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# INVESTIGATION OF THE WELL-POSEDNESS OF THE MIXED PROBLEM ON THE STABILITY OF FAST SHOCK WAVES IN MAGNETOHYDRODYNAMICS

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In the present work the question on the stability of fast magnetohydrodynamic shock waves is studied in the case of a polytropic gas under the assumption that the magnetic field is weak. With the help of techniques of dissipative energy integrals, an a priori estimate showing the well-posedness of the corresponding mixed problem on the stability of a fast shock wave is obtained.

## Introduction.

The equations of magnetohydrodynamics are widely applied in the description of actual processes in such fields of physics and technology as astrophysics, aerodynamics of high velocities etc. It is known that in these processes strong discontinuities (for example, shock waves) are present while a gas is in motion. In this connection, the problem of strong discontinuities (including shock waves) stability in magnetohydrodynamics is of great interest.

The term "stability of strong discontinuity" was introduced by physicists. This term means the following. Let a strong discontinuity front be slightly perturbed. The question is if this perturbation either increases or decreases with time. If this perturbation decreases, then the strong discontinuity is stable, otherwise it is unstable. Mathematically the question of the stability of strong discontinuity is reduced to the investigation of some linear mixed problem.

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One of the first investigations in this field was carried out by L. D. Landau and his disciples. Their approach was as follows. They formulated the above-mentioned mixed linear problem and sought exponential solutions to this problem. By the behaviour of these solutions they judged about the stability or instability of the strong discontinuity.

Our approach implies the investigation of the well-posedness of the above-mentioned mixed linear problem by means of techniques of dissipative energy integrals. Here we use the term "well-posedness of mixed problem" in a classical sense. In the case the mixed problem is well-posed the strong discontinuity is stable (as a physical structure). Otherwise, the strong discontinuity is unstable.

It should be noted that unlike the situation in gas dynamics, ordinary and relativistic (see [1], [2]), the question of the stability of strong discontinuities in magnetohydrodynamics has not been fully investigated. Actually, after the publication of classical works [3], [4] only a few works in which the question of the stability of strong discontinuities is discussed in one degree or an other may be named. They are, for example, [5], [6].

In the present work the so-called "equational" approach (see above) to the investigation of the stability of fast shock waves in magnetohydrodynamics is used. It contains the study of the well-posedness of the corresponding linear mixed problem of the stability of fast shock waves. The basic method of this investigation is the construction of an a priori estimate implying to the well-posedness of the linear mixed problem. This approach (in connection with problems of magnetohydrodynamics) is described in the recently published monograph [7]. It should be noted also that in this monograph two special cases of the above mentioned linear mixed problem on the stability of a fast shock wave are considered, and its well-posedness is proved by obtaining an a priori estimate with the help of techniques of dissipative energy integrals.

In the proposed work the case of general statement of linear mixed problem on the stability of fast magnetohydrodynamic shock wave is considered, and the well-posedness of the mixed problem is proved by obtaining a corresponding a priori estimate. While preparing the material of this work we used ideas and notations of the monograph [7].

## 1. Statement of problem on the stability of fast magnetohydrodynamic shock wave.

In this section we shall formulate the mathematical statement of the problem of the stability of a fast magnetohydrodynamic shock wave in the plane case.

This problem is obtained as a result of the linearization of magnetohydrodynamics equations and relationes on the strong discontinuity. The process of linearization is described in detail in the monograph [7].

So, following [7], we formulate the linear mixed problem on the stability of a fast magnetohydrodynamic shock wave in the case of two space variables.

*Problem $\mathcal{F}$.* We find the solution

$$\mathbf{U} = \mathbf{U}(t, \mathbf{x}), \quad \mathbf{x} = (x_1, x_2)$$

of the system

$$\begin{gathered} Lp + \operatorname{div} \mathbf{v} = 0, \\ LS = 0, \\ M^2 L\mathbf{v} + \nabla p - (\operatorname{rot} \mathbf{H}) \times \mathbf{h} = 0, \\ L\mathbf{H} - \operatorname{rot}(\mathbf{v} \times \mathbf{h}) = 0; \quad t > 0, \quad \mathbf{x} \in \mathbb{R}^2_+, \end{gathered} \tag{1.1}$$

satisfying the following boundary conditions

$$\begin{gathered} v_1 + dp - d_1 H_1 + d_2 H_2 = 0, \\ F_t = \mu p + \mu_1 H_1 - \mu_2 H_2, \\ v_2 = \lambda_0 F_{x_2} + \lambda_1 p + \lambda_2 H_1 + \lambda_3 H_2, \\ H_2 = mq(1-\chi) F_t + h_1 v_2 - h_2 v_1, \\ H_1 = mq(1-\chi) F_{x_2}, \\ S = \kappa p + \kappa_1 H_2 \end{gathered} \tag{1.2}$$

at $t > 0$, $x_1 = 0$, $x_2 \in \mathbb{R}^1$ and the initial data under $t = 0$:

$$\begin{gathered} \mathbf{U}|_{t=0} = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2_+, \\ F|_{t=0} = F_0(x_2), \quad x_2 \in \mathbb{R}^1. \end{gathered} \tag{1.3}$$

Here

$$L = \tau + \xi_1, \quad \nabla = (\xi_1, \xi_2)^*, \quad \tau = \frac{\partial}{\partial t}, \quad \xi_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2;$$

$t$ is the time, $\mathbf{x} = (x_1, x_2)$ are the Cartesian coordinates,

$$\mathbb{R}^2_+ = \{\mathbf{x} \mid x_1 > 0, \quad x_2 \in \mathbb{R}^1\}, \quad \mathbf{U} = (p, S, \mathbf{v}^*, \mathbf{H}^*)^*,$$

$$\mathbf{v} = (v_1, v_2)^*, \quad \mathbf{H} = (H_1, H_2)^*$$

$p, S, v_k, H_k$ $(k = 1, 2)$ stand for small perturbations of the pressure, entropy, components of the velocity vector and magnetic field strength reduced in the corresponding way to the dimensionless form;

$$\mathbf{h} = (h_1, h_2)^*, \quad q = |\mathbf{h}|;$$

$h_1, h_2, M$ are some constants characterising the piecewise constant solution of the initial magnetohydrodynamics equations and the relationes on the shock wave provided that the front of shock wave is stationary and described by the equation $x_1 = 0$, moreover $h_k \geq 0$ $(k = 1, 2)$, $M > 0$. For the polytropic gas with the isentropic exponent $\gamma > 1$ and under assumption that magnetic field is weak $(q \ll 1)$, the coefficients from boundary conditions (1.2) and the constant $M$ are determinated as follows:

