LE MATEMATICHE
Vol. XLIX (1994) – Fasc. I, pp. 149-168

COMMUTATORS OF INTEGRAL OPERATORS WITH POSITIVE KERNELS

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Let K be an integral operator on a homogeneous space (X, d, μ), defined by a positive, locally integrable kernel k, and assume that K is continuous from $\mathscr{L}^p$ to $\mathscr{L}^q$ for suitable p and q; let $a \in \mathrm{BMO}(X)$. Here we prove that, if k satisfies a *"pointwise Hörmander inequality"*, the operator

$$C_a f(x) = \int_{X\setminus\{x\}} k(x, y)|a(x) - a(y)| f(y)\, d\mu(y)$$

satisfies the $\mathscr{L}^p - \mathscr{L}^q$ estimate

$$\|C_a f\|_q \leq c\|a\|_* \|f\|_p$$

(with p, q as above). This estimate in particular implies an analogous one for the *commutator* of K with a.

0. Introduction.

This note deals with commutators of some integral operators on homogeneous spaces, and it is a continuation of [2] and [3]. In [2] it is proved that, if K is a Calderón-Zygmund operator on a homogeneous space (X, d, μ) and $a \in \mathrm{BMO}(X)$, then the *commutator*

$$C[K, a]f = K(af) - a \cdot Kf \tag{0.1}$$

Entrato in Redazione l'8 novembre 1994.

satisfies the $\mathscr{L}^p$-estimate:

$$\|C[K,a]f\|_p \leq c\|a\|_*\|f\|_p, \quad 1 < p < \infty. \tag{0.2}$$

(This generalizes a theorem by Coifman-Rochberg-Weiss [10], valid in the euclidean case).

In [3] a similar result is established for fractional integrals. More precisely, if

$$I_\alpha f(x) = \int_{X\setminus\{x\}} k_\alpha(x,y) f(y)\, d\mu(y) \tag{0.3}$$

is the fractional integral on (X, d, μ), that is $k_\alpha(x, y) = \mu(B(x, d(x, y)))^{\alpha-1}$, $0 < \alpha < 1$, and, for $a \in \mathrm{BMO}(X)$, we set

$$C_a^\alpha f(x) = \int_{X\setminus\{x\}} k_\alpha(x,y)|a(x) - a(y)| f(y)\, d\mu(y), \tag{0.4}$$

then it is proved that for $p \in (1, \frac{1}{\alpha})$, $\frac{1}{q} = \frac{1}{p} - \alpha$,

$$\|C_a^\alpha f\|_q \leq c\|a\|_*\|f\|_p. \tag{0.5}$$

Inequality (0.5), clearly, is a stronger estimate than the analogous one on the commutator of I_α, proved, in the euclidean case, by Chanillo [9].

The idea of the present paper is that what allows to put the absolute value inside the integral, in the definition of $C_a^\alpha f$, is the positivity of the fractional integral kernel k_α, and that for *any* integral operator with positive kernel the analogous estimate can be proved.

Theorem 0.1. *Let (X, d, μ) be a homogeneous space (see n.1 for the definition), K an integral operator of the kind*

$$Kf(x) = \int_{X\setminus\{x\}} k(x,y) f(y)\, d\mu(y) \tag{0.6}$$

with k non-negative measurable function, such that K is continuous from $\mathscr{L}^p(X)$ into $\mathscr{L}^p(X)$ for every $p \in (1, \infty)$. Moreover, assume k satisfies a pointwise Hörmander inequality:

there exist constants $c_K > 0$, $\beta > 0$, $M > 1$ such that for every $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,

$$|k(x_0, y) - k(x, y)| \leq \frac{c_K}{\mu(B(x_0, d(x_0, y)))} \cdot \frac{d(x_0, x)^\beta}{d(x_0, y)^\beta}. \tag{0.7}$$

In the following, we will briefly write $\mu(B(x_0; y))$ *for* $\mu(B(x_0, d(x_0, y)))$.
If X *is bounded, assume that also* $k^*(x, y) = k(y, x)$ *satisfies* (0.7). *For* $a \in \mathrm{BMO}(X)$, *set*

$$C_a f(x) = \int_{X\setminus\{x\}} k(x, y)|a(x) - a(y)| f(y)\, d\mu(y). \tag{0.8}$$

Then

$$\|C_a f\|_p \leq c\|a\|_* \|f\|_p \qquad \textit{for every } p \in (1, \infty). \tag{0.9}$$

Example 0.2. Consider the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$. For $x = (x', x_n) \in \mathbb{R}^n_+$, let $\tilde{x} = (x', -x_n)$ be the "reflected point". The operator

$$Kf(x) = \int_{\mathbb{R}^n{}_+} \frac{f(y)}{|\tilde{x} - y|^n}\, dy$$

is $\mathscr{L}^p \to \mathscr{L}^p$ continuous on $\mathbb{R}^n_+$. This operator and its commutator appear in [7] in connection with boundary estimates for solutions to elliptic equations, and this example was the first motivation for the present study. Analogous "parabolic" operators are studied in [1].

Note that if we want to bound the kernel $|\tilde{x} - y|^{-n}$ with some power of $|x - y|$, then the best possible exponent is $-n$; nevertheless, the kernel is not singular.

