Omega^{(m)+}_{r}(0). \end{aligned} \tag{5.28}$$

Hence, if we put $s = s_0\left(\frac{r}{r_0}\right)^{Q+m}$, from Lemma 5.1

$$\Omega^{(m)-}_{sc^-}(0) \subset \Omega^{(m)}_{s}(0) \subset \Omega^{(m)+}_{sc^+}(0) \subset H_r. \tag{5.29}$$

For every $\lambda > 0$, set

$$L_\lambda = \operatorname{div}\left(A(D(\lambda)z)D\right) + Y \tag{5.30}$$

and $u_\lambda(z) = u(D(\lambda)z)$. Then

$$Lu = 0 \text{ in } \Omega \iff L_\lambda u_\lambda = 0 \text{ in } D\left(\frac{1}{\lambda}\right)\Omega. \tag{5.31}$$

In the sequel we will use the following remark. For every λ_0, L_λ verifies Hypotheses H.2-H.4 uniformly in $\lambda \in]0, \lambda_0[$. Indeed, from Hypothesis H.3 and (1.9),

$$\begin{aligned} \left|a_{i,j}(D(\lambda)z) - a_{i,j}(D(\lambda)\zeta)\right| &\leq M\left\|(D(\lambda)\zeta)^{-1} \circ (D(\lambda)z)\right\|_B = \\ &= M\left\|D(\lambda)\left(\zeta^{-1}\circ z\right)\right\|_B = M\lambda\left\|\zeta^{-1}\circ z\right\|_B. \end{aligned}$$

A similar bound, with λ replaced by λ^2, holds for the derivatives $\partial_{x_j} a_{i,j}(D(\lambda)z)$, $1 \leq i, j \leq p_0$.

In order to prove Theorem 1.3 we need to show that, if we put $\lambda_0 = s_0^{\frac{1}{Q+m}}$, there exist two constants $c > 0$ and $\theta \in]0, 1[$ such that, for every $\lambda \in]0, \lambda_0]$ and for every non-negative solution v of $L_\lambda v = 0$ in $D\left(\frac{1}{\lambda}\right)\Omega$, we have

$$\sup_{H_\theta^-} v \leq c v(0). \tag{5.32}$$

Let us denote by $\Omega_{s,\lambda}^{(m)}$ the level set $\Omega_s^{(m)}$ corresponding to the operator L_λ and by M_λ, $N_{r,\lambda}$, $M_{r,\lambda}^{(m)}$ the functions defined in (5.2), (5.12) and (5.13), respectively, corresponding to L_λ. The fundamental solution of L_λ can be written as $\Gamma_\lambda = \lambda^Q \Gamma \circ D\left(\frac{1}{\lambda}\right)$. Hence, if we put $\lambda = s^{\frac{1}{Q+m}}$, (5.29) yields

$$\Omega_{c^-}^{(m)-}(0) \subset \Omega_{1,\lambda}^{(m)}(0) \subset \Omega_{c^+}^{(m)+}(0) \subset D\left(\frac{1}{\lambda}\right) H_r \subset D\left(\frac{1}{\lambda}\right) \Omega. \tag{5.33}$$

Then, by Proposition 5.2,

$$v(0) = \int_{\Omega_{1,\lambda}^{(m)}(0)} M_{1,\lambda}^{(m)}(0; z) v(z)\, dz. \tag{5.34}$$

Since, $M_\lambda(0; z) \geq 0$ (see (5.2)), the definition of $M_r^{(m)}$ yields

$$M_{1,\lambda}^{(m)}(0; z) \geq \omega_m \frac{m}{m+2} \frac{N_{1,\lambda}^{m+2}(0; z)}{4t^2}. \tag{5.35}$$

Moreover, from (5.11) and (5.12) we have, for $z \in \Omega_{\frac{3}{4},\lambda}^{(m)}(0)$,

$$N_{1,\lambda}(0; z) \geq \sqrt{-4t \log\left(\frac{4}{3}\right)}. \tag{5.36}$$

Then, from (5.34), (5.35) and (5.36) it follows that

$$v(0) \geq c_m \int_{\Omega_{\frac{3}{4},\lambda}^{(m)}(0)} (-t)^{\frac{m-2}{2}} v(x, t)\, dx dt, \tag{5.37}$$

for some positive constant c_m depending only on m.

Note that there exists $\theta \in]0, 1[$ such that

$$H_\theta^- \subset \Omega_{\frac{c^-}{2}}^{(m)-}(0),$$

and put

$$K = \Omega_{\frac{3}{4}c^-}^{(m)-}(0) \cap \left\{z \in \mathbb{R}^{N+1} : t \leq -\theta\right\}.$$

Since, from Lemma 5.1, it follows $K \subset \Omega_{\frac{3}{4}}^{(m)}(0)$, relation (5.37) yields

$$v(0) \geq c_m \theta^{\frac{m-2}{2}} \int_K v(z)\, dz. \tag{5.38}$$

In order to complete the proof of Theorem 1.3 we need the following Lemma, whose proof is postponed to the end of this Section. □

