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ITERATED DIRICHLET PROBLEM FOR THE HIGHER ORDER POISSON EQUATION

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Convoluting the harmonic Green function with itself consecutively leads to a polyharmonic Green function suitable to solve an iterated Dirichlet problem for the higher order Poisson equation. The procedure works in any regular domain and is not restricted to two dimensions. In order to get explicit expressions however the situation is studied in the complex plane and sometimes in particular the unit disk is considered.

1. Polyharmonic Green Functions

Let $D \subset \mathbb{C}$ be a regular domain in the complex plane, i.e. bounded with a (piecewise) smooth boundary. For such domains the Gauss theorem in the forms

$$\frac{1}{2\pi i}\int_{\partial D}\omega(z)dz = \frac{1}{\pi}\int_D \omega_{\overline{z}}(z)dxdy,$$

$$\frac{1}{2\pi i}\int_{\partial D}\omega(z)d\overline{z} = -\frac{1}{\pi}\int_D \omega_z(z)dxdy$$

holds for $\omega \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Moreover the harmonic Green function $G_1(z,\zeta)$ exists satisfying for any $\zeta \in D$

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This paper is dedicated to the memory of Gaetano Fichera.

- $G_1(\cdot,\zeta)$ is harmonic in $D\setminus\{\zeta\}$,
- $G_1(z,\zeta)+\log|\zeta-z|^2$ is harmonic for $z\in D$,
- $G_1(z,\zeta)=0$ for $z\in\partial D$,
- $G_1(z,\zeta)=G_1(\zeta,z)$ for $z,\zeta\in D$, $z\neq\zeta$.

As follows from the maximum principle for harmonic functions $G_1(z,\zeta)$ is uniquely defined by the first three properties.
Usually $\frac{1}{2}G_1(z,\zeta)$ is called the harmonic Green function.

The Green function is the tool to solve the Dirichlet problem for the Poisson equation

$$\partial_z\partial_{\bar z}\,\omega=f \text{ in } D,\ \omega=\gamma \text{ on } \partial D,$$

where $f\in L_1(D;\mathbb{C})$, $\gamma\in C(\partial D;\mathbb{C})$. The solution is unique and given by

$$\omega(z)=-\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}G_1(z,\zeta)\gamma(\zeta)ds_\zeta-\frac{1}{\pi}\int_D G_1(z,\zeta)f(\zeta)d\xi d\eta, \tag{1}$$

where ∂_{ν_ζ} denotes the outward normal derivative and s_ζ the arc length parameter on ∂D with respect to the variable ζ. The kernel $-\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta)$ is the Poisson kernel. In case of the unit disk $D=\mathbb{D}=\{|z|<1\}$ it is

$$g_1(z,\zeta)=\frac{1}{1-z\overline{\zeta}}+\frac{1}{1-\bar z\zeta}-1$$

as in that case

$$G_1(z,\zeta)=\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2.$$

Because of its weak singularity $G_1(z,\zeta)$ can be inserted instead of $f(z)$ in the area integral in (1). Denoting

$$\widehat{G}_2(z,\zeta)=-\frac{1}{\pi}\int_D G_1(z,\widetilde{\zeta})G_1(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta} \tag{2}$$

and comparing this with formula (1) obviously $\widehat{G}_2(\cdot,\zeta)$ is the solution to the Dirichlet problem

$$\partial_z\partial_{\bar z}\,\widehat{G}_2(z,\zeta)=G_1(z,\zeta) \text{ in } D,\ \widehat{G}_2(z,\zeta)=0 \text{ on } \partial D$$

for any $\zeta\in D$. That (2) in fact is the solution to this problem can be shown by considering

$$-\frac{1}{\pi}\int_{D_\varepsilon}G_1(z,\widetilde{\zeta})\partial_{\widetilde{\zeta}}\partial_{\overline{\widetilde{\zeta}}}\,\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},$$

where $D_\varepsilon = \{\widetilde{\zeta} \in D : \varepsilon_1 < |\widetilde{\zeta} - \zeta|, \varepsilon_2 < |\widetilde{\zeta} - z|\}$ for small enough, positive $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Applying the Gauss theorem and letting the ε's tend to zero then (2) follows.

Evaluating (2) for $D = \mathbb{D}$ shows $\widehat{G}_2(z,\zeta) =$

$$= |\zeta - z|^2 \log\Big|\frac{1 - z\overline{\zeta}}{\zeta - z}\Big|^2 + (1-|z|^2)(1-|\zeta|^2)\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big]. \quad (3)$$

This biharmonic Green function differs from [1, 2, 4–6]

$$G_2(z,\zeta) = |\zeta - z|^2 \log\Big|\frac{1 - z\overline{\zeta}}{\zeta - z}\Big|^2 - (1-|z|^2)(1-|\zeta|^2), \quad (4)$$

which is also a biharmonic Green function but not a primitive of $G_1(z,\zeta)$ with respect to the Laplacian $\partial_z\partial_{\overline{z}}$. Both these functions satisfy

- they are biharmonic in $z \in D\backslash\{\zeta\}$ for any $\zeta \in D$,
- adding $|\zeta - z|^2 \log|\zeta - z|^2$ produces a biharmonic function in $z \in D$ for any $\zeta \in D$,
- they are symmetric in z and ζ for $z \neq \zeta$.

