LE MATEMATICHE
Vol. XLIX (1994) – Fasc. I, pp. 107-121

ASYMPTOTICS FOR THE GREATEST ZEROS OF SOLUTIONS OF A PARTICULAR O.D.E.

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This article is dedicated to Prof. W.N. Everitt on the occasion of his seventieth birthday

This paper deals with the Liouville - Stekloff method for approximating solutions of homogeneous linear ODE and a general result due to Tricomi which provides estimates for the zeros of functions by means of the knowledge of an asymptotic representation. From those classical tools we deduce information about the asymptotics of the greatest zeros of a class of solutions of a particular ODE, including the classical Hermite polynomials.

1. Introduction.

Let $H_n(\xi)$ be the *nth* Hermite polynomial which belongs to the Orthogonal Polynomial Set in $(-\infty, \infty)$, with respect to the weight function $e^{-\xi^2}$.

Denote by $\xi_{1,n} > \xi_{2,n} > \ldots > \xi_{n,n}$ the zeros of $H_n(\xi)$, enumerated in decreasing order, and by $i_1 < i_2 < i_3 < \ldots$ the real zeros of the Airy's function $\mathscr{A}(x)$, $(i_1 \cong 3.37213\ldots)$.

Entrato in Redazione il 7 giugno 1994.

This research was partially supported by 60% funds of the M.U.R.S.T.

AMS Classification: 33C25 - 34E99.

Key Words: Ordinary differential equations. Orthogonal polynomials. Zeros distribution. Asymptotics.

Remark I. The Airy's function we consider here is a solution of the ODE $y'' + \frac{1}{3}xy = 0$ [6], pp. 18-19. In the sequel we mention also the more usual standardization, considering solutions $\mathscr{A}i(x)$, $\mathscr{B}i(x)$ of the ODE $y'' - xy = 0$ [2], pp. 253-256.

It is well known [4], p. 106, that by putting

$$z(\xi) = e^{-\frac{\xi^2}{2}} H_n(\xi), \tag{1.1}$$

$$h_n := (2n+1)^{1/2}, \qquad \xi = h_n - x, \tag{1.2}$$

$$y(x) = z(\xi(x)) = \exp\left(-\frac{(h_n - x)^2}{2}\right) H_n(h_n - x), \tag{1.3}$$

the function $y(x)$ satisfies the ODE

$$y'' + 2h_n xy = x^2 y. \tag{1.4}$$

The asymptotic behaviour of the greatest zero of $H_n(x)$, when $n \to \infty$, is given by $\xi_{1,n} \simeq (2n+1)^{1/2}$. More precisely the following asymptotic formula has been proved (P.E. Ricci [7])

$$\frac{\xi_{1,n}}{(2n+1)^{1/2}} = 1 - \frac{i_1}{6^{1/3}(2n+1)^{2/3}} - \frac{0.32823\ldots}{(2n+1)^{11/6}} + \tag{1.5}$$

$$+ \frac{0.0072757\ldots}{(2n+1)^3} + \mathcal{O}\left(n^{-25/6}\right).$$

This asymptotic formula represents an improvement of a very well known result, which can be found in the book of G. Szegö [6], p. 134.

Different estimates have been obtained by L. Gatteschi [3], and general results connected with the above problem are given by A. Matè - P. Nevai - V. Totik in [4].

It is shown below that it is possible to obtain information about the asymptotics for the greatest zeros $\Xi_{1,n}$ of a set of solutions $Z_n(\xi)$ ($\forall n \in \mathbb{N}$) of the ODE

$$Z_n'' - 2\xi Z_n' + \left(\xi^2 - 2h(n)\xi + 2h^2(n) - 1 - f(h(n) - \xi; n)\right) Z_n = 0, \tag{1.6}$$

where

$$
(1.7) \qquad \begin{cases} h(n) = \mathcal{O}(n^{\alpha}) & (n \to \infty), \\ f(x;n) = \left[\sum_{k=\gamma}^{\infty} A_k(n) x^k \right] \ell(n), & \\ \ell(n) = \mathcal{O}(n^{\beta});\ A_k(n) = \mathcal{O}(1) & (n \to \infty), \end{cases}
$$

and the further hypothesis

$$
(1.8) \qquad \beta < \alpha \left(\frac{5+\gamma}{3} \right)
$$

is assumed.