$$d = d^{(0)} + O(q^2), \quad d_{1,2} = d_{1,2}^{(1)} q + O(q^2),$$

$$\mu = \mu^{(0)} + O(q^2), \quad \mu_{1,2} = \mu_{1,2}^{(1)} q + O(q^2),$$

$$\lambda_0 = \lambda_0^{(0)} + O(q^2), \quad \lambda_1 = O(q^2),$$

$$\lambda_{2,3} = \lambda_{2,3}^{(1)} q + O(q^2), \quad \kappa = \kappa^{(0)} + O(q^2),$$

$$\kappa_1 = \kappa_1^{(1)} + O(q^2), \quad \chi = \chi_0 + O(q^2),$$

$$M^2 = M_0^2 + O(q^2),$$

where

$$d^{(0)} = \frac{3 - \gamma + (3\gamma - 1)M_0^2}{2M_0^2(2 + (\gamma - 1)M_0^2)}, \quad d_1^{(1)} = \frac{l}{2M_0^2},$$

$$d_2^{(1)} = \frac{m}{2M_0^2}\left(1 - \frac{2(\gamma - 1)(1 - M_0^2)^2}{(2 + (\gamma - 1)M_0^2)^2}\right),$$

$$\mu^{(0)} = -\frac{\gamma + 1}{4M_0^2}, \quad \mu_1^{(1)} = \frac{l(2 - (\gamma + 3)M_0^2)}{4M_0^2(1 - M_0^2)},$$

$$\mu_2^{(1)} = \frac{m}{4M_0^2}\left(1 + \frac{2(\gamma - 1)(1 - M_0^2)^2}{(2 + (\gamma - 1)M_0^2)^2}\right)\frac{2 + (\gamma - 1)M_0^2}{1 - M_0^2},$$

$$\lambda_0^{(0)} = \frac{2(1 - M_0^2)}{(\gamma + 1)M_0^2}, \quad \lambda_2^{(1)} = -\frac{m(\gamma + 1)}{2 + (\gamma - 1)M_0^2},$$

$$\lambda_3^{(1)} = \frac{l}{M_0^2}, \quad \kappa^{(0)} = \frac{(\gamma - 1)(1 - M_0^2)^2}{M_0^2(2 + (\gamma - 1)M_0^2)},$$
$$\kappa_1^{(1)} = \frac{2m(\gamma - 1)(1 - M_0^2)^2}{M_0^2(2 + (\gamma - 1)M_0^2)^2}, \quad \chi_0 = \frac{(\gamma + 1)M_0^2}{2 + (\gamma - 1)M_0^2},$$
$$l = h_1/q, \qquad m = h_2/q,$$

moreover the parameter $M_0$ (=const) for the fast magnetohydrodynamic shock wave satisfies the inequalities (see [7]):

$$0 < M_0 < 1.$$

We note also that $F = F(t, x_2)$ is a small perturbation of the shock wave front (with the equation $x_1 = F(t, x_2)$).

**Remark 1.1.** While solving the mixed problem $\mathscr{F}$, we also determine the function $F = F(t, x_2)$. For this purpose, one of boundary conditions (1.2) must be the equation for determination of the function $F$.

**Remark 1.2.** It is shown in [7] that for the case of weak magnetic fields ($q \ll 1$) the mixed problem $\mathscr{F}$ is correct (by the number of boundary conditions), i.e. the fast magnetohydrodynamic shock wave in this case is evolutionary by the one-dimensional attribute (for the concept of evolution see, for example, [8]).

**Remark 1.3.** The linear mixed problem $\mathscr{F}$ is formulated with regard to the fact that without loss of generality we may assume

$$\mathbf{U}(t, \mathbf{x}) \equiv 0 \quad \text{under} \quad x_1 < 0,$$

since for the fast shock wave in the one-dimensional case all the characteristics of the linearized magnetohydrodynamics equations system under $x_1 < 0$ are arriving (for the boundary $x_1 = 0$) and, thus, under $x_1 < 0$ the solution is completely determined by the initial data given at $t = 0$ (both in one-dimensional and multi-dimensional cases).

**Remark 1.4.** It is shown in [7] that on the smooth solutions of problem $\mathscr{F}$ the necessary for magnetohydrodynamics condition div $\mathbf{H} = 0$ (see, for example, [3]) is true for $t > 0$ if it holds for $t = 0$:

$$(\operatorname{div}\mathbf{H})|_{t=0} = 0, \quad \mathbf{x} \in \mathbb{R}_+^2. \tag{1.4}$$

That is, the condition

$$\operatorname{div}\mathbf{H} = 0, \quad t > 0, \quad \mathbf{x} \in \mathbb{R}_+^2 \tag{1.5}$$

becomes, as a matter of fact, a corollary of (1.4) on initial data (1.3).

**Remark 1.5.** We remind that boundary conditions (1.2) are obtained as a result of the linearization of the relationes on the strong discontinuity in magnetohydrodynamics (about these relationes see, for example, [8]).

**Remark 1.6.** In this section and in sections 2, 3 we suppose that problem $\mathscr{F}$ has the sufficiently smooth solution.

## 2. Some properties of problem $\mathscr{F}$.

In this section we shall write out another formulation of problem $\mathscr{F}$ and also discuss a number of properties of system (1.1). First of all, we rewrite system (1.1) in the following form:

$$
\begin{aligned}
&Lp + \operatorname{div} \mathbf{v} = 0,\\
&LS = 0,\\
&M^2 L\mathbf{v} + \nabla p - q(\xi_1 H_2 - \xi_2 H_1)\sigma = 0,\\
&LH_1 + q\xi_2 v_\sigma = 0,\\
&LH_2 - q\xi_1 v_\sigma = 0,
\end{aligned}
\tag{2.1}
$$

where $\sigma = (-m, l)^*$, $v_\sigma = (\mathbf{v}, \sigma)$.

The last two equations in (2.1), by virtue of (1.5), imply the existence of the function $\Phi = \Phi(t, \mathbf{x})$ such that

$$H_1 = -q\xi_2\Phi,\ H_2 = q\xi_1\Phi,\ L\Phi = v_\sigma.$$

Then, problem $\mathscr{F}$ may be rewritten as follows.