However their boundary behaviors differ. While

$$\cdot\ \widehat{G}_2(z,\zeta) = 0,\ \partial_z\partial_{\overline{z}}\,\widehat{G}_2(z,\zeta) = 0 \text{ for } z \in \partial D,\ \zeta \in D$$

instead

$$\cdot\ G_2(z,\zeta) = 0,\ \partial_{\nu_\zeta} G_2(z,\zeta) = 0 \text{ for } z \in \partial D,\ \zeta \in D$$

holds. From

$$G_2(z,\zeta) = -\frac{1}{\pi}\int_D \Big[\log|\widetilde{\zeta} - z|^2 \log|\widetilde{\zeta} - \zeta|^2 - \log|\widetilde{\zeta} - z| h_1(\widetilde{\zeta},\zeta) -$$

$$\log|\widetilde{\zeta} - \zeta|^2 h_1(z,\widetilde{\zeta}) + h_1(z,\widetilde{\zeta}) h_1(\widetilde{\zeta},\zeta)\Big] d\widetilde{\xi}\, d\widetilde{\eta}$$

it is seen that

$$G_2(z,\zeta) = -|\zeta - z|^2 \log|\zeta - z|^2 + h_2(z,\zeta)$$

with a biharmonic $h_2(z,\zeta)$. This follows because $-\log|\zeta - z|^2$ is a fundamental solution to the Laplace operator $\partial_z\partial_{\overline{z}}$ and

$$\partial_z\partial_{\overline{z}}\,|\zeta - z|^2[\log|\zeta - z|^2 - 2] = \log|\zeta - z|^2.$$

Hence, $\widehat{G}_2(z,\zeta)$ is a smooth function, moreover it is obviously symmetric. Proceeding with $\widehat{G}_2$ as before with G_1 leads to

$$\widehat{G}_3(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\widetilde{\zeta})\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}$$

being a solution to the Dirichlet problem

$$\partial_z\partial_{\overline{z}}\,\widehat{G}_3(z,\zeta) = \widehat{G}_2(z,\zeta) \text{ in } D,\ \widehat{G}_3(z,\zeta) = 0 \text{ on } \partial D$$

for any fixed $\zeta \in D$. As

$$(\partial_z\partial_{\overline{z}})^2\widehat{G}_3(z,\zeta) = G_1(z,\zeta)$$

its boundary behavior is

$$\cdot\ \widehat{G}_3(z,\zeta) = 0,\ \partial_z\partial_{\overline{z}}\,\widehat{G}_3(z,\zeta) = 0,\ (\partial_z\partial_{\overline{z}})^2\widehat{G}_3(z,\zeta) = 0,\ z \in \partial D,\ \zeta \in D.$$

Moreover

$$\cdot\ \widehat{G}_3(z,\zeta) - \frac{1}{4}|\zeta - z|^4 \log|\zeta - z|^2 \text{ is triharmonic in } z \in D,\ \zeta \in D.$$

But its symmetry in both variables is not obvious.

Inductively

$$\widehat{G}_n(z,\zeta) = -\frac{1}{\pi}\int_D G_1(z,\widetilde{\zeta})\,\widehat{G}_{n-1}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta} \tag{5}$$

is defined for $2 \leq n$. It has the properties

- $\widehat{G}_n(\,\cdot\,,\zeta)$ is polyharmonic of order n in $D\backslash\{\zeta\}$,
- $\widehat{G}_n(z,\zeta) + \dfrac{|\zeta - z|^{2(n-1)}}{(n-1)!^2}\log|\zeta - z|^2$ is polyharmonic of order n for $z \in D$,
- $(\partial_z\partial_{\overline{z}})^\mu\widehat{G}_n(z,\zeta) = 0$ for $0 \leq \mu \leq n-1$ and $z \in \partial D$

for any $\zeta \in D$.

For convenience $\widehat{G}_1(z,\zeta) = G_1(z,\zeta)$ is used.

Lemma 1.1. *The polyharmonic Green function $\widehat{G}_n$ is symmetric in its variables.*

Proof. That $\widehat{G}_1$ is symmetric follows from the harmonicity of

$$\widehat{G}_1(z,\zeta) - \widehat{G}_1(\zeta,z) \text{ for } z \in D.$$

As $\widehat{G}_1$ is a non-negative function, a consequence of the maximum principle

$$\lim_{z\to\partial D}[\widehat{G}_1(z,\zeta)-\widehat{G}_1(\zeta,z)]\le 0$$

for any $\zeta\in D$. Again using the maximum principle for harmonic functions the symmetry follows after interchanging the roles of both variables. This immediately shows also G_2 to be symmetric. Assuming $\widehat{G}_{n-1}$ is symmetric besides

$$\partial_z\partial_{\overline{z}}\,\widehat{G}_n(z,\zeta)=\widehat{G}_{n-1}(z,\zeta)$$