Theorem 0.1 implies in particular that the $\mathscr{L}^p$-estimate on the commutator of K (see (0.1) - (0.2)) still holds if the kernel k is replaced with any other *equivalent* function, $\tilde{k}$, that is

$$c_1\, k(x, y) \leq \tilde{k}(x, y) \leq c_2\, k(x, y)$$

for some positive constants c_1, c_2. Note that this fact *cannot* be assured for a singular integral of Calderón-Zygmund type (that is, which is not absolutely convergent).

The proof of Theorem 0.1, given in n. 2, is a variation of the one given in [2] for commutators of Calderón-Zygmund operators, which makes essentially use of an estimate on the "sharp function" of $C_a f$ and is, in turn, based on the analogous result proved in the euclidean case in [16], p. 418. The same technique can be adapted also to *fractional* integrals: this leads to another proof of the main result in [3] (see inequality (0.4) above), which holds for a completely general homogeneous space (whereas in [3] a geometric condition on the space is required). This is discussed in n. 3. Finally, n. 4 contains the proof of the inverse of Theorem 0.1 (and its analogous for fractional integrals): i.e., if the operator $f \mapsto C_a f$ is bounded on $\mathscr{L}^p$, then $a \in$ BMO.

Acknowledgements. I wish to thank Prof. Filippo Chiarenza for stimulating discussions on the subject of this paper.

1. Some basic facts about homogeneous spaces.

In this section we give precise definitions and recall some known results about homogeneous spaces.

Definition 1.1. *Let X be a set. A function $d : X \times X \to [0, \infty)$ is called quasidistance if:*

i) $d(x, y) = 0 \Leftrightarrow x = y$;
ii) $d(x, y) = d(y, x)$;
iii) there exists a constant $c_d \geq 1$ such that for every $x, y, z \in X$

$$d(x, y) \leq c_d[d(x, z) + d(z, y)]. \tag{1.1}$$

If (X, d) is a set endowed with a quasidistance, the "balls" $B_r(x) \equiv B(x, r) = \{y \in X : d(x, y) < r\}$ (for $x \in X$ and $r > 0$) form a base for a complete system of neighbourhoods of X, so that X is a Hausdorff space. Note that the balls are *not* in general open sets; if they are, then they form a base for the topology of X.

Definition 1.2. *We say that (X, d, μ) is a homogeneous space if:*

i) X is a set endowed with a quasidistance d, such that the balls are open sets in the topology induced by d;

ii) μ is a positive Borel measure on X, satisfying the doubling condition:

$$0 < \mu(B_{2r}(x)) \leq c_\mu \, \mu(B_r(x)) < \infty \tag{1.2}$$

for every $x \in X$, $r > 0$, some constant $c_\mu > 1$.

The numbers c_d, c_μ in (1.1) - (1.2) will be called "constants of X" and we will write $c(X)$ for a constant depending on the constants of X.

We start recalling two theorems due to Macias-Segovia (see [15]).

Theorem 1.3. *Let d be a quasidistance on a set X. Then there exists a quasidistance d' on X such that:*

(i) d and d' are equivalent, that is there exist positive constants c_1, c_2 such that for every $x, y \in X$:

$$c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y); \tag{1.3}$$

(ii) d' is "locally Hölder continuous"; more precisely there exist $\gamma \in (0, 1]$ and $c > 0$ such that for every $x, y, z \in X$:

$$|d'(x, z) - d'(y, z)| \leq c\, d'(x, y)^{\gamma} \left[d'(x, z) + d'(y, z)\right]^{1-\gamma}. \tag{1.4}$$

Theorem 1.4. *Let (X, d, μ) be a homogeneous space. Define:*

$$\delta(x, y) = \inf\left\{\mu(B) : x, y \in B, \text{ and } B \text{ is a ball with respect to } d\right\},$$

if $x \neq y$, and $\delta(x, x) = 0$.

Then:

(i) δ is a quasidistance;

(ii) δ and d are topologically equivalent
(note that they are not, in general, equivalent in the sense of (1.3)!);

(iii) the space (X, δ, μ) is normal, that is there exist positive constants c_1, c_2 such that

$$c_1 r \leq \mu(\mathscr{B}(x, r)) \leq c_2 r$$

for every $x \in X$ and every r such that $\mu(\{x\}) < r < \mu(X)$. Here $\mathscr{B}(x, r)$ denotes a ball with respect to δ;

(iv) for any δ-ball $\mathscr{B}$ there exist two d-balls B_1, B_2 such that $B_1 \subseteq \mathscr{B} \subseteq B_2$ and $\mu(B_2) \leq c\, \mu(B_1)$, for some constant c independent of $\mathscr{B}$.

The definition of standard real analysis tools, such that as the *maximal function* $\mathscr{M} f$, the *sharp function* $f^{\sharp}$ and the BMO seminorm $\|f\|_*$, naturally carries over to this context, namely:

$$\mathscr{M} f(x) = \sup_{x \in B} \fint_B |f(y)|\, d\mu(y);$$

where the sup is taken over all balls containing x and $\displaystyle\fint_B \cdots = \frac{1}{\mu(B)} \int_B \cdots;$

$$f^{\sharp}(x) = \sup_{x \in B} \fint_B |f(y) - f_B|\, d\mu(y)$$

where $f_B = \fint_B f(x)\,d\mu(x)$;

$$\|f\|_* = \sup_x f^\sharp(x) = \sup_B \fint_B |f(y) - f_B|\,d\mu(y);$$

$$\mathrm{BMO} \equiv \left\{ f \in \mathcal{L}^1_{\mathrm{loc}}(X) : \|f\|_* < \infty \right\}.$$

We recall three results related to these concepts, which have been proved in the context of homogeneous spaces in [5], [11].

