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OPERATOR EQUATIONS AND INVARIANT SUBSPACES

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Banach space operators acting on some fixed space X are considered. If two such operators A and B verify the condition $A^2 = B^2$ and if A has nontrivial hyperinvariant subspaces, then B has nontrivial invariant subspaces. If A and B commute and satisfy a special type of functional equation, and if A is not a scalar multiple of the identity, the author proves that if A has nontrivial invariant subspaces, then so does B.

1. Introduction.

This is a short note inspired by [1]. There the author shows that two nilpotent operators of index 2, having no nontrivial, common invariant subspaces are quasi -similar, provided the dimension of the space be larger than 3. Reviewing [1] for the American Mathematical Society this author observed that the condition regarding the dimension of the space is superfluous and gave an elementary proof for the result above.

In the following the term operator means linear, bounded operator acting on some complex Banach space X. Invariant subspace means closed linear subspace invariated by some operator. If T is an operator on X we denote Lat$T$ the lattice of all invariant subspaces of T. By $\{T\}'$ we denote the set of all operators commuting with T. If a subspace M is invariant for any operator in $\{T\}'$,

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we say M is a hyperinvariant subspace of T. If 1 denotes the identity on X, we call scalar multiple of the identity any operator A such that $A = \lambda 1$ for some complex number λ.

The authors of [2] show that the existence of some T such that Lat$T$ is trivial, that is Lat$T$ consists only of X and 0 is equivalent with the existence of two quadratic operators having no common, nontrivial invariant subspaces. There is a Banach space operator T such that Lat$T$ is trivial. The first published example is in [4]. It is still unknown if a Hilbert space operator acting on some complex Hilbert space having dimension larger than 1 without nontrivial invariant subspaces exists. This open problem is usually referred to as *the invariant subspace problem*. It makes the result in [1] interesting via its connection with [2]. This note contains the elementary proof we can give for it removing the unnecessary restriction regarding the dimension of the space along with related results which obtain the existence of nontrivial invariant subspaces from the fact that two operators satisfy some operator equation.

2. The main results.

An operator T is called a quasiafinity if it has null kernel and dense range. We say that T intertwines the operator pair (A, B) if $TA = BT$. If T intertwines both (A, B) and (B, A) we say T doubly intertwines the operator pair (A, B). A and B are quasi-similar if there exist the quasiafinities T and V such that T intertwines (A, B) and V intertwines (B, A).

Lemma. *If T doubly intertwines (A, B) and* Lat $A \cap$ Lat B *is trivial then T is either 0 or a quasiafinity. The same is true if T commutes with A and B and* Lat $A \cap$ Lat B *is trivial.*

Proof. Denote by Im T the closure of the range of T and by Ker T the kernel of T. Suppose T doubly intertwines (A, B). $TA = BT$ implies that Im $T \in$ Lat B and Ker $T \in$ Lat A. Since $TB = AT$ we deduce Im T and Ker T are in Lat $A \cap$ Lat B. If Im $T = 0$ then $T = 0$. If Im $T = X$ then Ker $T = 0$ and T is in this case a quasiafinity. If T commutes with A and B then the very same as above Im T and Ker T are in Lat $A \cap$ Lat B so by the same argument either T is 0 or T is a quasiafinity. □

Theorem 1. *If $A^2 = B^2$ and A has nontrivial hyperinvariant subspaces then B has nontrivial invariant subspaces.*

Proof. If we suppose Lat B is trivial then Lat $A \cap$ Lat B is trivial. Denote $T = A + B$. It is easy to verify that T doubly intertwines (A, B). If we suppose $T = 0$ then Lat A = Lat B, so that Lat B is nontrivial, which is a contradiction. We admit $T \neq 0$ so by the Lemma, T is a quasiafinity doubly intertwining (A, B), so that B is quasi-similar to A. By [3], Theorem 6.19, we deduce B has nontrivial hyperinvariant subspaces which is once more absurd. We conclude that Lat B is nontrivial □

Remark. *The trick in the proof above may be used to prove the main result in* [1], *namely if A and B are nilpotent operators of index 2 having no nontrivial common invariant subspaces then A and B are quasi-similar.*

Proof. If $\operatorname{Lat} A \cap \operatorname{Lat} B$ is trivial then $T = A + B$ is nonzero because if $T = 0$ then the two lattices would coincide and a nilpotent operator has a nontrivial invariant subspace lattice. Consequently A and B are quasi-similar. No assumption on the dimension of the space has been made. □