from the symmetry of $\widehat{G}_{n-1}$ also

$$\partial_\zeta\partial_{\overline{\zeta}}\,\widehat{G}_n(z,\zeta)=-\frac{1}{\pi}\int_D\widehat{G}_1(z,\widetilde{\zeta})G_{n-2}(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}=\widehat{G}_{n-1}(z,\zeta)$$

follows. Hence $\widehat{G}_n(z,\zeta)-\widehat{G}_n(\zeta,z)$ is harmonic in $z\in D$ for any $\zeta\in D$. Because $\widehat{G}_{n-1}(z,\zeta)$ vanishes for $\zeta\in\partial D$ this difference tends to zero if z approaches the boundary,

$$\lim_{z\to\partial D}[\widehat{G}_n(z,\zeta)-\widehat{G}_n(\zeta,z)]=0.$$

By the maximum principle the symmetry follows. □

Theorem 1.2. *Any* $\omega\in C^{2n}(D;\mathbb{C})\cap C^{2n-1}(\overline{D};\mathbb{C})$, $n\in\mathbb{N}$, *can be represented as*

$$\omega(z)=-\sum_{\mu=1}^{n}\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_\mu(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{\mu-1}\omega(\zeta)ds_\zeta- \\ \frac{1}{\pi}\int_D\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n\omega(\zeta)d\xi d\eta. \tag{6}$$

Proof. From the Gauss theorem

$$-\frac{1}{\pi}\int_D\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n\omega(\zeta)d\xi d\eta=$$

$$=-\frac{1}{2\pi}\int_D\Big\{\partial_{\overline{\zeta}}[\widehat{G}_n(z,\zeta)\partial_\zeta^n\partial_{\overline{\zeta}}^{n-1}\omega(\zeta)]+\partial_\zeta[\widehat{G}_n(z,\zeta)\partial_\zeta^{n-1}\partial_{\overline{\zeta}}^n\omega(\zeta)]+ \\ -\partial_\zeta[\partial_{\overline{\zeta}}\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}\omega(\zeta)]-\partial_{\overline{\zeta}}[\partial_\zeta\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}\omega(\zeta)]+ \\ +2\partial_\zeta\partial_{\overline{\zeta}}\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}\omega(\zeta)\Big\}d\xi d\eta=$$

$$=\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}\omega(\zeta)ds_\zeta+ \\ -\frac{1}{\pi}\int_D\widehat{G}_{n-1}(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}\omega(\zeta)d\xi d\eta$$

is seen. Continuing inductively

$$-\frac{1}{\pi}\int_D \widehat{G}_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n\omega(\zeta)d\xi d\eta =$$

$$=\sum_{\mu=2}^{n}\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_\mu(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{\mu-1}\omega(\zeta)ds_\zeta-\frac{1}{\pi}\int_D \widehat{G}_1(z,\zeta)\partial_\zeta\partial_{\overline{\zeta}}\omega(\zeta)d\xi d\eta$$

follows. Using formula (1) then (1.2) is a consequence. □

If the generalization of (4) for the $n-$harmonic operator $G_n(z,\zeta)=$

$$=\frac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2-\sum_{\mu=1}^{n-1}\frac{1}{\mu}|\zeta-z|^{2(n-1-\mu)}(1-|z|^2)^\mu(1-|\zeta|^2)^\mu$$

is generalized to an arbitrary regular domain D, it has the properties

- $G_n(\cdot,\zeta)$ is polyharmonic of order n in $D\backslash\{\zeta\}$, $\zeta\in D$,
- $G_n(z,\zeta)-\dfrac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\log|\zeta-z|^2$ is polyharmonic for $z\in D$, $\zeta\in D$,
- $(\partial_z\partial_{\overline{z}})^\mu G_n(z,\zeta)=0$, $0\le 2\mu\le n-1$, for $z\in\partial D$, $\zeta\in D$,
- $\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^\mu G_n(z,\zeta)=0$, $0\le 2\mu\le n-2$, for $z\in\partial D$, $\zeta\in D$,
- $G_n(z,\zeta)=G_n(\zeta,z)$ for $z,\zeta\in D$.

Using this function another representation formula is available.

Lemma 1.3. *The polyharmonic Green function G_n satisfies*

$$(\partial_z\partial_{\overline{z}})^{n-1}G_n(z,\zeta)=\log|\zeta-z|^2+\widetilde{h}(z,\zeta)$$

with a function $\widetilde{h}$ being harmonic in $z\in D$ for any $\zeta\in D$. Also

$$(\partial_\zeta\partial_{\overline{\zeta}})^{n-1}G_n(z,\zeta)=\log|\zeta-z|^2+h(z,\zeta)$$

with a function h being harmonic in $\zeta\in D$ for any $z\in D$.