Theorem 1.5. (Maximal Inequality). *For every $p \in (1, \infty]$ there exists a constant $c(X, p)$ such that for every $f \in \mathcal{L}^p$:*

$$\|\mathcal{M} f\|_p \le c\,\|f\|_p.$$

Theorem 1.6. (John-Nirenberg Lemma). *For every $p \in [1, \infty)$ there exists a constant $c(X, p)$ such that for every $f \in$ BMO, every ball B:*

$$\left(\fint_B |f(x) - f_B|^p\,d\mu(x) \right)^{1/p} \le c\,\|f\|_*.$$

Theorem 1.7. (Sharp Inequality). *For every $p \in [1, \infty)$ there exists a constant $c(X, p)$ such that for every $f \in \mathcal{L}^p$:*

$$\begin{aligned} &\text{if} \quad \mu(X) = \infty: && \|f\|_p \le c\|f^\sharp\|_p \\ &\text{if} \quad \mu(X) < \infty: && \left\| f - \fint_X f \right\|_p \le c\|f^\sharp\|_p\,. \end{aligned}$$

The following two lemmas about BMO functions can be proved as in the euclidean case:

Lemma 1.8. (See [16], p. 206). *Let $a \in$ BMO and $M > 1$. Then for every ball $B_r(x)$ and every positive integer j:*

$$\left| a_{B_{M^j r}} - a_{B_r} \right| \le c \cdot j\,\|a\|_*$$

where c depends on M and on the doubling constant c_μ.

Lemma 1.9. *Let* $f \in \mathrm{BMO}$*, and let:*

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \le n \\ n & \text{if} \quad f(x) \ge n \\ -n & \text{if} \quad f(x) \le -n. \end{cases}$$

Then $f_n \in \mathrm{BMO}$ *and :*

$$\|f_n\|_* \le c\,\|f\|_*$$

with c an absolute constant.

Finally, we will use the following:

Lemma 1.10. (See Lemma 1.9 in [2]). *Let* (X, d, μ) *be a homogeneous space. Then* $\mu(X) < \infty$ *if and only if* X *is bounded.*

2. $\mathscr{L}^p$-Estimates for integral operators with positive kernels.

Here we keep the same notations and assumptions that appear in Theorem 0.1, which we want to prove in this section. Moreover, let us define the space

$$\mathscr{D} = \left\{ f \in \mathscr{L}^\infty(X) \mid \text{ the support of } f \text{ is bounded} \right\}.$$

Note that $\mathscr{D}$ is dense in $\mathscr{L}^p$ for every $p \in [1, \infty)$. For $f \in \mathscr{L}^1_{\mathrm{loc}}$, let $f^\sharp$, $\mathscr{M} f$ denote the sharp function and maximal function of f, defined in n. 1. Then:

Theorem 2.1. *For every* $r \in (1, \infty)$ *there exists a constant* $c(r, K, X)$ *such that for every* $a \in \mathscr{L}^\infty(X)$, $f \in \mathscr{D}$,

$$\left|(C_a f)^\sharp(\bar{x})\right| \le c \cdot \|a\|_* \left\{ \left(\mathscr{M}([K(|f|)]^r)(\bar{x})\right)^{1/r} + \left(\mathscr{M}(|f|^r)(\bar{x})\right)^{1/r} \right\}$$

for every $\bar{x} \in X$.

Proof. Let $B_\delta = B_\delta(x_0)$, $x, \bar{x} \in B_\delta$.

$$(2.1) \qquad (C_a f)(x) = \int_X |a(x) - a(y)|\, k(x, y) f(y) \chi_{B_{M\delta}}(y)\, d\mu(y) +$$

$$+ \int_X |a(x) - a(y)| k(x, y) f(y) \chi_{X \setminus B_{M\delta}}(y)\, d\mu(y) \equiv \mathscr{A}(x) + \mathscr{B}(x).$$

$$(2.2) \qquad \fint_{B_\delta} |\mathscr{A}(x) - \mathscr{A}_{B_\delta}|\, d\mu(x) \le 2 \fint_{B_\delta} |\mathscr{A}(x)|\, d\mu(x) \le$$

$$\leq 2 \fint_{B_\delta} d\mu(x) \int_{B_{M\delta}} |a(x) - a(y)| k(x, y) |f(y)| \, d\mu(y) \leq$$

$$\leq 2 \fint_{B_\delta} d\mu(x) \int_{B_{M\delta}} \left(|a(x) - a_{B_\delta}| + |a(y) - a_{B_\delta}| \right) \cdot$$

$$\cdot \, k(x, y) |f(y)| \, d\mu(y) \equiv \mathscr{A}_1 + \mathscr{A}_2 .$$

$$\mathscr{A}_1 = 2 \fint_{B_\delta} |a(x) - a_{B_\delta}| d\mu(x) \int_{B_{M\delta}} k(x, y) |f(y)| d\mu(y) \leq \tag{2.3}$$