*Problem.* $\mathscr{F}'$. We find the solution for the system of equations

$$
\begin{aligned}
&Lp + \operatorname{div} \mathbf{v} = 0,\\
&LS = 0,\\
&M^2 L\mathbf{v} + \nabla p - q^2\Delta\Phi\sigma = 0,\\
&L\Phi = v_\sigma; \quad t > 0, \quad \mathbf{x} \in \mathbb{R}^2_+,
\end{aligned}
\tag{2.2}
$$

satisfying the boundary conditions

$$
\begin{aligned}
&v_1 + \tilde{d}p = N_1\xi_2 F,\\
&\tau F = \tilde{\mu}p + N_2\xi_2 F,\\
&v_2 = \tilde{\lambda}_0\xi_2 F + N_3 p,\\
&S = \tilde{\kappa}_0 p + N_4\xi_2 F,\\
&\Phi = -m(1-\chi)F
\end{aligned}
\tag{2.3}
$$

at $t > 0$, $x_1 = 0$, $x_2 \in \mathbb{R}^1$ and the initial data under $t = 0$:

$$\tilde{\mathbf{U}}|_{t=0} = \tilde{\mathbf{U}}_0(\mathbf{x}), \quad \mathbf{x} \in R^2_+,$$
$$F|_{t=0} = F_0(x_2), \quad x_2 \in R^1.$$

Here $\tilde{\mathbf{U}} = (p, S, \mathbf{v}^*, \Phi)^*$, $\Delta = \xi_1^2 + \xi_2^2$, and for the coefficients from boundary conditions (2.3), previous assumptions (i.e., gas is polytropic, magnetic field is weak) held, the expansions

$$\tilde{d} = d^{(0)} + O(q^2), \quad \tilde{\mu} = \mu^{(0)} + O(q^2),$$

$$\tilde{\lambda}_0 = \lambda_0^{(0)} + O(q^2), \quad \tilde{\kappa} = \kappa^{(0)} + O(q^2),$$

$$N_k = O(q^2), \quad \overline{1,4}$$

are valid.

**Remark 2.1.** Let initial data (1.3) be such that

$$H_k|_{t=0} = q\varphi_k(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2_+, \quad k = 1, 2.$$

Then the function $\Phi_0(\mathbf{x})(= \Phi|_{t=0})$ is found as the solution of the Dirichlet problem for the Poisson equation:

$$\Delta\Phi_0 = \xi_1\varphi_2 - \xi_2\varphi_1, \quad \mathbf{x} \in \mathbb{R}^2_+,$$
$$\Phi_0|_{x_1=0} = -m(1-\chi)F_0(x_2), \quad x_2 \in \mathbb{R}^1.$$

We introduce the differential operators

$$L_\sigma = (\sigma, \nabla), \quad L_\nu = (\nu, \nabla),$$

where $\nu = (l, m)^*$. Then system (2.2) can be rewritten as follows

$$\begin{aligned} &Lp + L_\nu v_\nu + L_\sigma v_\sigma = 0,\\ &LS = 0,\\ &M^2 L v_\nu + L_\nu p = 0,\\ &M^2 L v_\sigma + L_\sigma p - q^2 \Delta\Phi = 0,\\ &L\Phi = v_\sigma. \end{aligned} \tag{2.4}$$

Here $v_\nu = (\mathbf{v}, \nu)$. By simple manipulations, from system (2.4) we obtain that the functions $p$, $\Phi$ satisfy the following equations:

$$M^2L^2p - \Delta p + q^2\Delta L_\sigma \Phi = 0, \tag{2.5}$$

$$M^2L^2\Phi - q^2\Delta\Phi + L_\sigma p = 0. \tag{2.6}$$

At the end of this section we write down system (2.2) in the form of the following symmetric $t$-hyperbolic (by Friedrichs) system:

$$A\mathbf{V}_t + B\mathbf{V}_{x_1} + C\mathbf{V}_{x_2} + \Omega\mathbf{V} = 0. \tag{2.7}$$

Here

$$\mathbf{V} = (p, S, \mathbf{v}^*, Q, R, \Phi)^*, \quad Q = \xi_1\Phi, \quad R = \xi_2\Phi,$$

$A = \mathrm{diag}(1, 1, M^2, M^2, q^2, q^2, 1)$ is the diagonal matrix,

$$B = A + B_0,$$

$$B_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & mq^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -lq^2 & 0 & 0 \\ 0 & 0 & mq^2 & -lq^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & mq^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -lq^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & mq^2 & -lq^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & -l & 0 & 0 & 0 \end{pmatrix}.$$

## 3. Well-posedness of mixed problem $\mathscr{F}$ under $q \ll 1$.

In [7], as it was noted in introduction, the a priori estimates for the solution of problem $\mathscr{F}$ under $q \ll 1$ are obtained for two special cases: $m = 0, l = 1$ (parallel shock wave), $l = 0, m = 1$ (transversal shock wave). Here we shall consider the general case, i.e., we shall prove the well-posedness of problem $\mathscr{F}$, when a weak magnetic field is given.

**Theorem.** *Mixed problem $\mathscr{F}$ is well-posed under $q \ll 1$, and also the following a priori estimates take place for the solutions of this problem:*

$$\|\mathbf{U}(t)\|_{W_2^2(\mathbb{R}_+^2)} \le K_1, \quad 0 < t \le T < \infty, \tag{3.1}$$

$$\|F\|_{W_2^3((0,T)\times\mathbb{R}^1)} \le K_2, \tag{3.2}$$

*where $K_i > 0$, $i = 1, 2$ are constants depending on $T$ $(0 < T < \infty)$;*

$$\|\mathbf{U}(t)\|^2_{W_2^2(\mathbb{R}_+^2)} = \iint_{\mathbb{R}_+^2} \{(\mathbf{U},\mathbf{U}) + (\mathbf{U}_{x_1},\mathbf{U}_{x_1}) + (\mathbf{U}_{x_2},\mathbf{U}_{x_2}) +$$

$$+ (\mathbf{U}_{x_1x_1},\mathbf{U}_{x_1x_1}) + (\mathbf{U}_{x_1x_2},\mathbf{U}_{x_1x_2}) + (\mathbf{U}_{x_2x_2},\mathbf{U}_{x_2x_2})\}\, d\mathbf{x},$$

$$\|F\|^2_{W_2^3((0,T)\times\mathbb{R}^1)} = \int_0^T\int_{\mathbb{R}^1} \{F^2 + (F_t)^2 + (F_{x_2})^2 + \ldots + (F_{x_2x_2x_2})^2\}\, dx_2dt.$$