Proof. Rewriting

$$G_n(z,\zeta)=\frac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\Big[\log|\zeta-z|^2-2\sum_{\mu=1}^{n-1}\frac{1}{\mu}\Big]+h_n(z,\zeta)$$

with a polyharmonic function h_n of order n in both variables, shows

$$\partial_z \partial_{\overline{z}} G_n(z,\zeta) = \frac{|\zeta - z|^{2(n-2)}}{(n-2)!^2} \left[\log|\zeta - z|^2 - 2\sum_{\mu=1}^{n-2} \frac{1}{\mu} \right] + \partial_z \partial_{\overline{z}} h_n(z,\zeta)$$

and inductively

$$(\partial_z \partial_{\overline{z}})^{n-1} G_n(z,\zeta) = \log|\zeta - z|^2 + (\partial_z \partial_{\overline{z}})^{n-1} h_n(z,\zeta).$$

□

Theorem 1.4. *Any $\omega \in C^{2n}(D;\mathbb{C}) \cap C^{2n-1}(\overline{D};\mathbb{C})$, $n \in \mathbb{N}$, is representable by*

$$\omega(z) = -\sum_{\mu=0}^{[\frac{n}{2}]-1} \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1} G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{\mu} \omega(\zeta) ds_\zeta +$$

$$+\sum_{\mu=0}^{[\frac{n-1}{2}]} \frac{1}{4\pi} \int_{\partial D} (\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1} G_n(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{\mu} \omega(\zeta) ds_\zeta +$$

$$-\frac{1}{\pi} \int_D G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{n} \omega(\zeta) d\xi d\eta. \tag{7}$$

For the particular case of $D = \mathbb{D}$ this representation is explicitly given in [7, 9]. For this formula for the upper half plane see [8, 10].

Proof. Proceeding as in the proof of (1.2) gives

$$-\frac{1}{\pi} \int_D G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{n} \omega(\zeta) d\xi d\eta =$$

$$= -\frac{1}{4\pi} \int_{\partial D} G_n(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} \omega(\zeta) ds_\zeta +$$

$$+\frac{1}{4\pi} \int_{\partial D} \partial_{\nu_\zeta} G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} \omega(\zeta) ds_\zeta +$$

$$-\frac{1}{\pi} \int_D \partial_\zeta \partial_{\overline{\zeta}} G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} \omega(\zeta) d\xi d\eta =$$

$$= -\sum_{\mu=0}^{n-2} \frac{1}{4\pi} \int_{\partial D} \Big\{ (\partial_\zeta \partial_{\overline{\zeta}})^{\mu} G_n(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1} \omega(\zeta) +$$

$$-\partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{\mu} G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1} \omega(\zeta) \Big\} ds_\zeta$$

$$-\frac{1}{\pi} \int_D (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} G_n(z,\zeta) \partial_\zeta \partial_{\overline{\zeta}} \omega(\zeta) d\xi d\eta.$$

Using the result of Lemma 2

$$(\partial_\zeta \partial_{\overline{\zeta}})^{n-1} G_n(z,\zeta) = G_1(z,\zeta) + \widehat{h}(z,\zeta)$$

with some function $\widehat{h}(z,\cdot)$ harmonic in D. Then

$$-\frac{1}{\pi}\int_D (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} G_n(z,\zeta)\partial_\zeta \partial_{\overline{\zeta}}\omega(\zeta)d\xi d\eta =$$

$$= -\frac{1}{\pi}\int_D [G_1(z,\zeta)+\widehat{h}(z,\zeta)]\partial_\zeta \partial_{\overline{\zeta}}\omega(\zeta)d\xi d\eta .$$

Applying formula (1) and

$$-\frac{1}{\pi}\int_D \widehat{h}(z,\zeta)\partial_\zeta \partial_{\overline{\zeta}}\,\omega(\zeta)d\xi d\eta =$$

$$= -\frac{1}{2\pi}\int_D \Big\{ \partial_{\overline{\zeta}}[\widehat{h}(z,\zeta)\partial_\zeta \omega(\zeta)] + \partial_\zeta[\widehat{h}(z,\zeta)\partial_{\overline{\zeta}}\,\omega(\zeta)] +$$

$$-\partial_\zeta[\partial_{\overline{\zeta}}\widehat{h}(z,\zeta)\omega(\zeta)] - \partial_{\overline{\zeta}}[\widehat{h}(z,\zeta)\omega(\zeta)]\Big\} d\xi d\eta =$$

$$= -\frac{1}{4\pi}\int_D \Big\{ \widehat{h}(z,\zeta)\partial_{\nu_\zeta}\omega(\zeta) - \partial_{\nu_\zeta}\widehat{h}(z,\zeta)\omega(\zeta)\Big\} ds_\zeta$$

then as $G_1(z,\zeta)$ vanishes for ζ on ∂D, $\omega(z) =$

$$= -\frac{1}{4\pi}\int_{\partial D} \Big\{ \partial_{\nu_\zeta}[G_1(z,\zeta)+\widehat{h}(z,\zeta)]\omega(\zeta) - [G_1(z,\zeta)+\widehat{h}(z,\zeta)]\partial_{\nu_\zeta}\omega(\zeta)\Big\} ds_\zeta$$

$$-\frac{1}{\pi}\int_D (\partial_\zeta \partial_{\overline{\zeta}})^{n-1} G_n(z,\zeta)\partial_\zeta \partial_{\overline{\zeta}}\,\omega(\zeta)d\xi d\eta$$

follows. This proves

$$\omega(z) = -\sum_{\mu=0}^{n-1}\frac{1}{4\pi}\int_{\partial D}\Big\{ \partial_{\nu_\zeta}(\partial_\zeta \partial_{\overline{\zeta}})^{\mu} G_n(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1}\omega(\zeta) +$$

$$-(\partial_\zeta \partial_{\overline{\zeta}})^{\mu} G_n(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta \partial_{\overline{\zeta}})^{n-\mu-1}\omega(\zeta)\Big\} ds_\zeta +$$

$$-\frac{1}{\pi}\int_D G_n(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^{n}\omega(\zeta)d\xi d\eta . \tag{8}$$