(by Hölder)

$$\leq 2 \left(\fint_{B_\delta} |a(x) - a_{B_\delta}|^{r'} \, d\mu(x) \right)^{1/r'} \cdot \left(\fint_{B_\delta} |K(f)(x)|^r \, d\mu(x) \right)^{1/r} \leq$$

(by Theorem 1.6)

$$\leq c(r, X) \|a\|_* \left(\mathscr{M} \left(K(|f|)^r \right) (\bar{x}) \right)^{1/r} .$$

$$\mathscr{A}_2 \leq 2 \fint_{B_\delta} d\mu(x) \int_{B_{M\delta}} |a(y) - a_{B_\delta}| k(x, y) |f(y)| \, d\mu(y) = \tag{2.4}$$

$$= 2 \fint_{B_\delta} K \left(|a - a_{B_\delta}| \, |f| \chi_{B_{M\delta}} \right) \, d\mu(x) \leq$$

(for every $q > 1$)

$$\leq 2 \left(\frac{1}{\mu(B_\delta)} \int_X \left| K \left((a - a_{B_\delta}) f \chi_{B_{M\delta}} \right) (x) \right|^q \, d\mu(x) \right)^{1/q} \leq$$

(by the $\mathscr{L}^q$-continuity of K)

$$\leq c(q, K) \left(\frac{1}{\mu(B_\delta)} \int_{B_{M\delta}} \left(|a(x) - a_{B_\delta}| \, |f(x)| \right)^q \, d\mu(x) \right)^{1/q} \leq$$

(choosing $q < r$, by Hölder)

$$\leq c(q, K) \left\{ \frac{1}{\mu(B_\delta)} \left(\int_{B_{M\delta}} |f(x)|^r d\mu(x) \right)^{q/r} \cdot \right.$$

$$\cdot\left(\int_{B_{M\delta}} |a(x) - a_{B_\delta}|^{qr/(r-q)} d\mu(x)\right)^{(r-q)/r}\Bigg\}^{1/q} \leq$$

(by Lemma 1.8: $|a(x) - a_{B_\delta}| \leq |a(x) - a_{B_{M\delta}}| + |a_{B_{M\delta}} - a_{B_\delta}| \leq |a(x) - a_{B_{M\delta}}| + c\,\|a\|_*$)

$$\leq c(r, K)\mu(B_\delta)^{-1/q}\left(\int_{B_{M\delta}} |f(x)|^r d\mu(x)\right)^{1/r} \cdot$$

$$\cdot\left\{\left(\int_{B_{M\delta}} |a(x) - a_{B_{M\delta}}|^{qr/(r-q)} d\mu(x)\right)^{(r-q)/qr} + \|a\|_*\,\mu(B_{M\delta})^{(r-q)/qr}\right\} \leq$$

(by Theorem 1.6 and the doubling condition)

$$\leq c(r, K, X)\left(⨍_{B_{M\delta}} |f(x)|^r d\mu(x)\right)^{1/r} \cdot \|a\|_* \leq$$

$$\leq c(r, K, X)\|a\|_*\left(\mathscr{M}(|f|^r)(\bar{x})\right)^{1/r}.$$

$$\mathscr{B}(x) = \int_{X\setminus B_{M\delta}} |a(x) - a(y)| k(x, y) f(y)\, d\mu(y).$$

$$⨍_{B_\delta} |\mathscr{B}(x) - \mathscr{B}_{B_\delta}|\, d\mu(x) \leq 2⨍_{B_\delta} |\mathscr{B}(x) - c|\, d\mu(x) = \tag{2.5}$$

(with c to be chosen later)

$$= 2⨍_{B_\delta}\Bigg|\int_{X\setminus B_{M\delta}} |a_{B_\delta} - a(y)|\, k(x, y) f(y)\, d\mu(y) - c +$$

$$+ \int_{X\setminus B_{M\delta}} \big(|a(x) - a(y)| - |a_{B_\delta} - a(y)|\big)\, k(x, y) f(y)\, d\mu(y)\Bigg|\, d\mu(x) \leq$$

$$\leq 2⨍_{B_\delta} d\mu(x)\left\{\left|\int_{X\setminus B_{M\delta}} |a_{B_\delta} - a(y)|\, k(x, y) f(y)\, d\mu(y) - c\right| +\right.$$

$$\left. + \int_{X\setminus B_{M\delta}} \big||a(x) - a(y)| - |a_{B_\delta} - a(y)|\big|\, k(x, y) |f(y)|\, d\mu(y)\right\} \equiv \mathscr{B}_1 + \mathscr{B}_2.$$

In $\mathscr{B}_1$, choose

$$c = \int_{X\setminus B_{M\delta}} |a_{B_\delta} - a(y)| k(x_0, y) f(y)\, d\mu(y).$$

Then:

$$\mathscr{B}_1 \leq 2 \fint_{B_\delta} d\mu(x) \int_{X \setminus M\delta} |a_{B_\delta} - a(y)| \, |k(x_0, y) - k(x, y)| \, |f(y)| \, d\mu(y).$$