*Proof.* Following [1], [7], to obtain a priori estimates (3.1), (3.2) at the first stage we construct the following symmetric $t$-hyperbolic (by Friedrichs) system from system (2.7):

$$A_p(\mathbf{V}_p)_t + B_p(\mathbf{V}_p)_{x_1} + C_p(\mathbf{V}_p)_{x_2} + \Omega_p\mathbf{V}_p = 0. \tag{3.3}$$

Here

$$\mathbf{V}_p = (\mathbf{V}^*, \tau\mathbf{V}^*, \xi_1\mathbf{V}^*, \xi_2\mathbf{V}^*, \tau^2\mathbf{V}^*, \tau\xi_1\mathbf{V}^*, \tau\xi_2\mathbf{V}^*, \xi_1^2\mathbf{V}^*, \xi_1\xi_2\mathbf{V}^*, \xi_2^2\mathbf{V}^*)^*;$$

$A_p = \mathrm{diag}(A, A, A, A, A, A, A, A, A, A)$ is the block-diagonal matrix etc.

Writing down the energy integral for system (3.3) in the differential form (see [1], [7]) and, integrating it over the domain $\mathbb{R}_+^2$, we obtain:

$$\frac{d}{dt}J_0(t) - \int_{\mathbb{R}^1} (B_p\mathbf{V}_p, \mathbf{V}_p)|_{x_1=0}\, dx_2 + \iint_{\mathbb{R}_+^2} \left((\Omega_p + \Omega_p^*)\mathbf{V}_p, \mathbf{V}_p\right) d\mathbf{x} = 0, \tag{3.4}$$

where

$$J_0(t) = \iint_{\mathbb{R}^2_+} (A_p \mathbf{V}_p, \mathbf{V}_p)\, d\mathbf{x}, \quad (A_p \mathbf{V}_p, \mathbf{V}_p) = (A\mathbf{V}, \mathbf{V}) + (A\mathbf{V}_t, \mathbf{V}_t) +$$

$$+ (A\mathbf{V}_{x_1}, \mathbf{V}_{x_1}) + (A\mathbf{V}_{x_2}, \mathbf{V}_{x_2}) + (A\mathbf{V}_{tt}, \mathbf{V}_{tt}) + (A\mathbf{V}_{tx_1}, \mathbf{V}_{tx_1}) + (A\mathbf{V}_{tx_2}, \mathbf{V}_{tx_2}) +$$

$$+ (A\mathbf{V}_{x_1x_1}, \mathbf{V}_{x_1x_1}) + (A\mathbf{V}_{x_1x_2}, \mathbf{V}_{x_1x_2}) + (A\mathbf{V}_{x_2x_2}, \mathbf{V}_{x_2x_2}),$$

$$(A\mathbf{V}, \mathbf{V}) = p^2+S^2+M^2|\mathbf{v}|^2+q^2(Q^2+R^2)+\Phi^2 = p^2+S^2+M^2|\mathbf{v}|^2+|\mathbf{H}|^2+\Phi^2$$

etc. When deducing (3.4), we assume that $(\mathbf{V}_p, \mathbf{V}_p)^{\frac{1}{2}} = |\mathbf{V}_p| \to 0$ as $x_1 \to \infty$ or $|x_2| \to \infty$.

By virtue of boundary conditions (2.3) and system (2.2) under $x_1 = 0$, we estimate the second and the third terms in equality (3.4). As the result, we obtain the inequality:

$$\frac{d}{dt} J_0(t) - \int_{\mathbb{R}^1} \{M_1(p^2 + v_2^2 + p_t^2 + p_{x_1}^2 + p_{x_2}^2 + P) + \tag{3.5}$$

$$+ M_q(\Psi^2 + \Psi_t^2 + \Psi_{x_2}^2)\}\,|_{x_1=0}\, dx_2 \le M_2 J_0(t),$$

where $M_1,\ M_2 > 0$ are some constants independent of $q$;

$$M_q = O(q^2), \quad \Psi = F_{x_2x_2},$$

$$P = p_{tt}^2 + p_{tx_1}^2 + p_{tx_2}^2 + p_{x_1x_1}^2 + p_{x_1x_2}^2 + p_{x_2x_2}^2 .$$

Considering again system (2.2) under $x_1 = 0$, after cumbersome calculations, we obtain from it, with the help of boundary conditions (2.3), the following equality:

$$\Psi = (c_1 p_t + c_2 p_{x_1} + N_5 p_{x_2})|_{x_1=0} . \tag{3.6}$$

Here

$$c_1 = -\frac{1+\tilde{d}}{\tilde{\lambda}_0} + O(q^2), \quad c_2 = -\frac{1}{M^2\tilde{\lambda}_0} + O(q^2), \quad N_5 = O(q^2).$$

Estimating the integral

$$\int_{\mathbb{R}^1} \{M_1(p^2 + v_2^2 + p_t^2 + p_{x_1}^2 + p_{x_2}^2) + M_q\Psi^2\}\,|_{x_1=0}\, dx_2$$

with the help of the property of the function trace from $W_2^1(R_+^2)$ at the line $x_1 = 0$ (see [10]), by virtue of (3.6), we reduce inequality (3.5) to the form:

$$\frac{d}{dt} J_0(t) - \tilde{M}_1 \int_{\mathbb{R}^1} P|_{x_1=0}\, dx_2 \le \tilde{M}_2 J_0(t), \tag{3.7}$$

where $\tilde{M}_1,\ \tilde{M}_2 > 0$ is some constant independent of $q$.