This is

$$\omega(z) = -\sum_{\mu=0}^{n-1}\frac{1}{4\pi}\int_{\partial D}\Big\{ \partial_{\nu_\zeta}(\partial_\zeta \partial_{\overline{\zeta}})^{n-1-\mu} G_n(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^{\mu}\omega(\zeta) +$$

$$-(\partial_\zeta \partial_{\overline{\zeta}})^{n-1-\mu} G_n(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^\mu \omega(\zeta) \Big\} ds_\zeta +$$

$$-\frac{1}{\pi} \int_D G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^n \omega(\zeta) d\xi d\eta,$$

which is

$$\omega(z) = -\sum_{\mu=[\frac{n+1}{2}]}^{n-1} \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{n-1-\mu} G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^\mu \omega(\zeta) ds_\zeta +$$

$$+\sum_{\mu=[\frac{n}{2}]}^{n-1} \frac{1}{4\pi} \int_{\partial D} (\partial_\zeta \partial_{\overline{\zeta}})^{n-1-\mu} G_n(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^\mu \omega(\zeta) ds_\zeta +$$

$$-\frac{1}{\pi} \int_D G_n(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^n \omega(\zeta) d\xi d\eta$$

because of the boundary behavior of G_n. This is (7). □

Obviously there are many more polyharmonic Green functions and related integral representation formulas, see e.g. [3, 6]. An example of a tetraharmonic hybrid Green function is the convolution of G_2 with $\widehat{G}_2$

$$H_4(z,\zeta) = -\frac{1}{\pi} \int_D G_2(z,\widetilde{\zeta}) \widehat{G}_2(\widetilde{\zeta},\zeta) d\widetilde{\xi} d\widetilde{\eta}.$$

Because G_2 is a fundamental solution of $(\partial_z \partial_{\overline{z}})^2$ it is a solution to

$$(\partial_z \partial_{\overline{z}})^2 H_4(z,\zeta) = \widehat{G}_2(z,\zeta) \text{ in } D \text{ for any } \zeta \in D.$$

Moreover

$$H_4(z,\zeta) = 0,\ \partial_{\nu_z} H_4(z,\zeta) = 0 \text{ for } z \in \partial D,\ \zeta \in D.$$

As $\widehat{G}_2$ is a fundamental solution of $(\partial_\zeta \partial_{\overline{\zeta}})^2$ it solves

$$(\partial_\zeta \partial_{\overline{\zeta}})^2 H_4(z,\zeta) = G_2(z,\zeta) \text{ in } D \text{ for any } \zeta \in D$$

and satisfies

$$H_4(z,\zeta) = 0,\ \partial_\zeta \partial_{\overline{\zeta}} H_4(z,\zeta) = 0 \text{ for } \zeta \in \partial D, z \in D.$$

Hence $H_4(z,\zeta)$ has the properties

- $H_4(z,\zeta)$ is tetraharmonic for $z \in D\backslash\{\zeta\}$ and for $\zeta \in D\backslash\{z\}$,
- $H_4(z,\zeta) + \dfrac{|\zeta - z|^6}{3!} \log|\zeta - z|^2$ is tetraharmonic for $z,\ \zeta \in D$,

- $H_4(z,\zeta)=0,\ \partial_{\nu_z}H_4(z,\zeta)=0,\ (\partial_z\partial_{\overline{z}})^2H_4(z,\zeta)=0,\ (\partial_z\partial_{\overline{z}})^3H_4(z,\zeta)=0$ for $z\in\partial D$, $\zeta\in D$,
- $H_4(z,\zeta)=0$, $\partial_\zeta\partial_{\overline{\zeta}}H_4(z,\zeta)=0$, $(\partial_\zeta\partial_{\overline{\zeta}})^2H_4(z,\zeta)=0$, $\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2H_4(z,\zeta)=0$ for $\zeta\in\partial D$, $z\in D$.

Obviously $H_4(z,\zeta)$ is not symmetric in its variables.

Theorem 1.5. *Any $\omega\in C^4(D;\mathbb{C})\cap C^3(\overline{D};\mathbb{C})$ can be represented as*

$$\omega(z)=-\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(z,\zeta)\omega(\zeta)+$$
$$-(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(z,\zeta)\partial_{\nu_\zeta}\omega(\zeta)+\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)$$
$$+\partial_{\nu_\zeta}H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^3\omega(\zeta)\Big\}ds_\zeta-\frac{1}{\pi}\int_D H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta \tag{9}$$

and as

$$\omega(z)=-\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(\zeta,z)\omega(\zeta)+\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2H_4(\zeta,z)\partial_\zeta\partial_{\overline{\zeta}}\omega(\zeta)+$$
$$+\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)+$$
$$-\partial_\zeta\partial_{\overline{\zeta}}H_4(\zeta,z)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)\Big\}ds_\zeta-\frac{1}{\pi}\int_D H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta. \tag{10}$$