By (0.7), the inner integral in $\mathscr{B}_1$ is bounded by:

$$c_K \cdot d(x_0, x)^\beta \int_{X \setminus B_{M\delta}} \frac{|a(y) - a_{B_\delta}| \, |f(y)|}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) \leq \tag{2.6}$$

(by Hölder)

$$\leq c_K \cdot \delta^\beta \left(\int_{X \setminus B_{M\delta}} \frac{|a(y) - a_{B_\delta}|^{r'}}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) \right)^{1/r'} \cdot$$

$$\cdot \left(\int_{X \setminus B_{M\delta}} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) \right)^{1/r}.$$

Now:

$$\int_{X \setminus B_{M\delta}} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) = \tag{2.7}$$

$$= \sum_{j=1}^{\infty} \int_{M^j\delta \leq d(x_0, y) < M^{j+1}\delta} \frac{|f(y)|^r}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) \leq$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{[M^j\delta]^\beta} \cdot \frac{1}{\mu(B(x_0, M^j\delta))} \int_{B(x_0, M^{j+1}\delta)} |f(y)|^r d\mu(y) \leq$$

(by the doubling condition)

$$\leq c(X, M, \beta) \, \frac{1}{\delta^\beta} \, \mathscr{M}(|f|^r)(\bar{x}).$$

Analogously:

$$\int_{X \setminus B_{M\delta}} \frac{|a(y) - a_{B_\delta}|^{r'}}{\mu(B(x_0; y)) \cdot d(x_0, y)^\beta} \, d\mu(y) \leq \tag{2.8}$$

$$\leq c(X, M) \sum_{j=1}^{\infty} \frac{1}{[M^j d]^\beta} \fint_{B(x_0, M^{j+1}\delta)} |a(y) - a_{B_\delta}|^{r'} d\mu(y) \leq$$

(by Theorem 1.6 and Lemma 1.8)

$$\leq \frac{1}{\delta^\beta}\sum_{j=1}^{\infty} c(r,K,X)\|a\|_*^{r'}\left(\frac{1+j^{r'}}{M^{\beta j}}\right) = c(r,K,X)\frac{1}{\delta^\beta}\|a\|_*^{r'}.$$

From (2.7) - (2.8):

$$\mathscr{B}_1 \leq c(r,K,X)\,\|a\|_*\left(\mathscr{M}(|f|^r)(\bar{x})\right)^{1/r}. \tag{2.9}$$

$$\mathscr{B}_2 \leq 2 ⨍_{B_\delta} |a(x)-a_{B_\delta}|\,d\mu(x)\int_{X\setminus B_{M\delta}} k(x,y)|f(y)|\,d\mu(y) \leq \tag{2.10}$$

$$\leq 2\left(⨍_{B_\delta}\left|K\left(|f|\chi_{X\setminus B_{M\delta}}\right)(x)\right|^r d\mu(x)\right)^{1/r}\cdot\left(⨍_{B_\delta}|a(x)-a_{B_\delta}|^{r'}d\mu(x)\right)^{1/r'} \leq$$

(by Theorem 1.6, and since $K\left(|f|\chi_{X\setminus B_{M\delta}}\right)(x) \leq K(|f|)(x)\ \forall x$)

$$\leq c(r,X)\|a\|_*\left(\mathscr{M}([K(|f|)]^r)(\bar{x})\right)^{1/r}.$$

Collecting inequalities (2.1) - (2.10), we get the theorem. □

By the $\mathscr{L}^p$-continuity of K and the maximal inequality (Theorem 1.5), we get from Theorem 2.1 the following

Corollary 2.2. *Under the same assumptions of Theorem 2.1, for every $p \in (1,\infty)$ there exists a constant $c(p,K,X)$ such that for every $a \in \mathscr{L}^\infty(X)$, $f \in \mathscr{D}$,*

$$\left\|(C_a f)^\sharp\right\|_p \leq c\,\|a\|_*\,\|f\|_p.$$

Proof of Theorem 0.1. In view of Lemma 1.9 and the density of $\mathscr{D}$ in $\mathscr{L}^p$ for every $p<\infty$, it is enough to prove the theorem for $a \in \mathscr{L}^\infty$ and $f \in \mathscr{D}$.

Taking $a \in \mathscr{L}^\infty$ and $f \in \mathscr{D}$, we have that $C_a f \in \mathscr{L}^p$ for every $p \in (1,\infty)$.

Now, if $\mu(X)=\infty$, that is X is unbounded (see Lemma 1.10), the theorem follows from Corollary 2.2 by the "sharp inequality" (Theorem 1.7). If $\mu(X) < \infty$, from Corollary 2.2 and Theorem 1.7 we get:

$$\left\|C_a f - ⨍_X C_a f\right\|_p \leq \|a\|_*\|f\|_p. \tag{2.11}$$

Then it is enough to prove that

$$\left|\int_X C_a f(x)\, d\mu(x)\right| \le c\mu(X)^{1/p'} \|a\|_* \|f\|_p . \tag{2.12}$$

This can be done repeating the proof of Theorem 3.2 in [2]. For convenience of the reader, the proof is included below.