Note that the case of Hermite polynomials is obtained by assuming $h(n) := h_n = (2n+1)^{1/2}$, $(\alpha = 1/2)$, $f(x;n) := x^2$, $(\gamma = 2;\ \beta = 0)$.

We use the following mathematical tools

– a method described by F.G. Tricomi [8] - [9] and improved by L. Gatteschi [2], which allows to deduce estimates (and bounds for the error) for the zeros of a function F by means of knowledge of an asymptotic representation of F;

– the Liouville - Stekloff method, by means of which asymptotic expansions of the solutions of an homogeneous linear ODE are obtained in terms of the known solutions of a basic equation.

After a short describing of the above mentioned tools, given in the paragraphes 2 and 3, the application to the asymptotics for the greatest zeros of particular solutions of the ODE (1.6) is given in the subsequent paragraphes 4 and 5.

2. A result due to F.G. Tricomi.

In the article [9], F.G. Tricomi proved the following result

Proposition I. *Suppose the continuous function $F(x)$ admits (uniformly with respect to x), the asymptotic representation*

$$
(2.1) \qquad F(x) = \sum_{k=0}^{m} g_k(x)\mu^k + \mathcal{O}(\mu^{m+1}), \qquad (\mu \to 0),
$$

where the functions $g_k(x)$ are differentiable $m-k+1$ times in a neighbourhood of a point x_0 which is a simple zero of the function $g_0(x)$ $(g_0(x_0) = 0,\ g_0'(x_0) \neq$

0). *Suppose further that* $g_m(x) \in C^1$ *in the same neighbourhood. Then* $\forall \varepsilon > 0$ *and for* $|\mu|$ *less than a suitable* $\delta > 0$, *the equation* $F = 0$ *is satisfied at least by a value* x_0^* *s.t.:* $|x_0^* - x_0| < \varepsilon$, *and the following expansion holds*

$$x_0^* = x_0 + \sum_{k=1}^{m} \omega_{k-1}\mu^k + \mathcal{O}(\mu^{m+1}), \tag{2.2}$$

where the coefficients $\omega_0, \omega_1, \omega_2, \ldots$ *are rational functions of the values*

$$G_{k,\ell} := \frac{1}{\ell!} g_k^{(\ell)}(x_0),$$

and are determined recursively as solution of a suitable system of equations (F.G. Tricomi [9]*).*

The first expressions for the ω_k are given by

$$\omega_0 = -\frac{G_{10}}{G_{01}} = -\frac{g_1(x_0)}{g_0'(x_0)}, \tag{2.3}$$

$$\omega_1 = -\frac{G_{10}^2 G_{02} - G_{10}G_{01}G_{11} + G_{01}^2 G_{20}}{G_{01}^3} = \tag{2.4}$$

$$= -\frac{\frac{1}{2}g_1^2(x_0)g_0''(x_0) - g_1(x_0)g_0'(x_0)g_1'(x_0) + \left(g_0'(x_0)\right)^2 g_2(x_0)}{\left(g_0'(x_0)\right)^3}.$$

Note that formula (2.2) provides an asymptotic estimate for a zero x_0^* of the function F in terms of the zero x_0 of g_0, provided that the representation (2.1) is known.

A result of L. Gatteschi [2], pp. 372-373, provides an extension of Proposition I, in order to obtain a bound for the error of the asymptotic formula for the zeros through the knowledge of a bound for the error of the asymptotic expansion of the function.