Proof. In the same way as (8) the representations

$$\omega(z)=-\sum_{\mu=0}^{3}\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^\mu H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{3-\mu}\omega(\zeta)+$$
$$-(\partial_\zeta\partial_{\overline{\zeta}})^\mu H_4(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^{3-\mu}\omega(\zeta)\Big\}ds_\zeta+$$
$$-\frac{1}{\pi}\int_D H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta$$

and

$$\omega(z)=-\sum_{\mu=0}^{3}\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^\mu H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^{3-\mu}\omega(\zeta)+$$
$$-(\partial_\zeta\partial_{\overline{\zeta}})^\mu H_4(\zeta,z)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^{3-\mu}\omega(\zeta)\Big\}ds_\zeta+$$

$$-\frac{1}{\pi}\int_D H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta$$

follows. Taking the boundary behavior of H_4 into account

$$\omega(z) = -\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^3\omega(\zeta)+\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)+$$

$$+\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(z,\zeta)\omega(\zeta)-(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(z,\zeta)\partial_{\nu_\zeta}\omega(\zeta)\Big\}ds_\zeta+$$

$$-\frac{1}{\pi}\int_D H_4(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta$$

and $\omega(z) =$

$$= -\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)-\partial_\zeta\partial_{\overline{\zeta}}H_4(\zeta,z)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2\omega(\zeta)+$$

$$+\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2H_4(\zeta,z)\partial_\zeta\partial_{\overline{\zeta}}\omega(\zeta)+\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^3H_4(\zeta,z)\omega(\zeta)\Big\}ds_\zeta+$$

$$-\frac{1}{\pi}\int_D H_4(\zeta,z)(\partial_\zeta\partial_{\overline{\zeta}})^4\omega(\zeta)d\xi d\eta$$

are seen. □

2. Dirichlet Problems.

The above representation formulas leads to different boundary value problems. Inserting the respective boundary values in the boundary integrals and the inhomogeneity of the higher order Poisson differential equation obviously provides a weak solution to the Poisson equation. However the boundary behavior has to be verified. As is well known e.g. from the Cauchy integral not every boundary integral representation formula attains its layer function as boundary values.

Theorem 2.1. *The unique solution to the Dirichlet problem*

$$(\partial_z\partial_{\overline{z}})^n\omega = f \text{ in } D,\ (\partial_z\partial_{\overline{z}})^\mu\omega = \gamma_\mu,\ 0\le\mu\le n-1 \text{ on } \partial D,$$

$f\in L_1(D;\mathbb{C})$, $\gamma_\mu\in C(\partial D;\mathbb{C})$, $0\le\mu\le n-1$, *is given by*

$$\omega(z) = -\sum_{\mu=1}^{n}\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_\mu(z,\zeta)\gamma_{\mu-1}(\zeta)ds_\zeta-\frac{1}{\pi}\int_D\widehat{G}_n(z,\zeta)f(\zeta)d\xi d\eta. \tag{11}$$

Proof. For the Poisson kernel

$$\lim_{z\to\zeta_0} -\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_1(z,\zeta)\gamma(\zeta)ds_\zeta = \gamma(\zeta_0),\ \zeta_0\in\partial D$$

is known for $\gamma\in C(\partial D;\mathbb{C})$ and smooth ∂D.

From

$$(\partial_z\partial_{\overline{z}})^\rho\omega(z) = -\sum_{\mu=\rho+1}^{n} T\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}\widehat{G}_{\mu-\rho}(z,\zeta)\gamma_{\mu-1}(\zeta)ds_\zeta +$$

$$-\frac{1}{\pi}\int_D \widehat{G}_{n-\rho}(z,\zeta)f(\zeta)d\xi d\eta$$

it follows

$$\lim_{z\to\zeta_0}(\partial_z\partial_{\overline{z}})^\rho\omega(z) = \gamma_\rho(\zeta_0),\ \zeta_0\in\partial D.$$

□

Theorem 2.2. *The Dirichlet problem* $(\partial_z\partial_{\overline{z}})^n\omega = f$ *in* D, $(\partial_z\partial_{\overline{z}})^\mu\omega == \gamma_\mu$, $0\leq 2\mu\leq n-1$, $\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^\mu\omega = \widehat{T\gamma_\mu}$, $0\leq 2\mu\leq n-2$, *on* ∂D, *for* $f\in L_1(D;\mathbb{C})$, $\gamma_\mu\in C^{n-2\mu}(\partial D;\mathbb{C})$, $0\leq 2\mu\leq n-1$, $\widehat{\gamma}_\mu\in C^{n-1-2\mu}(\partial D;\mathbb{C})$, $0\leq 2\mu\leq n-2$ *is uniquely solvable by*

$$\omega(z) = -\sum_{\mu=0}^{[\frac{n}{2}]-1}\frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^{n-\mu-1}G_n(z,\zeta)\gamma_\mu(\zeta)ds_\zeta +$$

$$+\sum_{\mu=0}^{[\frac{n-1}{2}]}\frac{1}{4\pi}\int_{\partial D}(\partial_\zeta\partial_{\overline{\zeta}})^{n-\mu-1}G_n(z,\zeta)\widehat{\gamma}_\mu(\zeta)ds_\zeta +$$

$$-\frac{1}{\pi}\int_D G_n(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n\omega(\zeta)d\xi d\eta \tag{12}$$

As G_n are not related by a recursion scheme the proof is not obvious. That (12) provides a weak solution to the differential equation follows from the properties of G_n. Uniqueness also follows immediately from (12). If the solution exists it is of the form (12) as follows from the representation formula (7). In particular the solution to the related homogeneous problem is identically zero. The existence proof is achieved by verifying (12) to satisfy the boundary conditions. In the particular case of the unit disk [8] and the upper half plane [10] this is done.