Let us consider:

$$\int_X (C_a f)(x)\, d\mu(x) = \int_X d\mu(x) \int_X k(x,y)[a(x)-a(y)] f(y)\, d\mu(y) .$$

Since a, f are bounded functions and $\mu(X) < \infty$, the integral on the right hand side of the last equation converges absolutely and equals

$$-\int_X f(y)\, d\mu(y) \int_X k(x,y)[a(y)-a(x)]\, d\mu(x) = -\int_X f(y) C_a^* 1(y)\, d\mu(y) .$$

Here we denote by C_a^* the operator defined as in (0.8), with $k(x,y)$ replaced by $k^*(x,y) \equiv k(y,x)$.

Hence:

$$\left|\int_X (C_a f)(x)\, d\mu(x)\right| \le \|f\|_p \, \|C_a^* 1\|_{p'} . \tag{2.13}$$

Since also k^* satisfies (0.7), (2.11) holds for C_a^*, too, and

$$\|C_a^* 1\|_{p'} \le c\, \|a\|_*\, \mu(X)^{1/p'} + \mu(X)^{-1/p} \left|\int_X (C_a^* 1)(x)\, d\mu(x)\right| . \tag{2.14}$$

Again by Fubini's theorem, we get:

$$\int_X (C_a^* 1)(x)\, d\mu(x) = \int_X (Ka)(y)\, d\mu(y) - \int_X (K^* a)(x)\, d\mu(x) . \tag{2.15}$$

By Hölder and the $\mathcal{L}^p$ estimate on K:

$$\left|\int_X (Ka)(y)\, d\mu(y)\right| \le \|Ka\|_p \cdot \mu(X)^{1/p'} \le c\, \|a\|_p \cdot \mu(X)^{1/p'} . \tag{2.16}$$

Without loss of generality, we can assume

$$\int_X a(x)\, d\mu(x) = 0$$

since the commutator we are estimating is not affected by adding a constant to a. We claim that

$$\|a\|_p \leq c_p \|a\|_* \mu(X)^{1/p}. \tag{2.17}$$

To see this, recall that, by Lemma 1.10, X is bounded, so it coincides with some ball B. Then, by Theorem 1.6,

$$\int_X |a(x)|^p d\mu(x) = \mu(X) \cdot ⨍_B |a(x) - a_B|^p d\mu(x) \leq c\,\mu(X)\,\|a\|_*^p .$$

Therefore from (2.16) we get:

$$\left|\int_X (Ka)(y)\,d\mu(y)\right| \leq c\,\|a\|_*\,\mu(X)\,. \tag{2.18}$$

An analogous estimate holds for the second term in (2.15) and from (2.13) - (2.18) the result follows. □

3. Fractional integrals.

As we said in the Introduction, Theorem 0.1 can be rephrased for *fractional* integrals on homogeneous spaces; this leads to another proof of the results in [3].

Define the *fractional maximal function*:

$$\mathscr{M}_\beta f(x) = \sup_{x\in B} \frac{1}{\mu(B)^{1-\beta}} \int_B |f(y)|\,d\mu(y) \quad \text{for } 0 \leq \beta < 1 \tag{3.1}$$

(i.e., the sup is taken for all the balls containing x). When $\beta = 0$, we have the standard maximal function. Reasoning as in [16], p. 153, the following can be proved:

Proposition 3.1. *If* $0 \leq \beta < 1$, $1 < p \leq \frac{1}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \beta$, *then there exists* $c = c(x, \beta, p) > 0$ *such that for every* $f \in \mathscr{L}^p$

$$\|\mathscr{M}_\beta f\|_q \leq c\,\|f\|_p\,. \tag{3.2}$$

Moreover, we know that

Proposition 3.2. (See [13]). *If*

$$I_\alpha f(x) = \int_{X\setminus\{x\}} k_\alpha(x,y) f(y)\, d\mu(y)$$

is the fractional integral on (X, d, μ), *that is* $k_\alpha(x,y) = \mu(B(x, d(x,y)))^{\alpha-1}$, $0 < \alpha < 1$, *and* $1 < p < \frac{1}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, *then there exists* $c = c(x, \alpha, p) > 0$ *such that for every* $f \in \mathscr{L}^p$

$$\|I_\alpha f\|_q \leq c\, \|f\|_p . \tag{3.3}$$

Theorem 3.3. *Let* k_α *be as in Proposition 3.2 and assume that if satisfies the following "Hörmander inequality":*
there exist constants $c > 0$, $\beta > 0$, $M > 1$ *such that for every* $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,

$$|k_\alpha(x_0,y) - k_\alpha(x,y)| \leq \frac{c}{\mu\big(B(x_0, d(x_0,y))\big)^{1-\alpha}} \cdot \frac{d(x_0,x)^\beta}{d(x_0,y)^\beta} . \tag{3.4}$$

If X *is bounded, assume that also* $k^*_\alpha(x,y) = k_\alpha(y,x)$ *satisfies* (3.4). *For* $a \in \mathrm{BMO}(X)$, *set*

$$C^\alpha_a f(x) = \int_{X\setminus\{x\}} k_\alpha(x,y)\, |a(x) - a(y)| f(y)\, d\mu(y) . \tag{3.5}$$