Proposition II. *Suppose that the same hypotheses as in Proposition I hold, let* $m = 0$ *and* $m = 1$ *in the expansion* (2.1) *and write*

$$F(x) = g_0(x) + \varepsilon_1(x)\mu,$$

$$F(x) = g_0(x) + g_1(x)\mu + \varepsilon_2(x)\mu^2.$$

Suppose that in a fixed interval I, containing the above mentioned zeros x_0^* of $F(x)$ and x_0 of $g_0(x)$, the following estimates hold true

$$|\varepsilon_1(x)| < A_1, \qquad |\varepsilon_2(x)| < A_2, \tag{2.5}$$

then the expansion (2.2), *when $m = 1$, can be written in the following form*

$$x_0^* = x_0 - \frac{g_1(x_0)}{g_0'(x_0)}\mu + \rho, \tag{2.6}$$

with an upper bound of the error term given by

$$|\rho| < \frac{\mu^2}{|g_0'(x_0)|}\left(\frac{\tilde{A}_1^2}{2}\sup|g_0''(x')| + \tilde{A}_1 \sup|g_1'(x'')| + A_2\right), \tag{2.7}$$

where

$$\tilde{A}_1 := \frac{A_1}{\inf|g_0'(x)|}, \quad \text{and} \quad |x_0 - x'| < \tilde{A}_1\mu, \quad |x_0 - x''| < \tilde{A}_1\mu.$$

3. The general form of the Liouville - Stekloff method.

In a book of F.G. Tricomi [10] can be found a method, ascribed to G. Fubini [1], which can be interpreted as a general form of the Liouville - Stekloff method.

Tricomi considers a second order homogeneous linear ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{3.1}$$

with $P_0(x) \neq 0$ in $(a, b) \subseteq \mathbb{R}$.

Suppose that by splitting (3.1) in the form

$$y'' + p_1(x)y' + p_2(x)y = \alpha(x)y'' + \beta(x)y' + \gamma(x)y, \tag{3.2}$$

the following conditions are satisfied

I. – the approximate ODE

$$y'' + p_1(x)y' + p_2(x)y = 0, \tag{3.3}$$

can be integrated explicitly. Denote by $F_1(x)$, $F_2(x)$ two of its linearly independent integrals.

II. – the remainder functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ satisfy, with respect to a parameter λ, and for a suitable $r > 0$, the conditions

$$\alpha(x) = \mathcal{O}\left(\lambda^{-r}\right), \; \beta(x) = \mathcal{O}\left(\lambda^{-r}\right), \; \gamma(x) = \mathcal{O}\left(\lambda^{-r}\right). \tag{3.4}$$

Denote by $W(x)$ the wronskian of the above integrals and let

$$\mathcal{G}_h(x) := \frac{\alpha(x)F_h''(x) + \beta(x)F_h'(x) + \gamma(x)F_h(x)}{[1 - \alpha(x)]W(x)}, \qquad (h = 1, 2) \tag{3.5}$$

$$L(x, \xi) := \begin{vmatrix} F_1(\xi) & F_2(\xi) \\ F_1(x) & F_2(x) \end{vmatrix}. \tag{3.6}$$

Then the following proposition holds true

Proposition III. *Anyone of the independent solutions* $Y_1(x)$, $Y_2(x)$ *of the original equation* (3.1) *admits the representation*

$$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x, \xi)\mathcal{G}_h(\xi)\, d\xi + \tag{3.7}$$
$$+ \sum_{s=1}^{\infty} \int_{a_0}^{x} L(x, \xi) \left[\int_{a_0}^{\xi} K_s(\xi, \eta)\mathcal{G}_h(\eta)\, d\eta \right] d\xi,$$

where $a_0 \in (a, b)$, $h = 1, 2$, *and the kernels* $K_s(x, \eta)$ *are defined by the induction formulas*

$$K_1(\xi, \eta) := K(\xi, \eta) := \begin{vmatrix} F_1(\eta) & F_2(\eta) \\ \mathcal{G}_1(\xi) & \mathcal{G}_2(\xi) \end{vmatrix} \tag{3.8}$$

$$K_s(\xi, \eta) = \int_{\eta}^{\xi} K(\xi, z)K_{s-1}(z, \eta)\, dz, \qquad (s > 1). \tag{3.9}$$

By Proposition III, and using hypotheses (3.4), the approximation formulas, of increasing precision, for the integrals of equation (3.1) follow

$$Y_h(x) = F_h(x) + \mathcal{O}\left(\lambda^{-r}\right), \tag{3.10}$$

(3.11) $$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x,\xi)\mathcal{G}_h(\xi)\,d\xi + \mathcal{O}\left(\lambda^{-2r}\right),$$

(3.12) $$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x,\xi)\mathcal{G}_h(\xi)\,d\xi +$$

$$+ \int_{a_0}^{x} L(x,\xi)\left[\int_{a_0}^{\xi} K_1(\xi,\eta)\mathcal{G}_h(\eta)\,d\eta\right]d\xi + \mathcal{O}\left(\lambda^{-3r}\right),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

$(h = 1, 2)$, and so on.

