LE MATEMATICHE
Vol. XLIX (1994) – Fasc. I, pp. 169-173

COMPLETIONS OF RATIONALS

JANUSZ J. CHARATONIK - ALFONSO VILLANI

Spaces which are metrizable completions of the space $\mathbb{Q}$ of rationals are described. A characterization of metrizable spaces having the same family of metrizable completions as $\mathbb{Q}$ is deduced.

Given a metric space (X, d), we denote by $\text{compl}(X, d)$ the *metric completion* of (X, d), i.e., that metric space (X', d') which is uniquely determined, up to an isometry, by the conditions of being complete and of containing a dense subspace isometric to (X, d).

Let X be a metrizable space. Then a metrizable space X' is said to be a *metrizable completion*, or simply a *completion*, of X provided that there exist a compatible metric d on X, and a compatible metric d' on X', such that $(X', d') = \text{compl}(X, d)$. In other words, a metrizable space X' is a metrizable completion of X if and only if X' is completely metrizable and X is densely embeddable in X'. We denote by $\text{Compl}(X)$ the family of all metrizable completions of X.

In a previous paper [1] we showed that it is possible to give a characterization in terms of $\text{Compl}(X)$ of some topological properties of X, such as compactness, local compactness, and some other covering properties. In view of such a kind of results the following question arose, which seemed to us to be natural

Entrato in Redazione l'8 novembre 1994.
1980 Mathematics Subject Classification (1985 Revision): 54E52, 54E45, 54E65.
Key words and phrases. Completion, dense, homeomorphic, irrationals, metric, metrizable, rationals, reals, zero-dimensional.

enough to pay some attention to it and to look for an answer. *Given two metrizable spaces having the same families of metrizable completions, must the spaces be homeomorphic* ? In this paper we give a negative answer to this question by showing that the spaces $\mathbb{Q}$ of rationals and $\mathbb{R} \setminus \mathbb{Q}$ of irrationals (in the real line $\mathbb{R}$) have the same families of metrizable completions. In fact, we get a stronger result (Theorem 5), which moreover gives a complete description of the family $\mathrm{Compl}(\mathbb{Q})$. Also, the above mentioned result enables us to obtain some characterizations of those metrizable spaces X for which the equality $\mathrm{Compl}(X) = \mathrm{Compl}(\mathbb{Q})$ holds (Theorem 7).

The authors are obliged to Professors A. Arkhangel'skiĭ and C. Ryll-Nardzewski for fruitful discussions on the topic of this paper.

Before starting our investigation it is worth to point out that, given a metrizable space X, all metrizable completions of X are (homeomorphic to) subspaces of a suitable metric space; (as a consequence of this, we see that there is no trouble in considering $\mathrm{Compl}(X)$ as a set). Indeed, it is well known that all metrizable completions of a metrizable space X have the same weight as X has (see [2], Theorem 4.3.19, p. 272). Since the Cartesian product of countably many copies of the hedgehog space of spininess $\mathfrak{m}$ is universal for all metrizable spaces of weight $\mathfrak{m}$ ([2], Theorem 4.4.9, p. 282); see also Example 4.1.5, p. 251), it follows that for a metrizable space X each completion X' of X can be considered as a subspace of that product (even as a closed subspace; see [2], 4.4.B, p. 286).

We begin with an easy proposition of a general nature. Its proof is left to the reader.

Proposition 1. *Let X be a metrizable space and let Y be a dense subspace of X. Then each metrizable completion of X is also a metrizable completion of Y, i.e.*

$$\mathrm{Compl}(X) \subset \mathrm{Compl}(Y).$$

We need the following lemma.

Lemma 2. *Let a dense subspace Y of a Hausdorff space X be given. If a compact subset of Y is open with respect to Y, then it is also open with respect to X.*

Proof. Let a compact set $A \subset Y$ be open with respect to Y. Take any set B which is open with respect to X and such that $B \cap Y = A$. We shall prove that $B = A$. Let $b \in B$. For each open neighbourhood U of b in X the intersection $B \cap U$ is again an open neighbourhood of b in X, and thus it contains a point $y \in Y$. For such a point y we have $y \in U \cap (B \cap Y) = U \cap A$, whence $b \in \mathrm{cl}\, A = A$, since A is compact. Thus $B \subset A$, and the proof is complete.

Next we prove

Theorem 3. *Let Y be a separable and zero-dimensional metrizable space which does not contain any nonempty compact open subset. Also, let X be any completely metrizable space which contains Y as a subspace. Then there exists a set W, homeomorphic to the space $\mathbb{R} \setminus \mathbb{Q}$ of irrationals, such that*

$$Y \subset W \subset \operatorname{cl} Y \subset X.$$

Proof. Since $\operatorname{cl} Y$ is completely metrizable, it is clear that there is no loss of generality in assuming that Y is dense in X. Then, also the entire space X is separable. Let M be a countable (possibly empty) set such that $M \subset X \setminus Y \subset \operatorname{cl} M$, and let

$$X_1 = X \setminus M = \cap \{X \setminus \{m\} : m \in M\}.$$

Then X_1 is a G_δ-set, and it is dense in X because $Y \subset X_1$. Further, since complete metrizability is hereditary with respect to G_δ-sets ([2], Theorem 4.3.23, p. 274), X_1 is completely metrizable.

We claim that the subspace X_1 does not contain any nonempty compact set which is open with respect to X_1. Indeed, let B be a nonempty subset of X_1 that is both compact and open. Since $\operatorname{cl} X_1 = X$, then B is also open with respect to X by Lemma 2. Thus, by the assumptions concerning Y, we see that B is not contained in Y, i.e., $B \cap (X \setminus Y) \neq \emptyset$, which implies $B \cap M \neq \emptyset$, a contradiction.