Then, if p, q *are as in Proposition 3.2,*

$$\left\|C^\alpha_a f\right\|_q \leq c\, \|a\|_* \, \|f\|_p . \tag{3.6}$$

Sketch of the proof. We claim that:
for every $\alpha \in (0,1)$, $r \in (1, \frac{1}{\alpha})$ there exists $c = c(r, \alpha, X)$ such that for every $x \in X$

$$\left|(C^\alpha_a f)^\sharp(x)\right| \leq c\, \|a\|_* \left\{ \left(M\left([I_\alpha(|f|)]^r\right)(x)\right)^{1/r} + \left(M_{\alpha r}(|f|^r)(x)\right)^{1/r} \right\} . \tag{3.7}$$

(3.7) can be proved similarly to Theorem 2.1, using (3.3) and (3.4), with a careful choice of the exponents. We do not give account of the details. By Theorem 1.5, Proposition 3.1 and Proposition 3.2, inequality (3.7) implies:

$$\left\|C^\alpha_a f)^\sharp\right\|_q \leq c\, \|a\|_* \, \|f\|_p . \tag{3.8}$$

Then Theorem 3.3 follows from (3.8) like Theorem 0.1 follows from Corollary 2.2. □

The previous Theorem requires that the kernel k_α satisfies inequality (3.4). This is a "geometric" assumption on the space (X, d, μ), which can be false, for instance when the space has "atoms": in this case the function $x \mapsto \mu(B_r(x))$ is discontinuous, while (3.4) requires its continuity. Moreover, even in non-pathological cases, the proof of property (3.4) can be difficult: for instance, if X is a manifold with variable curvature, the function $x \mapsto \mu(B_r(x))$ has not a simple explicit form. On the other hand, a nontrivial example where (3.4) can be proved is $\mathbb{R}^N$ with the euclidean distance and a weighted measure $d\mu = w(x)dx$, where w ($w \in \mathscr{L}^1_{\text{loc}}$, $w \geq 0$) is such that μ satisfies a doubling condition (for instance, w is an A_p weight of Muckenhoupt). These homogeneous spaces are studied in [12], in relation with fractional integrals; in particular, they prove that, under the above assumptions,

$$\mu\big(B(x, l+r)\big) - \mu\big(B(x, l)\big) \leq c\,\mu\big(B(x, r)\big)^{1-\beta} \cdot \mu\big(B(x, l)\big)^{\beta}$$

for every $l \geq r > 0$, $x \in \mathbb{R}^N$, some $c > 0$ and $\beta \in (0, 1)$. From this result, with some computation, (3.4) can be proved.

However, it is possible to remove assumption (3.4) without loss of generality, using some known results on homogeneous spaces: what follows is an explanation of this extension.

Let (X, d, μ) be *any* homogeneous space; let us construct the quasidistance δ as in Theorem 1.4; then, starting from δ, let us construct the quasidistance δ' as in Theorem 1.3. The fractional integral kernel k_α, in the space (X, δ', μ), can be defined as:

$$k'_\alpha(x, y) = \delta'(x, y)^{\alpha - 1}. \tag{3.9}$$

Lemma 3.6. *The kernel k'_α defined as in* (3.9) *satisfies a Hörmander inequality* (3.4).

Proof.

$$\left|k'_\alpha(x, y) - k'_\alpha(x_0, y)\right| \leq$$

$$(\text{for } \delta'(x_0, y) \geq 2\delta'(x, x_0))$$

$$\leq c\,\frac{|\delta'(x, y)^{1-\alpha} - \delta'(x_0, y)^{1-\alpha}|}{\delta'(x_0, y)^{2(1-\alpha)}} \leq c\,\frac{|\delta'(x, y) - \delta'(x_0, y)|^{1-\alpha}}{\delta'(x_0, y)^{2(1-\alpha)}} \leq$$

$$(\text{by } (1.4))$$

$$\leq c\,\frac{\delta'(x_0, x)^{(1-\alpha)\gamma}}{\delta'(x_0, y)^{2(1-\alpha)}}\left(\delta'(x, y) + \delta'(x_0, y)\right)^{(1-\alpha)(1-\gamma)} \leq$$

$$\leq c\,\frac{\delta'(x_0,x)^{(1-\alpha)\gamma}}{\delta'(x_0,y)^{(1-\alpha)(1+\gamma)}} = c\,k'_\alpha(x_0,y)\cdot\left(\frac{\delta'(x_0,x)}{\delta'(x_0,y)}\right)^{\beta}$$

with $\beta = (1-\alpha)\gamma$. □

Theorem 3.7. *Let (X, d, μ) be any homogeneous space. Then the conclusion of Theorem 3.3 holds.*

Proof. By Theorem 3.3 and Lemma 3.6, estimate (3.6) holds in (X, δ', μ). By definition of δ and since δ and δ' are equivalent, we see that

$$k_\alpha(x,y) = \mu(B(x,d(x,y)))^{\alpha-1} \quad \text{and} \quad k'_\alpha(x,y) = \delta'(x,y)^{\alpha-1}$$

are equivalent (where B denotes a d-ball). Therefore we can write:

$$\left\|C_a^\alpha f\right\|_q \leq c\,\|a\|_{\mathrm{BMO}(X,\delta',\mu)}\,\|f\|_p \tag{3.10}$$

with $C_a^\alpha f$ defined as in (0.4) by k_α or k'_α. We now claim that:

$$\|a\|_{\mathrm{BMO}(X,\delta',\mu)} \leq c\,\|a\|_{\mathrm{BMO}(X,d,\mu)}\,. \tag{3.11}$$