4. Application of the generalized Liouville - Stekloff method to the present problem.

Starting from the differential equation (1.6), by putting

(4.1) $$\xi = h(n) - x,$$

(4.2) $$z(\xi) = e^{-\frac{\xi^2}{2}} Z_n(\xi),$$

(4.3) $$y(x) = z(\xi(x)) = \exp\left(-\frac{(h(n)-x)^2}{2}\right) Z_n(h(n)-x),$$

we obtain the differential equation

(4.4) $$y'' + 2h(n)xy = f(x;n)y,$$

under the hypotheses (1.7) - (1.8).

Writing (4.4) in the form

(4.5) $$\frac{1}{h(n)}y'' + 2xy = \frac{1}{h(n)}f(x;n)y = \phi(x;n)y,$$

where $\phi(x;n) = \left[\sum_{k=\gamma}^{\infty} A_k(n)x^k\right]\mathcal{O}\left(n^{\beta-\alpha}\right)$, $(n \to \infty)$, it is possible to apply the Liouville - Stekloff method to the approximating equation

(4.6) $$\frac{1}{h(n)}y'' + 2xy = 0,$$

which is equivalent to the following

$$y'' + 2h(n)xy = 0. \tag{4.7}$$

The last equation is satisfied by the Airy's functions

$$F_1(x) = \mathscr{A}\left((6h(n))^{1/3}x\right), \quad F_2(x) = \mathscr{B}\left((6h(n))^{1/3}x\right).$$

Remark II. Note that in the present case the coefficients, and consequently the solutions of the "approximate" differential equation (4.7) are depending on n. Nevertheless the methods exposed in the preceding paragraphes 2 and 3 remain true. The only difference will be the necessity of a more careful evaluation of the infinitesimal orders in the asymptotic estimates.

By recalling [2], p. 254, $W(\mathscr{A}i(x), \mathscr{B}i(x)) = \frac{1}{\pi}$, and using the relationes between two different standardizations

$$\mathscr{A}(x) = \frac{\pi}{3^{1/3}}\mathscr{A}i\left(-\frac{x}{3^{1/3}}\right);$$
$$\mathscr{B}(x) = \frac{\pi}{3^{1/3}}\mathscr{B}i\left(-\frac{x}{3^{1/3}}\right),$$

we can write

$$W\left(F_1(x), F_2(x)\right) = -\frac{\pi}{3}(6h(n))^{1/3} = \mathscr{O}\left(n^{\alpha/3}\right).$$

Furthermore, recalling an asymptotic property for the Airy's functions [2], p. 256, we obtain

$$L(x,\xi) := \mathscr{A}\left((6h(n))^{1/3}\xi\right)\mathscr{B}\left((6h(n))^{1/3}x\right) + \tag{4.8}$$
$$- \mathscr{A}\left((6h(n))^{1/3}x\right)\mathscr{B}\left((6h(n))^{1/3}\xi\right) = \mathscr{O}\left(n^{-\alpha/6}\right).$$

Comparing (4.4) with (3.2) we deduce

$$\alpha(x) = \beta(x) \equiv 0; \quad \gamma(x) = \phi(x;n) = \mathscr{O}\left(n^{\beta-\alpha}\right). \tag{4.9}$$

Then

$$\mathscr{G}_1(x) = -\frac{3\phi(x;n)\mathscr{A}\left((6h(n))^{1/3}x\right)}{\pi(6h(n))^{1/3}} = \mathscr{O}\left(n^{-\frac{17}{12}\alpha+\beta}\right), \tag{4.10}$$

$$\mathscr{G}_2(x) = -\frac{3\phi(x;n)\mathscr{B}\left((6h(n))^{1/3}x\right)}{\pi(6h(n))^{1/3}} = \mathscr{O}\left(n^{-\frac{17}{12}\alpha+\beta}\right). \tag{4.11}$$