Now we proceed to the second, more complicated, stage in the construction of the expanded system. Following [1], [7], we rewrite equation (2.5) as follows:

$$(\tilde{L}_1^2 - L_2^2 - \tilde{L}_3^2)p + \frac{q^2}{\beta^2}\Delta L_\sigma \Phi = 0. \tag{3.8}$$

Here

$$\tilde{L}_1 = ML_1, \quad L_1 = \tau/\beta^2, \quad L_2 = \xi_1 - M^2 L_1,$$
$$\tilde{L}_3 = L_3/\beta, \quad L_3 = \xi_2, \quad \beta = \sqrt{1 - M^2},$$

moreover, $M < 1$ for weak magnetic field (see section 1). If the function $p$ satisfies equation (3.8), then the vector

$$\mathbf{W} = (\mathbf{Y}_1^*, \mathbf{Y}_2^*, \mathbf{Y}_3^*)^*,$$

where $\mathbf{Y}_1 = \tilde{L}_1\mathbf{Y},\ \mathbf{Y}_2 = L_2\mathbf{Y},\ \mathbf{Y}_3 = \tilde{L}_3\mathbf{Y},\ \mathbf{Y} = \tilde{\nabla}p,\ \tilde{\nabla} = (\tilde{L}_1, L_2, \tilde{L}_3)^*$, satisfies the system of the form (see [7]):

$$\left\{\hat{A}\tilde{L}_1 - \hat{B}L_2 - \hat{C}\tilde{L}_3\right\}\mathbf{W} + \frac{q^2}{\beta^2}\begin{pmatrix}\mathcal{K}\\ \mathcal{L}\\ \mathcal{M}\end{pmatrix}\Delta L_\sigma(\tilde{\nabla}\Phi) = 0. \tag{3.9}$$

Here

$$\hat{A} = \begin{pmatrix}\mathcal{K} & \mathcal{L} & \mathcal{M}\\ \mathcal{L} & \mathcal{K} & i\mathcal{N}\\ \mathcal{M} & -i\mathcal{N} & \mathcal{K}\end{pmatrix}, \quad \hat{B} = \begin{pmatrix}\mathcal{L} & \mathcal{K} & i\mathcal{N}\\ \mathcal{K} & \mathcal{L} & \mathcal{M}\\ -i\mathcal{N} & \mathcal{M} & -\mathcal{L}\end{pmatrix}$$

$$\hat{C} = \begin{pmatrix}\mathcal{M} & -i\mathcal{N} & \mathcal{K}\\ i\mathcal{N} & -\mathcal{M} & \mathcal{L}\\ \mathcal{K} & \mathcal{L} & \mathcal{M}\end{pmatrix};$$

$\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are as yet arbitrary Hermitian matrices of order 3. Returning in (3.9) to the differential operators $\tau$, $\xi_1$, $\xi_2$, we obtain the system:

$$\left\{D\tau - \hat{B}\xi_1 - \frac{1}{\beta}\hat{C}\xi_2\right\}\mathbf{W} + \frac{q^2}{\beta^2}\begin{pmatrix}\mathcal{K}\\ \mathcal{L}\\ \mathcal{M}\end{pmatrix}\Delta L_\sigma(\tilde{\nabla}\Phi) = 0, \tag{3.10}$$

where $D = \dfrac{M}{\beta^2}(\hat{A} + M\hat{B})$.

**Remark 3.1.** The following relationes are valid (see [4]):

$$\hat{A} = T_0^*\{I_2 \times \tilde{H}\}T_0,$$
$$\hat{B} = T_0^*\left\{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \tilde{H}\right\} T_0, \tag{3.11}$$
$$\hat{C} = T_0^*\left\{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \tilde{H}\right\} T_0.$$

Here

$$T_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times I_3, \quad \tilde{H} = \begin{pmatrix} \mathscr{K} - \mathscr{M} & -\mathscr{L} - i\mathscr{N} \\ -\mathscr{L} + i\mathscr{N} & \mathscr{K} + \mathscr{M} \end{pmatrix}$$

$I_2 \times \tilde{H}$ is the Kronecker product of the matrices $I_2$ and $\tilde{H}$ etc.; $I_2$ is the unit matrix of order 2 etc. By virtue of (3.11),

$$D = \frac{M}{\beta^2} T_0^* \left\{\begin{pmatrix} 1 & -M \\ -M & 1 \end{pmatrix} \times \tilde{H}\right\} T_0.$$

Let us obtain the boundary conditions for system (3.10). To this end, we multiply system (3.10) scalarwise by the vector $(M^2\tau, 0, -\tau, 0, 0)^*$. Considering the obtained expression at $x_1 = 0$ and making use of boundary conditions (2.3), we obtain the following relationes:

$$\left\{M^2(1 + \tilde{d}\rho_1)\tau^2 - \beta^2\rho_2\tau\xi_1 + M^2\lambda\xi_2^2 + N_6\tau\xi_2\right\} p = 0, \quad x_1 = 0, \tag{3.12}$$

where $\lambda = \tilde{\lambda}_0\tilde{\mu}$, $\rho_i = 1 + O(q^2)$, $i = 1, 2$; $N_6 = O(q^2)$. We consider also equation (3.8) at $x_1 = 0$. Using boundary conditions (2.3) (see also (3.6)), we reduce it to the form:

$$\begin{aligned}(\rho_3\tilde{L}_1^2 - \rho_4 L_2^2 - \rho_5\tilde{L}_3^2)p + \\ +(N_7\tilde{L}_1 L_2 + N_8\tilde{L}_1 L_3 + N_9 L_2\tilde{L}_3)p = 0, \quad x_1 = 0,\end{aligned} \tag{3.13}$$

where $\rho_i = 1 + O(q^2)$, $i = \overline{3,5}$; $N_k = O(q^2)$, $k = \overline{7,9}$. Accounting (3.12), (3.13), we take the following expressions as the boundary conditions at $x_1 = 0$ for system (3.10) (see [7]):

$$\rho_3\tilde{L}_1(\tilde{L}_1 p) - \rho_4 L_2(L_2 p) - \rho_5\tilde{L}_3(\tilde{L}_3 p) +$$

$$+\alpha\{\rho_6\tilde{L}_1(L_2p)-\rho_7L_2(\tilde{L}_1p)\}+N_8\tilde{L}_1(\tilde{L}_3p)+N_9L_2(\tilde{L}_3p)=0,$$

$$\tilde{L}_3(L_2p)-L_2(\tilde{L}_3p)=0,$$

$$\rho_8\tilde{L}_1(L_2p)-\rho_9M\tilde{d}L_2(L_2p)-\frac{M}{\beta}\tilde{m}\tilde{L}_3(\tilde{L}_3p)+$$

$$+N_{10}\tilde{L}_1(\tilde{L}_3p)+N_{11}L_2(\tilde{L}_3p)+N_{12}\tilde{L}_1(\tilde{L}_1p)=0,$$

which we shall rewrite in the following form

$$A_1\mathbf{Y}_1+B_1\mathbf{Y}_2+C_1\mathbf{Y}_3=0. \tag{3.14}$$

Here

$$A_1=\begin{pmatrix}\rho_3 & \alpha\rho_6 & N_8\\ 0 & 0 & 0\\ N_{12} & \rho_8 & N_{10}\end{pmatrix},\quad B_1=\begin{pmatrix}-\alpha\rho_7 & -\rho_4 & N_9\\ 0 & 0 & -1\\ 0 & -\rho_9M\tilde{d} & N_{11}\end{pmatrix},$$

$$C_1=\begin{pmatrix}0 & 0 & -\rho_5\\ 0 & 1 & 0\\ 0 & 0 & -\frac{M\tilde{m}}{\beta}\end{pmatrix},$$