Theorem 2.3. *The Dirichlet problem*

$$(\partial_z\partial_{\overline{z}})^4\omega = f \text{ in } D,\ \omega=\gamma_0,\ \partial_\nu\omega=\gamma_1,\ (\partial_z\partial_{\overline{z}})^2\omega=\gamma_2,\ (\partial_z\partial_{\overline{z}})^3\omega=\gamma_3, \text{ on } \partial D,$$

is uniquely solvable for $f\in L_1(D;\mathbb{C})$, $\gamma_0\in C^2(\partial D;\mathbb{C})$, $\gamma_1\in C^1(\partial D;\mathbb{C})$, $\gamma_2,\gamma_3\in C(\partial D;\mathbb{C})$ *by*

$$\omega(z) = -\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^3 H_4(z,\zeta)\gamma_0(\zeta) - (\partial_\zeta\partial_{\overline{\zeta}})^3 H_4(z,\zeta)\gamma_1(\zeta)$$

$$+\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} H_4(z,\zeta)\gamma_2(\zeta) + \partial_{\nu_\zeta} H_4(z,\zeta)\gamma_3(\zeta)\Big\}ds_\zeta - \frac{1}{\pi}\int_D H_4(z,\zeta)f(\zeta)d\xi d\eta. \tag{13}$$

Proof. The proof is given only for the particular case of the unit disk $\mathbb{D}$. The solution (13) can be written as

$$\omega(z) = \omega_0(z) - \frac{1}{\pi}\int_D H_4(z,\zeta)f(\zeta)d\xi d\eta,$$

where

$$\omega_0(z) = -\frac{1}{4\pi}\int_{\partial D}\Big\{\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta)\gamma_0(\zeta) - \partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta)\gamma_1(\zeta)+$$

$$+\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} H_4(z,\zeta)\gamma_2(\zeta) + \partial_{\nu_\zeta} H_4(z,\zeta)\gamma_3(\zeta)\Big\}ds_\zeta \tag{14}$$

and $\omega-\omega_0$ is a particular solution to the inhomogeneous equation satisfying the respective homogeneous boundary conditions.

1. As

$$\partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta) = \log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right| - g_1(z,\zeta)(1-|z|^2),$$

$$\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta) =$$

$$= -\frac{\zeta}{\zeta-z} - \frac{\overline{\zeta}}{\overline{\zeta-z}} - \frac{1}{1-z\overline{\zeta}} - \frac{1}{1-\overline{z}\zeta} + 2 - [g_2(z,\zeta)-g_1(z,\zeta)](1-|z|^2),$$

which for $|\zeta|=1$ are

$$\partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta) = -g_1(z,\zeta)(1-|z|^2),$$

$$\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} G_2(z,\zeta) = -(1+|z|^2)g_1(z,\zeta) - (1-|z|^2)g_2(z,\zeta),$$

and

$$\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}} H_4(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} G_2(z,\widetilde{\zeta})\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},$$

$$\partial_{\nu_\zeta} H_4(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}} G_2(z,\widetilde{\zeta})\partial_{\nu_\zeta}\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},$$

$$\lim_{z\to\zeta_0}\omega(z) = \gamma_0(\zeta_0) + \lim_{z\to\zeta_0}\frac{1}{4\pi i}\int_{\partial\mathbb{D}}(1-|z|^2)g_2(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} = \gamma_0(\zeta_0)$$

for $|\zeta_0| = 1$. Here

$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(1-|z|^2)g_2(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta} = 0$$

because

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1-|z|^2}{(1-\overline{z}\zeta)^2}\gamma_0(\zeta)\frac{d\zeta}{\zeta} = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1-|z|^2}{\overline{z}(1-\overline{z}\zeta)}\partial_\zeta\Big(\frac{\gamma_0(\zeta)}{\zeta}\Big)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{1-|z|^2}{|1-\overline{z}\zeta|^2}\frac{\zeta-z}{\overline{z}}\partial_\zeta\Big(\frac{\gamma_0(\zeta)}{\zeta}\Big)\frac{d\zeta}{\zeta}$$

with the Poisson kernel $\dfrac{1-|z|^2}{|1-\overline{z}\zeta|^2}$.