As a consequence of Proposition I, we have

Theorem I. *Using preceding hypotheses and notations, a solution $Y_1(x)$ of the* ODE (4.4) *admits the following representation*

$$Y_1(x) = \mathscr{A}\left((6h(n))^{1/3}x\right) + \tag{4.12}$$

$$- \frac{3}{\pi(6h(n))^{1/3}} \int_0^x \phi(\xi; n) L(x, \xi) \mathscr{A}\left((6h(n))^{1/3}\xi\right) d\xi +$$

$$+ \sum_{s=1}^{\infty} \left[\frac{-3}{\pi(6h(n))^{1/3}} \right]^{s+1} \int_0^x \phi(\xi; n) L(x, \xi) \cdot$$

$$\cdot \left[\int_0^{\xi} \phi(\eta; n) L_s(\xi, \eta) \mathscr{A}\left((6h(n))^{1/3}\eta\right) d\eta \right] d\xi,$$

where the iterated kernels $L_s(\xi, \eta)$ are defined by the induction formulas:

$$\begin{cases} L_1(\xi, \eta) := L(\xi, \eta) \\ L_s(\xi, \eta) := \displaystyle\int_{\eta}^{\xi} \phi(z; n) L(\xi, z) L_{s-1}(z, \eta)\, dz, \qquad (s > 1). \end{cases} \tag{4.13}$$

Proof. It is sufficient to write the representation defined by formulas (3.7) - (3.8) - (3.9), remarking that, in this case, we have

$$K_1(\xi, \eta) = \begin{vmatrix} \mathscr{A}\left((6h(n))^{1/3}\eta\right) & \mathscr{B}\left((6h(n))^{1/3}\eta\right) \\ \dfrac{-3\phi(\xi; n)\mathscr{A}\left((6h(n))^{1/3}\xi\right)}{\pi(6h(n))^{1/3}} & \dfrac{-3\phi(\xi; n)\mathscr{B}\left((6h(n))^{1/3}\xi\right)}{\pi(6h(n))^{1/3}} \end{vmatrix} =$$

$$= -\frac{3\phi(\xi; n)}{\pi(6h(n))^{1/3}} L(\xi, \eta) = \left[\sum_{k=\gamma}^{\infty} A_k(n) \xi^k \right] \mathscr{O}\left(n^{-\frac{3}{2}\alpha+\beta}\right),$$

and, by induction

$$K_s(\xi, \eta) = \left[-\frac{3}{\pi(6h(n))^{1/3}} \right]^s \phi(\xi; n) L_s(\xi, \eta).$$

We can write, indeed

$$K_s(\xi, \eta) = \int_{\eta}^{\xi} K(\xi, z) K_{s-1}(z, \eta)\, dz =$$

$$= \frac{-3}{\pi(6h(n))^{1/3}}\,\phi(\xi;n)\int_\eta^\xi L(\xi,z)K_{s-1}(z,\eta)\,dz =$$

$$= \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^s \phi(\xi;n)\int_\eta^\xi \phi(z;n)L(\xi,z)L_{s-1}(z,\eta)\,dz =$$

$$= \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^s \phi(\xi;n)L_s(\xi,\eta).$$

Then, for $s = 1$

$$\int_0^x L(x,\xi)\left[\int_0^\xi K_1(\xi,\eta)\mathscr{G}_1(\eta)\,d\eta\right]d\xi =$$

$$\int_0^x L(x,\xi)\left[\int_0^\xi \frac{3\phi(\xi;n)}{\pi(6h(n))^{1/3}}L(\xi,\eta)\frac{3\phi(\eta;n)}{\pi(6h(n))^{1/3}}\mathscr{A}\left((6h(n))^{1/3}\eta\right)d\eta\right]d\xi =$$

$$= \left[\frac{3}{\pi(6h(n))^{1/3}}\right]^2 \int_0^x \phi(\xi;n)L(x,\xi)\,\cdot$$