To prove (3.11), it is enough to show that

$$\|a\|_{\mathrm{BMO}(X,\delta,\mu)} \leq c\,\|a\|_{\mathrm{BMO}(X,d,\mu)}\,, \tag{3.12}$$

since δ and δ' are equivalent. By Theorem 1.4 *(iv)*, we know that for any δ-ball $\mathscr{B}$ there exist two d-balls B_1, B_2 such that $B_1 \subseteq \mathscr{B} \subseteq B_2$ and $\mu(B_2) \leq c\mu(B_1)$, for some constant c independent of $\mathscr{B}$. From this fact and Theorem 1.6:

$$\frac{1}{\mu(\mathscr{B})}\int_{\mathscr{B}} |a(x) - a_{\mathscr{B}}|^2\,d\mu(x) \leq$$

$$\text{(for any } \lambda \in \mathbb{R})$$

$$\leq \frac{1}{\mu(\mathscr{B})}\int_{\mathscr{B}} |a(x)-\lambda|^2\,d\mu(x) \leq \frac{c}{\mu(B_2)}\int_{B_2} |a(x)-\lambda|^2\,d\mu(x) \leq$$

$$\left(\text{for } \lambda = \frac{1}{\mu(B_2)}\int_{B_2} a(y)\,d\mu(y)\right)$$

$$\leq c\,\|a\|^2_{\mathrm{BMO}(X,d,\mu)}\,.$$

So (3.12) is proved. From (3.10) - (3.11), the theorem follows. □

Remark 3.8. Theorem 3.7 improves the analogous result in [2], where a geometrical property of the space (X, d, μ) is assumed. Let us call (P) this assumption: it means, roughly speaking, that X has not too many empty spherical shells. Condition (P) could be removed also from the proof given in [2] noting that, whatever the space (X, d, μ) is, the space (X, δ, μ) satisfies *always* condition (P): this fact was proved by [6]. Therefore the result holds for (X, δ, μ) and so, reasoning as in the proof of Theorem 3.7, also for (X, d, μ).

4. The inverse theorem.

We conclude pointing out that the inverse theorems of Theorem 0.1 and Theorem 3.3 hold. Namely:

Theorem 4.1. (Inverse of Theorem 3.3). *Let (X, d, μ) be a homogeneous space, $a \in \mathcal{L}^1_{\text{loc}}(X)$, $C^\alpha_a f$ be defined as in Theorem 2.3, p, q, α as in Proposition 2.2 and assume that following inequality holds*

$$\left\| C^\alpha_a f \right\|_q \leq H \, \| f \|_p \tag{4.1}$$

for some positive constant H, every $f \in \mathcal{L}^p$. Then $a \in$ BMO and

$$\| a \|_* \leq c \cdot H, \tag{4.2}$$

for some constant c independent of a.

Theorem 4.2. (Inverse of Theorem 0.1). *Let (X, d, μ) be a homogeneous space, $a \in \mathcal{L}^1_{\text{loc}}(X)$, $C_a f$ be defined as in Theorem 0.1 and assume that the kernel k satisfies the growth condition:*

$$|k(x, y)| \leq \frac{c}{\mu\left(B(x, d(x, y))\right)}. \tag{4.3}$$

If, for some $p \in (1, \infty)$, the following inequality holds

$$\| C_a f \|_p \leq H \, \| f \|_p$$

for some positive constant H, every $f \in \mathcal{L}^p$, then $a \in$ BMO and

$$\| a \|_* \leq c \cdot H,$$

for some constant c independent of a.

Theorem 4.1 has been proved in [4]; the proof of Theorem 4.2 can be carried out similarly, setting $\alpha = 0$ in the proof of Theorem 4.1 and using assumption (4.3). For convenience of the reader, the proof of Theorem 4.1 is included below.
Proof of Theorem 4.1. Let $f = \chi_B$ with B any fixed ball. Then

$$C^\alpha_a f(x) = \int_{B \setminus \{x\}} k_\alpha(x, y) |a(x) - a(y)| \, d\mu(y).$$

If $x, y \in B$,

$$\mu\big(B(x, d(x, y))\big) \le c(X) \cdot \mu(B), \tag{4.4}$$

then

$$|k_\alpha(x, y)| \ge c\,\mu(B)^{\alpha-1}, \tag{4.5}$$

and

$$C_a^\alpha f(x) \ge c\mu(B)^{\alpha-1} \int_B |a(x) - a(y)|\, d\mu(y) \ge \tag{4.6}$$

$$\ge c\,\mu(B)^\alpha \left| a(x) - \frac{1}{\mu(B)}\int_B a(y)\, d\mu(y) \right| .$$

Raising to the q both sides of (4.6) and integrating on B:

$$c\,\mu(X)^{\alpha q} \int_B |a(x) - a_B|^q \, d\mu(x) \le \int_B \left|C_a^\alpha f(x)\right|^q \, d\mu(x) \le$$

(by (4.1))

$$\le H^q \left\|\chi_B\right\|_p^q = H^q\,\mu(X)^{q/p}.$$

Then from the relation between p, q and α we get

$$\left(\frac{1}{\mu(B)}\int_B |a(x) - a_B|^q \, d\mu(x)\right)^{1/q} \le cK.$$

Since B is generic, by Theorem 1.6, (4.2) follows. $\square$

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