We have failed trying to prove similar results for nilpotent operators of higher index. Suppose now that A and B are commuting operators. Chose $f \neq 0$ analytic on a Cauchy domain containing the spectrum of A, $\varphi \neq 0$ analytic on a Cauchy domain containing the spectrum of B, φ_j analytic and one-to-one on an open set containing the full spectrum of B, f_j analytic on the full spectrum of A, $j = 1, 2, 3, \ldots, n$. We intend to use the Riesz - Dunford functional calculus. We refer the reader to [3], Chapter 2, for this calculus and the setting above. We recall however that full spectrum means the union of the spectrum and the bounded components of the resolvent set. Under these assumptions we can prove

Theorem 2. *If A and B are commuting operators which satisfy the equation*

$$f(A)\,\varphi(B) \prod_{1\leq j\leq n} (f_j(A) + \varphi_j(B)) = 0 \tag{1}$$

A is not a scalar multiple of the identity, and Lat A *is not trivial then* Lat B *is not trivial.*

Proof. First observe that each operator above commutes both wit A and B. Consequently the operators involved in (1) commute with each other. If we suppose that Lat B is trivial, then A and B have no nontrivial common invariant subspaces. By the Lemma, each factor in (1) is either 0 or a quasiafinity. Consequently at least one factor must be 0.

If we suppose $f(A) = 0$ and denote by σ the spectrum, then $\sigma(f(A)) = f(\sigma(A))$ implies that $\sigma(A) \subseteq Z(f)$ where $Z(f)$ denotes the zeros of f. Since $Z(f)$ consists of isolated points and $\sigma(A)$ is compact we deduce $\sigma(A)$ is finite. We can chose g analytic in a Cauchy domain containing $\sigma(A)$ such that g has no zeros and $f = pg$ where p is a nonzero polynomial. $f(A) = p(A)g(A) = 0$ and $g(A)$ is invertible because if we suppose $0 \in \sigma(g(A)) = g(\sigma(A))$, we obtain

g has zeros in $\sigma(A)$. Consequently $p(A) = 0$ that is A is an algebraic operator. Chose any nonzero vector x in X, observe that the cyclic invariant subspace spanned by x, $\bigvee_{n\geq 0} A^n x$ is finite dimensional and deduce that A has nonvoid point spectrum. Consequently A has proper hyperinvariant subspaces. Hence Lat B is nontrivial which is a contradiction.

If we suppose $\varphi(B) = 0$ we obtain as above that B has nontrivial invariant subspaces.

If we suppose $f_j(A) + \varphi_j(B) = 0$ for some j, observe that f_j is chosen such that Lat $A \subseteq$ Lat $f_j(A)$ [3], Corollary 2.13, so Lat $f_j(A)$ is nontrivial. φ_j is chosen such that Lat B = Lat $\varphi_j(B)$ [3], Theorem 2.14, and Lat $f_j(A)$ = Lat $\varphi_j(B)$ which imply Lat B is nontrivial. We admit that B has nontrivial invariant subspaces. □

In the following Remark, neither A nor B is a scalar multiple of identity.

Remark. *If A and B are commuting operators which satisfy* (1) *then either* Lat A *is trivial or A and B have common nontrivial invariant subspaces.*

Proof. Suppose A and B have no common, nontrivial invariant subspaces. In that case (1) holds if and only if at least one factor in it is 0. If $f(A) = 0$ then like in the proof of Theorem 2 we obtain that A has nontrivial hyperinvariant subspaces. Consequently Lat $A \cap$ Lat B cannot be trivial. The same holds if $\varphi(B) = 0$. If for some j, $f_j(A) + \varphi_j(B) = 0$ then Lat $A \subseteq$ Lat B and in that case Lat A is trivial. □

Corollary 3. *Suppose A and B are commuting operators p, q and r_j are nonzero polynomials with complex coefficients and a_j is a nonzero complex number for each $j = 1, 2, 3, \ldots, n$. If A is not a scalar multiple of the identity,*

$$p(A)q(B)\prod_{j=1}^{n}(a_j B + r_j(A)) = 0 \tag{2}$$

and Lat A *is nontrivial then* Lat B *is nontrivial.*

Consequently if for some n, $A^n = B^n$ and $AB = BA$ then either both Lat A and Lat B are trivial or A and B have common, nontrivial invariant subspaces. Indeed, in this case $\prod_{j=1}^{n}(A - a_j B) = 0$ with $a_1, \ldots, a_n$ the n distinct complex roots of order n of 1.

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