$$\cdot\left[\int_0^\xi \phi(\eta;n)L_1(\xi,\eta)\mathscr{A}\left((6h(n))^{1/3}\eta\right)d\eta\right]d\xi;$$

and, in general

$$\int_0^x L(x,\xi)\left[\int_0^\xi K_s(\xi,\eta)\mathscr{G}_1(\eta)\,d\eta\right]d\xi =$$

$$= \int_0^x L(x,\xi)\left[\int_0^\xi\left[\int_\eta^\xi K(\xi,z)K_{s-1}(z,\eta)\,dz\right]\mathscr{G}_1(\eta)\,d\eta\right]d\xi =$$

$$= \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^s \int_0^x \phi(\xi;n)L(x,\xi)\,\cdot$$

$$\cdot\left[\int_0^\xi\left[\int_\eta^\xi \phi(z;n)L(\xi,z)L_{s-1}(z,\eta)\,dz\right]\mathscr{G}_1(\eta)\,d\eta\right]d\xi =$$

$$= \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^s \int_0^x \phi(\xi;n)L(x,\xi)\left[\int_0^\xi L_s(\xi,\eta)\mathscr{G}_1(\eta)\,d\eta\right]d\xi =$$

$$= \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^{s+1} \int_0^x \phi(\xi;n)L(x,\xi)\,\cdot$$

$$\cdot \left[\int_0^{\xi} \phi(\eta; n) L_s(\xi, \eta) \mathscr{A}\left((6h(n))^{1/3}\eta\right) d\eta\right] d\xi. \qquad \square$$

By Theorem I, we deduce, in particular, the following asymptotic formulas for the solution $Y_1(x)$ of the ODE equation (4.4)

$$Y_1(x) = \mathscr{A}\left((6h(n))^{1/3}x\right) + \left[\sum_{k=\gamma}^{\infty} A_k(n)x^k\right] \mathscr{O}\left(n^{-\frac{19}{12}\alpha+\beta}\right), \tag{4.14}$$

$$\begin{aligned} Y_1(x) &= \mathscr{A}\left((6h(n))^{1/3}x\right) + \\ &- \frac{3}{\pi(6h(n))^{1/3}} \int_0^x \phi(\xi; n) L(x, \xi) \mathscr{A}\left((6h(n))^{1/3}\xi\right) d\xi + \\ &+ \left[\sum_{k=\gamma}^{\infty} A_k(n)x^k\right]^2 \mathscr{O}\left(n^{-\frac{37}{12}\alpha+2\beta}\right), \end{aligned} \tag{4.15}$$

and in general

$$\begin{aligned} Y_1(x) &= \mathscr{A}\left((6h(n))^{1/3}x\right) + \\ &- \frac{3}{\pi(6h(n))^{1/3}} \int_0^x \phi(\xi; n) L(x, \xi) \mathscr{A}\left((6h(n))^{1/3}\xi\right) d\xi + \\ &+ \sum_{s=2}^{m} \left[\frac{-3}{\pi(6h(n))^{1/3}}\right]^{s+1} \int_0^x \phi(\xi; n) L(x, \xi) \cdot \\ &\cdot \left[\int_0^{\xi} \phi(\eta; n) L_s(\xi, \eta) \mathscr{A}\left((6h(n))^{1/3}\eta\right) d\eta\right] d\xi + \\ &+ \left[\sum_{k=\gamma}^{\infty} A_k(n)x^k\right]^{m+1} \mathscr{O}\left(n^{-\frac{(18m+19)}{12}\alpha+(m+1)\beta}\right). \end{aligned} \tag{4.16}$$

Remark III. The last relationes provide representation formulas for the function $Y_1(x)$ of the same kind needed in Proposition I. It is sufficient to assume

$$\mu = \frac{1}{h(n)} \simeq n^{-\alpha}, \quad (\text{for } n \to \infty).$$

Since hypotheses of this Proposition are clearly satisfied in our case, we can infer asymptotic representations of any order of accuracy for the zeros of $Y_1(x)$ in terms of the zero of the Airy's function $\mathscr{A}\left((6h(n))^{1/3}x\right)$ and using the function $L(x, \xi)$, by means of which the coefficients of formula (4.16) are expressed.