2. Differentiating (14) gives on $|z| = 1$

$$\partial_{\nu_z}\omega(z) = -\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\Big\{\partial_{\nu_z}\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}G_2(z,\zeta)\gamma_0(\zeta) - \partial_{\nu_z}\partial_\zeta\partial_{\overline{\zeta}}G_2(z,\zeta)\gamma_1(\zeta) +$$

$$+\partial_{\nu_z}\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(z,\zeta)\gamma_2(\zeta) + \partial_{\nu_z}\partial_{\nu_\zeta}H_4(z,\zeta)\gamma_3(\zeta)\Big\}\frac{d\zeta}{\zeta}. \tag{15}$$

From

$$\partial_{\nu_z}\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}G_2(z,\zeta) = -2g_1(z,\zeta) - (1+|z|^2)[g_2(z,\zeta) - g_1(z,\zeta)] + 2g_2(z,\zeta) +$$

$$-2(1-|z|^2)[g_3(z,\zeta) - g_2(z,\zeta)] =$$

$$= -(1-|z|^2)g_1(z,\zeta) + 3(1-|z|^2)g_2(z,\zeta) - 2(1-|z|^2)g_3(z,\zeta),$$

$$\partial_{\nu_z}\partial_\zeta\partial_{\overline{\zeta}}G_2(z,\zeta) = -(1-|z|^2)[g_2(z,\zeta) - g_1(z,\zeta)] + 2|z|^2 g_1(z,\zeta) =$$

$$= (1+|z|^2)g_1(z,\zeta) - (1-|z|^2)g_2(z,\zeta),$$

$$\partial_{\nu_z}\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}H_4(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}}\partial_{\nu_z}N_2(z,\widetilde{\zeta})\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta},$$

$$\partial_{\nu_z}\partial_{\nu_\zeta}H_4(z,\zeta) = -\frac{1}{\pi}\int_{\mathbb{D}}\partial_{\nu_z}G_2(z,\widetilde{\zeta})\partial_{\nu_\zeta}\widehat{G}_2(\widetilde{\zeta},\zeta)T\,d\widetilde{\xi}d\widetilde{\eta}$$

with on $|z| = 1$

$$\partial_{\nu_z}G_2(z,\zeta) = -[z(\overline{\zeta-z}) + \overline{z}(\zeta-z)]\log\Big|\frac{1-z\overline{\zeta}}{\zeta-z}\Big|^2 +$$

$$+|\zeta-z|^2\Big[\frac{z}{\zeta-z}+\frac{\overline{z}}{\overline{\zeta-z}}-\frac{z\overline{\zeta}}{1-z\overline{\zeta}}-\frac{\overline{z}\zeta}{1-\overline{z}\zeta}\Big]+2|z|^2(1-|\zeta|^2)=0$$

follows for $|\zeta_0|=1$

$$\lim_{z\to\zeta_0}\partial_{\nu_z}\omega(z)=\gamma_1(\zeta_0).$$

Here because of $\gamma_1\in C^1(\partial\mathbb{D};\mathbb{C})$

$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(1-|z|^2)g_2(z,\zeta)\gamma_1(\zeta)\frac{d\zeta}{\zeta}=0$$

and

$$\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(1-|z|^2)g_3(z,\zeta)\gamma_0(\zeta)\frac{d\zeta}{\zeta}=0$$

as $\gamma_0\in C^2(\partial\mathbb{D};\mathbb{C})$.

3. From (14) and because of $(\partial_z\partial_{\overline{z}})^2H_4(z,\zeta)=\widehat{G}_2(z,\zeta)$

$$(\partial_z\partial_{\overline{z}})^2\omega(z)=-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\Big\{\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}\widehat{G}_2(z,\zeta)\gamma_2(\zeta)+\partial_{\nu_\zeta}\widehat{G}_2(z,\zeta)\gamma_3(\zeta)\Big\}\frac{d\zeta}{\zeta}$$

$$=-\frac{1}{4\pi i}\int_{\partial 0\mathbb{D}}\Big\{\partial_{\nu_\zeta}G_1(z,\zeta)\gamma_2(\zeta)+\partial_{\nu_\zeta}\widehat{G}_2(z,\zeta)\gamma_3(\zeta)\Big\}\frac{d\zeta}{\zeta}\tag{16}$$

follows. As

$$\partial_{\nu_\zeta}G_1(z,\zeta)=-2g_1(z,\zeta)$$

and for $|\zeta|=1$

$$\partial_{\nu_\zeta}\widehat{G}_2(z,\zeta)=-2|\zeta-z|^2g_1(z,\zeta)-2(1-|z|^2)\Big[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}+\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}\Big]$$

taking limits shows

$$\lim_{z\to\zeta_0}(\partial_z\partial_{\overline{z}})^2\omega(z)=\gamma_2(\zeta_0).$$

4. Finally, differentiating (14) again gives

$$(\partial_z\partial_{\overline{z}})^3\omega(z)=-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}\partial_{\nu_\zeta}G_1(z,\zeta)\gamma_3(\zeta)\frac{d\zeta}{\zeta}$$

from what

$$\lim_{z\to\zeta_0}(\partial_z\partial_{\overline{z}})^3\omega(z)=\gamma_3(\zeta_0)$$

follows. □

Remark 2.4. The special assumption $D=\mathbb{D}$ is only used in steps 1 and 2. The last two steps hold in a similar way for any regular domain.

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