$\alpha>1$ is some constant; $\tilde{m}=\frac{\rho_5}{\rho_3}\rho_1\beta\tilde{d}+\frac{M^2\lambda}{\beta}$, $\rho_i=1+O(q^2)$, $i=\overline{6,9}$; $N_k=O(q^2)$, $k=\overline{10,12}$. Moreover for polytropic gas and weak magnetic field

$$\tilde{m}=\tilde{m}^{(0)}+O(q^2),\quad \lambda=\lambda^{(0)}+O(q^2),$$

$$\tilde{m}^{(0)}=\beta d^{(0)}+\frac{M^2\lambda^{(0)}}{\beta}>0,\quad \lambda^{(0)}=\mu^{(0)}\lambda_0^{(0)}<0.$$

Let

$$\Lambda=\begin{pmatrix}\Lambda_I\\ \Lambda_{II}\end{pmatrix}=T_0\mathbf{W},$$

where

$$\Lambda_I=\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix},\quad \Lambda_{II}=\begin{pmatrix}\Lambda_3\\ \Lambda_4\end{pmatrix},$$

$\Lambda_k$, $k=\overline{1,4}$ are the vectors of dimension 3. As

$$\mathbf{Y}_1=\frac{\sqrt{2}}{2}(\Lambda_1+\Lambda_4),\ \mathbf{Y}_2=-\sqrt{2}\Lambda_2=-\sqrt{2}\Lambda_3,\ \mathbf{Y}_3=\frac{\sqrt{2}}{2}(\Lambda_4-\Lambda_1),$$

then conditions (3.14) may be also presented in such form

$$\Lambda_I = G\Lambda_{II}, \tag{3.15}$$

where

$$G = \begin{pmatrix} G_1 & -G_2 \\ I_3 & 0 \end{pmatrix}, \quad G_1 = 2(A_1 - C_1)^{-1}B_1, \quad G_2 = (A_1 - C_1)^{-1}(A_1 + C_1).$$

Let all the eigenvalues of the matrix $G$ lie strictly in the left semi-plane, i.e., $\mathrm{Re}\,\lambda_j(G) < 0,\ j = \overline{1,6}$. The latter is valid if $\tilde{m} > 0$, $\lambda < 0$ (i.e., gas is polytropic, and $q \ll 1$, see [7])! Now we compose the Lyapunov matrix equation

$$G^*\tilde{H} + \tilde{H}G = -G_0 \tag{3.16}$$

to find the matrix $\tilde{H}$, which appears in the formulas (3.11). Equation (3.16), as it is known, has the unique solution (see, for example, [11])

$$\tilde{H} = \begin{pmatrix} \tilde{H}_1 & \tilde{H}_2 \\ \tilde{H}_2^* & \tilde{H}_3 \end{pmatrix} > 0, \quad \tilde{H}_1 = \tilde{H}_1^*, \quad \tilde{H}_3 = \tilde{H}_3^*,$$

for every real symmetric positive definite matrix $G_0$. Therewith, the matrix $\tilde{H}$ is also real and symmetric, and the matrices $\mathscr{K}$, $\mathscr{L}$, $\mathscr{M}$, $\mathscr{N}$ are the following:

$$\mathscr{K} = \frac{1}{2}(\tilde{H}_1 + \tilde{H}_3), \quad \mathscr{M} = \frac{1}{2}(\tilde{H}_3 - \tilde{H}_1),$$

$$\mathscr{L} = -\frac{1}{2}(\tilde{H}_2 + \tilde{H}_2^*), \quad i\mathscr{N} = \frac{1}{2}(\tilde{H}_2^* - \tilde{H}_2).$$

As $\tilde{H} > 0$, then $D > 0$ (see Remark 3.1).

Let us write out the energy integral in the differential form (see [12]) for system (3.10):

$$(D\mathbf{W}, \mathbf{W})_t - (\hat{B}\mathbf{W}, \mathbf{W})_{x_1} - \frac{1}{\beta}(\hat{C}\mathbf{W}, \mathbf{W})_{x_2} + \frac{q^2}{\beta^2}\Big\{2(\mathbf{Y}_1, \mathscr{K}\Delta L_\sigma(\tilde{\nabla}\Phi)) + \tag{3.17}$$
$$+ 2(\mathbf{Y}_2, \mathscr{L}\Delta L_\sigma(\tilde{\nabla}\Phi)) + 2(\mathbf{Y}_3, \mathscr{M}\Delta L_\sigma(\tilde{\nabla}\Phi))\Big\} = 0.$$

With reference to (2.6) and the last equation of system (2.2), we obtain the following equations:

$$\begin{aligned} L_\sigma \mathbf{Y} &= q^2\mathbf{X} - M^2 L\mathbf{Z}, \\ L\mathbf{X} &= \Delta\mathbf{Z}, \end{aligned} \tag{3.18}$$

where $\mathbf{X} = \Delta(\tilde{\nabla}\Phi)$, $\mathbf{Z} = \tilde{\nabla}v_\sigma$. Then, in view of (3.18), the expression in curly braces in equality (3.17) may be transformed as:

$$\{\cdots\} = \tau\Omega_0 + \xi_1\Omega_1 + \xi_2\Omega_2. \tag{3.19}$$

Here

$$\begin{aligned}
\Omega_0 =& -q^2(\mathbf{X}, \mathscr{K}_1\mathbf{X}) + M^2\{2(\tilde{L}_1\mathbf{Z}, \mathscr{K}\mathbf{X}) + 2(L_2\mathbf{Z}, \mathscr{L}\mathbf{X}) + \\
& + 2(\tilde{L}_3\mathbf{Z}, \mathscr{M}\mathbf{X}) + (\xi_1\mathbf{Z}, \mathscr{K}_1\xi_1\mathbf{Z}) + (\xi_2\mathbf{Z}, \mathscr{K}_1\xi_2\mathbf{Z})\},\\
\Omega_1 =& -2m\{(\mathbf{Y}_1, \mathscr{K}\mathbf{X}) + (\mathbf{Y}_2, \mathscr{L}\mathbf{X}) + (\mathbf{Y}_3, \mathscr{M}\mathbf{X})\} - \\
& - q^2(\mathbf{X}, \mathscr{L}\mathbf{X}) + M^2\{2(\tilde{L}_1\mathbf{Z}, \mathscr{K}\mathbf{X}) + 2(L_2\mathbf{Z}, \mathscr{L}\mathbf{X}) + \\
& + 2(\tilde{L}_3\mathbf{Z}, \mathscr{M}\mathbf{X}) - 2(\tau\mathbf{Z}, \mathscr{K}_1\xi_1\mathbf{Z}) + \\
& + (\xi_2\mathbf{Z}, \mathscr{L}\xi_2\mathbf{Z}) - (\xi_1\mathbf{Z}, \mathscr{L}\xi_1\mathbf{Z}) - 2(\xi_2\mathbf{Z}, \mathscr{M}_1\xi_1\mathbf{Z})\},\\
\Omega_2 =& 2l\{(\mathbf{Y}_1, \mathscr{K}\mathbf{X}) + (\mathbf{Y}_2, \mathscr{L}\mathbf{X}) + (\mathbf{Y}_3, \mathscr{M}\mathbf{X})\} - \\
& - q^2(\mathbf{X}, \mathscr{M}_1\mathbf{X}) - M^2\{2(\tau\mathbf{Z}, \mathscr{K}_1\xi_2\mathbf{Z}) + \\
& + 2(\xi_1\mathbf{Z}, \mathscr{L}\xi_2\mathbf{Z}) + +(\xi_2\mathbf{Z}, \mathscr{M}_1\xi_2\mathbf{Z}) - (\xi_1\mathbf{Z}, \mathscr{M}_1\xi_1\mathbf{Z})\},\\
& \mathscr{M}_1 = \frac{1}{\beta}\mathscr{M}, \quad \mathscr{K}_1 = \frac{M}{\beta^2}(\mathscr{K} - M\mathscr{L}).
\end{aligned}$$

Let us integrate identity (3.17) (with reference to (3.19)) over the domain $\mathbb{R}^2_+$ assuming that $|\mathbf{W}| \to 0$ as either $x_1 \to \infty$ or $|x_2| \to \infty$ etc. Finally, we obtain:

$$\frac{d}{dt}J_1(t) + \int_{\mathbb{R}^1}\left\{(\hat{B}\mathbf{W}, \mathbf{W}) - \frac{q^2}{\beta^2}\Omega_1\right\}\Big|_{x_1=0} dx_2 = 0. \tag{3.20}$$

Here

$$J_1(t) = \iint_{\mathbb{R}^2_+}\left\{(D\mathbf{W}, \mathbf{W}) + \frac{q^2}{\beta^2}\Omega_0\right\} d\mathbf{x}.$$

Note that, by virtue of (3.11), (3.15), the quadratic form

$$(\hat{B}\mathbf{W}, \mathbf{W})|_{x_1=0} = (G_0\Lambda_{II}, \Lambda_{II})|_{x_1=0} > 0.$$

Since

$$\Lambda_{II} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\mathbf{Y}_2 \\ \mathbf{Y}_1 + \mathbf{Y}_3, \end{pmatrix}$$

then

$$(\hat{B}\mathbf{W}, \mathbf{W})|_{x_1=0} > M_3\Big\{(\tilde{L}_1^2 p)^2 + (\tilde{L}_1 L_2 p)^2 + (\tilde{L}_1\tilde{L}_3 p)^2 + \tag{3.21}$$

$$+ (L_2^2 p)^2 + (L_2\tilde{L}_3 p)^2 + (\tilde{L}_3^2 p)^2\Big\}\Big|_{x_1=0} > \tilde{M}_3 P|_{x_1=0}\,,$$

where $M_3,\ \tilde{M}_3 > 0$ are some independent of $q$ constants. Note that with the help of boundary conditions (2.3), system (2.2) at $x_1 = 0$ and (3.6), we may obtain the inequality

$$-\frac{q^2}{\beta^2}\Omega_1|_{x_1=0} > N_{13} P|_{x_1=0}, \tag{3.22}$$

where $N_{13} = O(q^2)$.

In view of the smallness of $q$, quadratic form

$$(A_p\mathbf{V}_p, \mathbf{V}_p) + (D\mathbf{W}, \mathbf{W}) + \frac{q^2}{\beta^2}\Omega_0 > 0 \qquad (A_p > 0, \quad D > 0)\,.$$

So, adding equality (3.20) to inequality (3.7) and accounting that, by the choice of matrix $G_0$ (see inequality (3.21)) and in view of (3.22), it is possible to achieve the positive definiteness of the form

$$\mathscr{A} = \left\{(\hat{B}\mathbf{W}, \mathbf{W}) - \frac{q^2}{\beta}\Omega_1 - \tilde{M}_1 P\right\}\Big|_{x_1=0} > (\tilde{M}_3 - \tilde{M}_1 + N_{13})P|_{x_1=0} > 0\,,$$

we finally obtain the following inequality:

$$\frac{d}{dt}J(t) \le M_4 J(t), \quad t > 0,$$

where $J(t) = J_0(t) + J_1(t)$; $M_4 > 0$ is a constant independent of $q$. From this inequality the a priori estimate for problem $\mathscr{F}'$ follows:

$$J(t) \le e^{M_4 t} J(0), \quad t > 0, \tag{3.23}$$

which shows that mixed problem $\mathscr{F}'$ is well-posed.