5. Asymptotic estimates for the greatest zero of $Z_n(\xi)$.

Putting

$$g_0(x) = g_0(x; n) := \mathscr{A}\left((6h(n))^{1/3}x\right), \tag{5.1}$$

$$g_1(x) = g_1(x; n) := \tag{5.2}$$

$$= \frac{-3(6h(n))^{2/3}}{\pi} \int_0^x \phi(\xi; n) L(x, \xi) \mathscr{A}\left((6h(n))^{1/3}\xi\right) d\xi\,,$$

$$g_2(x) = g_2(x; n) := \left[\frac{-3(6h(n))^{2/3}}{\pi}\right]^2 \int_0^x \phi(\xi; n) L(x, \xi) \cdot \tag{5.3}$$

$$\cdot \left[\int_0^\xi \phi(\eta; n) L(\xi, \eta) \mathscr{A}\left((6h(n))^{1/3}\eta\right) d\eta\right] d\xi\,,$$

$$g_m(x) = g_m(x; n) := \left[\frac{-3(6h(n))^{2/3}}{\pi}\right]^m \int_0^x \phi(\xi; n) L(x, \xi) \cdot \tag{5.4}$$

$$\cdot \left[\int_0^\xi \phi(\eta; n) L_{m-1}(\xi, \eta) \mathscr{A}\left((6h(n))^{1/3}\eta\right) d\eta\right] d\xi,$$

writing (4.16) in the form

$$Y_1(x) = g_0(x; n) + (g_1(x; n)) \frac{1}{h(n)} + \tag{5.5}$$

$$+ (g_2(x; n)) \left(\frac{1}{h(n)}\right)^2 + \cdots + (g_m(x; n)) \left(\frac{1}{h(n)}\right)^m +$$

$$+ \left[\sum_{k=\gamma}^{\infty} A_k(n) x^k\right]^{m+1} \mathcal{O}\left(n^{-\frac{(18m+19)}{12}\alpha + (m+1)\beta}\right),$$

assuming $\mu = \frac{1}{h(n)}$ in Proposition I, and remarking that the coefficients $g_k(x; n)$ in formula (5.5) are depending on n, we deduce the following asymptotic estimates for the greatest zero x_0^* of $Y_1(x)$, which make use of the nearest zero $x_0 := \frac{i_1}{(6h(n))^{1/3}}$ of the approximating function $\mathscr{A}\left((6h(n))^{1/3}x\right)$

– From formula (4.14)

$$x_0^* = \frac{i_1}{(6h(n))^{1/3}} + \mathcal{O}\left(n^{-\alpha\left(\frac{6+\gamma}{3}\right)+\beta}\right). \tag{5.6}$$

– From formula (4.15)

$$x_0^* = \frac{i_1}{(6h(n))^{1/3}} + \omega_0 \frac{1}{h(n)} + \mathcal{O}\left(n^{-\alpha\left(\frac{11+2\gamma}{3}\right)+2\beta}\right), \tag{5.7}$$

where

$$\omega_0 = \frac{3}{\pi} \frac{\mathscr{B}(i_1)}{\mathscr{A}'(i_1)} \int_0^{i_1} \phi\left(\frac{x}{(6h(n))^{1/3}}; n\right) \mathscr{A}^2(x)\, dx = \tag{5.8}$$

$$= \mathcal{O}\left(n^{-\alpha(1+\frac{\gamma}{3})+\beta}\right).$$

– From formula (4.16), assuming $m = 2$

$$x_0^* = \frac{i_1}{(6h(n))^{1/3}} + \omega_0 \frac{1}{h(n)} + \omega_1 \left(\frac{1}{h(n)}\right)^2 + \tag{5.9}$$

$$+ \mathcal{O}\left(n^{-\alpha\left(\frac{16+3\gamma}{3}\right)+3\beta}\right)$$

with ω_0 given by formula (5.8), and

$$\omega_1 = -\frac{g_1^2(x_0)g_0''(x_0) - 2g_1(x_0)g_0'(x_0)g_1'(x_0) + 2\left(g_0'(x_0)\right)^2 g_2(x_0)}{2\left(g_0'(x_0)\right)^3} = \tag{5.10}$$

$$= \mathcal{O}\left(n^{-\alpha\left(\frac{5+2\gamma}{3}\right)+2\beta}\right),$$

where

$$g_1(x_0) = g_1(x_0; n) = \tag{5.11}$$

$$= -\frac{3}{\pi}(6h(n))^{1/3}\mathscr{B}(i_1) \int_0^{i_1} \phi\left(\frac{x}{(6h(n))^{1/3}}; n\right) \mathscr{A}^2(x)\, dx,$$