Lemma 5.2. *For every $m \in \mathbb{N}$, $m \geq 3$ there exist two positive constants $\sigma, k > 0$, depending only on m, r_0 and on the constants of Hypotheses H.1-H.4, such that, for every $\lambda \in]0, \lambda_0]$*

i) $$\Omega_{\sigma,\lambda}^{(m)}(\zeta) \subset K \qquad \forall \zeta \in H_\theta^-$$

ii) $$M_{\sigma,\lambda}^{(m)}(\zeta, z) \leq k \qquad \forall \zeta \in H_\theta^-, z \in \Omega_{\sigma,\lambda}^{(m)}(\zeta).$$

Then, for every $\zeta \in H_\theta^-$, from Lemma 5.2 (i) and from (5.38) it follows

$$v(0) \geq c_m \theta^{\frac{m-2}{2}} \int_{\Omega_{\sigma,\lambda}^{(m)}(\zeta)} v(z)\, dz \geq$$

(Lemma 5.2 (ii))

$$\geq \frac{c_m}{k} \theta^{\frac{m-2}{2}} \int_{\Omega_{\sigma,\lambda}^{(m)}(\zeta)} M_{\sigma,\lambda}^{(m)}(\zeta; z) v(z)\, dz = \frac{\sigma\, c_m}{k} \theta^{\frac{m-2}{2}} v(\zeta)$$

This proves (5.32), and hence, Theorem 1.3. □

We end this section with the proof of Lemma 5.2 above.

Proof of Lemma 5.2. We begin by observing that the level set $\Omega_1^{(m)+}(0)$ is bounded and that

$$\Omega_\sigma^{(m)+}(z_0) = z_0 \circ \left(D\left(\sigma^{\frac{1}{Q+m}}\right)\Omega_1^{(m)+}(0)\right). \tag{5.39}$$

Equation (5.39) follows from the fact that L^+ is invariant with respect to the dilation and the translations groups. Then, since the group operations are continuous functions, and H_θ^- and ∂K are compact sets, there exists $\delta > 0$ such that

$$\Omega_\delta^{(m)+}(\zeta) \subset K \qquad \forall \zeta \in H_\theta^-. \tag{5.40}$$

Then, setting $\sigma \leq \delta c^+$, (i) follows from Lemma 5.1.

In order to prove (ii), we note that, from (5.10) and (5.12),

$$N_{r,\lambda}^2(z_0, z) \leq 2(t_0 - t)\log\left(\frac{r^2}{(4\pi)^{m+n}(t_0 - t)^{m+Q}\det C_{z,\lambda}(1)}\right).$$

Moreover (see Proposition 2.4)

$$\det C^-(1) \leq \det C_{z,\lambda}(1) \leq \det C^+(1),$$

for every $\lambda > 0$ and for every $z \in \mathbb{R}^{N+1}$. Therefore there exists a positive constant c_1, that does not depend on λ, z, z_0 and such that

$$N_{r,\lambda}(z_0, z) \leq c_1\sqrt{t_0 - t}\sqrt{\log\left(\frac{r}{t_0 - t}\right)}. \tag{5.41}$$

On the other hand, Hypothesis H.1 yields

$$< A\big(D(\lambda)z\big)\eta, \eta > \leq \Lambda|\eta|^2 \qquad \forall \eta \in \mathbb{R}^{p_0}, \tag{5.42}$$

for every $\lambda > 0$ and for every $z \in \mathbb{R}^{N+1}$. Moreover, using (5.16), there exist $\delta, c > 0$, such that, for every $\sigma \in]0, \delta]$

$$\frac{|\partial_{x_i}\Gamma(z_0; z)|}{\Gamma(z_0; z)} \leq c\left[\frac{\left|D\big((t_0 - t)^{-1/2}\big)\big(x_0 - E(t_0 - t)x\big)\right|}{\sqrt{t_0 - t}} + 1\right] \tag{5.43}$$

for every $z_0, z \in \mathbb{R}^{N+1}$, $z \in \Omega_{\sigma,\lambda}^{(m)}(z_0)$ and for every $\lambda \in]0, \lambda_0]$. Substituting (5.42) and (5.43) in definition (5.2) of M_λ, we then obtain

$$M_\lambda(z_0; z) \leq c^2\Lambda\left[\frac{\left|D\big((t_0 - t)^{-1/2}\big)\big(x_0 - E(t_0 - t)x\big)\right|^2}{t_0 - t} + 1\right] \tag{5.44}$$

for every $\lambda \in]0, \lambda_0]$, for every $\sigma \in]0, \delta]$ and for any $z \in \Omega^{(m)}_{\sigma,\lambda}(z_0)$. Finally, since $z \in \Omega^{(m)}_{\sigma,\lambda}(z_0)$, there exists a constant $c_2 > 0$ such that

$$\left|D\big((t_0 - t)^{-1/2}\big)\big(x_0 - E(t_0 - t)x\big)\right|^2 \leq c_2 \log\left(\frac{1}{t_0 - t}\right),$$

and then

$$M_\lambda(z_0; z) \leq \frac{c_3}{t_0 - t} \log\left(\frac{1}{t_0 - t}\right). \tag{5.45}$$

This estimate, together with (5.41) and (5.13), for $m \geq 3$, gives (ii). □

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