**Remark 3.2.** Let the functions $\varphi_k(\mathbf{x})$ (see Remark 2.1), $k = 1, 2$, $\mathbf{x} \in \mathbb{R}^2_+$ be finite, with the compact supports lying within the bounded domain $\Omega \subset \mathbb{R}^2_+$ with the smooth boundary $\partial\Omega$. Then we determine the function $\Phi_0(\mathbf{x})$ as follows. In the domain $\bar{\mathbb{R}}^2_+ \setminus \Omega$ $\Phi_0(\mathbf{x}) \equiv -m(1-\chi)F_0(x_2)$; and in the domain $\Omega$ it is found as the solution of the Dirichlet problem:

$$\Delta\Phi_0 = \xi_1\varphi_2 - \xi_2\varphi_1, \quad \mathbf{x} \in \Omega,$$

$$\Phi_0|_{\partial\Omega} = -m(1-\chi)F_0(x_2).$$

Then for thus constructed function $\Phi_0(\mathbf{x})$ the following estimate is valid (see [13])

$$\|\Phi_0\|_{W_2^2(\mathbb{R}_+^2)} \leq M_5 \left\{ \|\varphi_1\|_{W_2^2(\mathbb{R}_+^2)} + \|\varphi_2\|_{W_2^2(\mathbb{R}_+^2)} + \|F_0\|_{W_2^2(\mathbb{R}^1)} \right\},$$

where $M_5 > 0$ is a constant independent of $\varphi_{1,2}$, $F_0$. From the last inequality with the help of boundary conditions (2.3) we easily deduce the following estimate:

$$\|\Phi_0\|_{W_2^2(\mathbb{R}_+^2)} \leq \tilde{M}_5 \left\{ \|\varphi_1\|_{W_2^2(\mathbb{R}_+^2)} + \|\varphi_2\|_{W_2^2(R_+^2)} + \right.$$
$$\left. + \left\| p_0|_{x_1=0} \right\|_{W_2^1(\mathbb{R}^1)} + \left\| v_{2,0}|_{x_1=0} \right\|_{W_2^1(\mathbb{R}^1)} \right\},$$

where $\tilde{M}_5 > 0$ is a constant, $p_0 = p|_{t=0}$, $v_{2,0} = v_2|_{t=0}$. Using the property of the function trace from $W_2^1(\mathbb{R}_+^2)$ at the line $x_1 = 0$ we finally obtain:

$$\|\Phi_0\|_{W_2^2(\mathbb{R}_+^2)} \leq M_6 \left\{ \|\varphi_1\|_{W_2^2(\mathbb{R}_+^2)} + \|\varphi_2\|_{W_2^2(\mathbb{R}_+^2)} + \|p_0\|_{W_2^2(\mathbb{R}_+^2)} + \left\|v_{2,0}\right\|_{W_2^2(\mathbb{R}_+^2)} \right\}.$$

Here $M_6 > 0$ is some constant independent of $\varphi_{1,2}$, $p_0$, $v_{2,0}$.

Considering the introduction of the function $\Phi$ as an auxiliary action, with reference to Remark 3.2, from inequality (3.23) we deduce the desired a priori estimate for the solution of problem $\mathscr{F}$:

$$\|\mathbf{U}(t)\|_{W_2^2(\mathbb{R}_+^2)} \leq K_1, \quad 0 < t \leq T < \infty,$$

Adding equality (3.21) to inequality (3.7), we obtain:

$$\frac{d}{dt} J(t) + \int_{\mathbb{R}^1} \mathscr{A}\, dx_2 \leq M_4 J(t). \tag{3.24}$$

Integrating (3.24) over the interval $(0, T)$ and accounting that $J(t) > 0$, $\mathscr{A} > 0$, we obtain the following inequality:

$$\int_0^T \int_{\mathbb{R}^1} \left\{ (F_t)^2 + (F_{x_2})^2 + (F_{tt})^2 + (F_{tx_2})^2 + \right.$$
$$\left. + \cdots + (F_{x_2x_2x_2})^2 \right\} dx_2 dt \leq M_7, \tag{3.25}$$

where $M_7 > 0$ is a constant depending on $T$. From the second and the third boundary conditions of (2.3) we obtain the equality

$$F_t = \left( \tilde{\mu} - \frac{N_2 N_3}{\tilde{\lambda}_0} \right) p + \frac{N_2}{\tilde{\lambda}_0} v_2, \quad x_1 = 0.$$

Multiplying it by $2F$ and integrating with respect to $x_2 \in \mathbb{R}^1$, using the Hölder inequality, we obtain the following estimate:

$$\frac{d}{dt}\|F(t)\|^2_{L_2(\mathbb{R}^1)} \leq c\,\|F(t)\|_{L_2(\mathbb{R}^1)}\left\{\left\|p|_{x_1=0}\right\|_{L_2(\mathbb{R}^1)} + \left\|v_2|_{x_1=0}\right\|_{L_2(\mathbb{R}^1)}\right\},$$

where $c > 0$ is a constant, $\|F(t)\|^2 = \int_{\mathbb{R}^1} F^2\,dx_2$ etc. The last inequality, if we use the property of the function trace at the line $x_1 = 0$ (see [14]), is rewritten as:

$$\frac{d}{dt}\|F(t)\|_{L_2(\mathbb{R}^1)} \leq \frac{cM_b}{2}\left\{\|p(t)\|_{W_2^1(\mathbb{R}^2_+)} + \|v_2(t)\|_{W_2^1(\mathbb{R}^2_+)}\right\}. \tag{3.26}$$

Here $M_b > 0$ is some constant. From inequality (3.26), with reference to proved estimate (3.1), we obtain

$$\|F\|_{L_2((0,T)\times\mathbb{R}^1)} \leq M_8\,, \tag{3.27}$$

where $M_8 > 0$ is a constant depending on $T$. Then, combining (3.25) and (3.27), we obtain finally the desired a priori estimate for the function $F$:

$$\|F\|_{W_2^3((0,T)\times\mathbb{R}^1)} \leq K_2\,,$$

and that completes the proof of Theorem.

Thus, on the basis of proved theorem we may conclude the well-posedness of linear mixed problem on the stability of fast magnetohydrodynamic shock wave in polytropic gas under the weak magnetic field. It means that the given type of strong discontinuity in magnetohydrodynamics is stable (with respect to small perturbations).

**Remark 3.3.** We remind that we find the unique solution of system (1.1) with boundary conditions (1.2) and initial data (1.3) and for which a priori estimates (3.1), (3.2) take place. These estimates specify the behaviour at infinity of the perturbations of the pressure, entropy, velocity, magnetic field and the front of fast shock wave $F$.

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