(5.12)
$$g_1'(x_0) = g_1'(x_0; n) =$$
$$= \frac{3}{\pi}(6h(n))^{2/3}\left[\mathscr{A}'(i_1)\int_0^{i_1}\phi\left(\frac{x}{(6h(n))^{1/3}}; n\right)\mathscr{A}(x)\mathscr{B}(x)\,dx + \right.$$
$$\left. - \mathscr{B}'(i_1)\int_0^{i_1}\phi\left(\frac{x}{(6h(n))^{1/3}}; n\right)\mathscr{A}^2(x)\,dx\right],$$

(5.13)
$$g_0'(x_0) = g_0'(x_0; n) = \mathscr{A}'(i_1)(6h(n))^{1/3},$$

(5.14)
$$g_0''(x_0) = g_0''(x_0; n) = \mathscr{A}''(i_1)(6h(n))^{2/3},$$

(5.15)
$$g_2(x_0) = g_2(x_0; n) = \frac{9}{\pi^2}(6h(n))^{2/3}\mathscr{B}(i_1)\cdot$$
$$\cdot\left[\int_0^{i_1}\phi\left(\frac{x}{(6h(n))^{1/3}}; n\right)\mathscr{A}(x)\mathscr{B}(x)\int_0^{x}\phi\left(\frac{\xi}{(6h(n))^{1/3}}; n\right)\mathscr{A}^2(\xi)\,d\xi\,dx + \right.$$
$$\left. - \int_0^{i_1}\phi\left(\frac{x}{(6h(n))^{1/3}}; n\right)\mathscr{A}^2(x)\int_0^{x}\phi\left(\frac{\xi}{(6h(n))^{1/3}}; n\right)\mathscr{A}(\xi)\mathscr{B}(\xi)\,d\xi\,dx\right].$$

Remark IV. Note that, at each step, the order of accuracy increases with a constant rate given by $-\alpha\left(\frac{5+\gamma}{3}\right) + \beta$, and by the hypothesis (1.8) this quantity is negative.

Returning to the original variable ξ, we can lastly proclaim the result below

Theorem II. *For the greatest zero $\Xi_{1,n}$ of the solution $Z_n(\xi)$ of the* ODE, *which corresponds to $Y_1(x)$, we can write the asymptotic estimates*

(5.16)
$$\Xi_{1,n} = h(n) - \frac{i_1}{(6h(n))^{1/3}} + \mathscr{O}\left(n^{-\alpha\left(\frac{6+\gamma}{3}\right)+\beta}\right),$$

(5.17)
$$\Xi_{1,n} = h(n) - \frac{i_1}{(6h(n))^{1/3}} - \omega_0\frac{1}{h(n)} + \mathscr{O}\left(n^{-\alpha\left(\frac{11+2\gamma}{3}\right)+2\beta}\right)$$

(5.18)
$$\Xi_{1,n} = h(n) - \frac{i_1}{(6h(n))^{1/3}} - \omega_0\frac{1}{h(n)} - \omega_1\left(\frac{1}{h(n)}\right)^2 +$$
$$+ \mathscr{O}\left(n^{-\alpha\left(\frac{16+3\gamma}{3}\right)+3\beta}\right).$$

Remark V. The method due to Gatteschi (Proposition II) provides a theoretical estimate for the error term in formula (5.17). However, in the present case, owing to the general form of the function $\phi(x; n) := \left[\sum_{k=\gamma}^{\infty} A_k(n) x^k \right] \mathcal{O}\left(n^{\beta-\alpha}\right)$, it is not possible to obtain an estimate which could be of some practical interest.

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