At this point we can assume without loss of generality (replacing X with X_1 if necessary) that also the space X does not contain any nonempty compact open subset. Now, it follows from a theorem of Tumarkin ([2], 7.4.17, p. 422) that there exists a zero-dimensional G_δ-set W in X such that $Y \subset W \subset X$. Of course W is separable and completely metrizable, too. Moreover, W does not contain any nonempty compact open subset by Lemma 2, because X does not and W is dense in X. So, to complete the proof, it is enough to apply the Alexandroff - Urysohn theorem saying that every separable zero-dimensional completely metrizable space, which does not contain any nonempty compact open subset, is homeomorphic to the space of irrational numbers ([2], 6.2.A (b), p. 370).

Corollary 4. *Every separable and zero-dimensional metrizable space Y, which does not contain any nonempty compact open subset, can be densely embedded into the irrationals.*

Proof. Take any completion X of Y and apply Theorem 3.

Now we are ready to prove our main result.

Theorem 5. *All separable and zero-dimensional metrizable spaces, which do not contain any nonempty compact open subset, have the same family of metrizable completions. In fact, for every such a space X the family* $\mathrm{Compl}(X)$ *consists exactly of all separable, dense in themselves and completely metrizable spaces.*

Proof. If $Z \in \mathrm{Compl}(X)$, then it is clear that Z is separable, dense in itself and completely metrizable. Conversely, let us prove that for every separable, dense in itself and completely metrizable space Z we have $Z \in \mathrm{Compl}(X)$. To this aim, consider a countable dense subset Y of Z. Since Z is dense in itself, so is Y. Thus Y is homeomorphic to the rationals by the Sierpinski characterization which says that any countable and dense in itself metrizable space is homeomorphic to $\mathbb{Q}$ (see e.g. [2], 6.2.A (d), p. 370). Therefore Y is a separable and zero-dimensional subspace of a completely metrizable space Z, and it does not contain any nonempty compact open subset. Thus, by Theorem 3, there is a set W homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ and such that $Y \subset W \subset \mathrm{cl}\, Y = Z$. Hence $Z \in \mathrm{Compl}(W) = \mathrm{Compl}(\mathbb{R} \setminus \mathbb{Q})$. By Corollary 4 the space X is densely embeddable into $\mathbb{R} \setminus \mathbb{Q}$, whence $\mathrm{Compl}(\mathbb{R} \setminus \mathbb{Q}) \subset \mathrm{Compl}(X)$ by Proposition 1. Thus $Z \in \mathrm{Compl}(X)$, and the theorem is proved.

As an obvious consequence of Theorem 5 we have the following corollary which solves in negative the problem stated at the beginning of the paper.

Corollary 6. *The spaces of rationals and of irrationals have the same families of metrizable completions, i.e.,*

$$\mathrm{Compl}(\mathbb{Q}) = \mathrm{Compl}(\mathbb{R} \setminus \mathbb{Q}).$$

In fact, Theorem 5 also enables us to get a complete characterization of those metrizable spaces X for which $\mathrm{Compl}(X) = \mathrm{Compl}(\mathbb{Q})$.

Theorem 7. *For each space X the following conditions are equivalent:*

(i) *X is a metrizable space which has the same family of metrizable completions as the space of rationals, i.e.,*

$$\mathrm{Compl}(X) = \mathrm{Compl}(\mathbb{Q});$$

(ii) *X is densely embeddable into the irrationals;*

(iii) *X is homeomorphic to a dense, zero-dimensional subset of reals;*

(iv) *X is metrizable, separable, zero-dimensional, and it does not contain any nonempty compact open subset.*

Proof. Assume (i). Since $\mathbb{R} \setminus \mathbb{Q}$ is completely metrizable ([2], Theorem 4.3.23, p. 274), we have $\mathbb{R} \setminus \mathbb{Q} \in \mathrm{Compl}(\mathbb{R} \setminus \mathbb{Q})$. Thus, by Corollary 6 and (i), we get $\mathbb{R} \setminus \mathbb{Q} \in \mathrm{Compl}(X)$, so (ii) holds. The implication from (ii) to (iii) is obvious. If (iii) holds, then X surely is metrizable, separable and zero-dimensional. Moreover, it does not contain any nonempty compact open subset by Lemma 2. So (iv) is satisfied. Finally, implication from (iv) to (i) is just an immediate consequence of Theorem 5.

Remark 8. Recall that a point p of a space X is called a point of local compactness if there exists in X a compact neighbourhood of p. It can easily be observed that a zero-dimensional space X does not contain any nonempty compact open subset if and only if it does not contain any point of local compactness. Hence condition (iv) of Theorem 7 can be formulated according to the above mentioned equivalence.

We end the paper by noticing that also the following known result (see [2], 6.2.A (e), p. 371) is a corollary to Theorem 5.

Corollary 9. *Every dense in itself, separable, completely metrizable space contains a dense subspace homeomorphic to the space of irrational numbers.*

REFERENCES

[1] J.J. Charatonik - A. Villani, *Metrizable completions and covering properties,* Expositiones Math., 5 (1987), pp. 275 - 281.

[2] R. Engelking, *General topology,* Heldermann, Berlin, 1989.

Janusz J. Charatonik,
Mathematical Institute,
University of Wroclaw,
pl. Grunwaldzki 2/4,
50-384 Wroclaw (POLAND)

Alfonso Villani,
Dipartimento di Matematica,
Università di Catania,
Viale A. Doria 6,
95125 Catania (ITALIA)