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A NEW APPROACH TO THE EXISTENCE OF ZEROS FOR NONLINEAR OPERATORS

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In this paper we present a necessary and sufficient condition for the existence of zeros for a nonlinear operator from on a compact topological space X into the topological dual E^* of a real Banach space E. Some applications are derived.

1. Preliminary remarks.

The aim of this communication is just to expose the present state of our current research. Thus, the results presented in the sequel must be intended as the "state of art" of a research that we are currently developing. More specifically, we shall present a result which provides a necessary and sufficient condition for the existence of zeros for an operator A defined in a compact topological space, which takes its values in the topological dual E^* of a real Banach space E. Then we derive sufficient conditions for the existence of zeros. We point out that our approach is absolutely new. Thus, we hope that our result will serve as a spur in order to investigate consequences and applications.

2. Results.

Our main result is the following.

Theorem 1. *Let X be a topological space, E a real Banach space, with* $\dim(E) > 1$, *$r > 0$, $Y = \{y \in E : \|y\| = r\}$, $A: X \to E^*$ a weakly-star continuous operator.*
Then, one has

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle \leq 0.$$

When X is compact, $A^{-1}(0_{E^*})$ is nonempty if (and only if) one has

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle = 0.$$

Proof. Arguing by contradiction, assume that

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle > 0.$$

Hence, there are $(u, \varphi) \in C^0([0, 1], X) \times C^0(Y, [0, 1])$ and $\eta > 0$ such that

$$\langle A(u(\varphi(y))), y\rangle \geq \eta$$

for all $y \in Y$. Now, put

$$S = \{(y, t) \in Y \times [0, 1] : \langle A(u(t)), y\rangle < \eta\}.$$

Observe what follows:
(a) the projection of S on $[0, 1]$ is equal to $[0, 1]$;
(b) for each $y \in Y$, the set $\{t \in [0, 1] : \langle A(u(t)), y\rangle < \eta\}$ is open in $[0, 1]$ since the function $t \to \langle A(u(t)), y\rangle$ is continuous;
(c) for each $t \in [0, 1]$, the set $\{y \in Y : \langle A(u(t)), y\rangle < \eta\}$ is connected.

Then, by Theorem 2.5-(δ_2) of [1], the graph of φ must intersect S, that is to say

$$\langle A(u(\varphi(y_0))), y_0\rangle < \eta$$

for some $y_0 \in Y$, a contradiction. So, the first part of our thesis is proved. Now, assume that X is compact and that

$$\sup_{(u,\varphi)\in C^0([0,1],X)\times C^0(Y,[0,1])} \inf_{y\in Y} \langle A(u(\varphi(y))), y\rangle = 0.$$

Again arguing by contradiction, let $A(x) \neq 0_{E^*}$ for all $x \in X$. Since the function $T \to \|T\|_{E^*}$ is weakly-star lower semicontinuous, the function $x \to \|A(x)\|_{E^*}$

is lower semicontinuous in X. By assumption, this latter function is everywhere positive in X which is compact. So, if $\beta = \inf_{x\in X} \|A(x)\|_{E^*}$, we have $\beta > 0$. Fix $\varepsilon \in]0, \beta[$. Furthermore, choose $(u, \varphi) \in C^0([0, 1], X) \times C^0(Y, [0, 1])$ in such a way that

$$\langle A(u(\varphi(y))), y\rangle \geq -r\varepsilon$$

for all $y \in Y$. Now, we get a contradiction proceeding exactly as in the first part of the proof, the role of η being assumed by $-r\varepsilon$. □

Now we derive some consequences of Theorem 1. The next result is due to B. Ricceri (see [2]).

Theorem 2. *Let X be a topological space, and let $f: X \times [0, 1] \to \mathbb{R}$, $\rho \in \mathbb{R}$. Assume that:*

(i) *the function f is upper semicontinuous on $X \times [0, 1]$;*
(ii) *the function $f(\cdot, t)$ is continuous in X for each $t \in [0, 1]$;*
(iii) *for each $x \in X$, one has $\{t \in [0, 1]: f(x, t) > \rho\} \neq \emptyset$ and*

$$\inf\{t \in [0, 1]: f(x, t) \geq -\rho\} = \inf\{t \in [0, 1]: f(x, t) > \rho\}.$$

Then, the function $x \to \inf\{t \in [0, 1]: f(x, t) \geq \rho\}$ is continuous.

By Theorems 1 and 2 we obtain the following result.

Theorem 3. *Let E be a real Banach space, with $\dim(E) > 1$, $Y = \{y \in E: \|y\| = 1\}$, $A: [0, 1] \to E^*$ a weakly-star continuous operator. Assume that:*

(i) *for each $y \in Y$ one has $\{t \in [0, 1]: \langle A(t), y\rangle \geq 0\} \neq \emptyset$;*
(ii) *there exists a sequence $\{\varepsilon_n\}$ of positive real numbers such that $\{\varepsilon_n\} \to 0$ and for each $n \in \mathbb{N}$ and each $y \in Y$ one has*

$$\inf\{t \in [0, 1]: \langle A(t), y\rangle \geq -\varepsilon_n\} = \inf\{t \in [0, 1]: \langle A(t), y\rangle > -\varepsilon_n\}.$$

Then there exists $t^* \in [0, 1]$ such that $A(t^*) = 0_{E^*}$.

Proof. For each fixed $n \in \mathbb{N}$, let $\Phi_n: Y \to 2^{[0,1]}$ be the multifunction defined by setting

$$\Phi_n(y) = \{t \in [0, 1]: \langle A(t), y\rangle \geq -\varepsilon_n\}.$$

By an easy application of Theorem 2 we have that there exists a continuous $\varphi_n: Y \to [0, 1]$ such that $\varphi_n(y) \in \Phi_n(y)$ for all $y \in Y$, hence $\langle A(\varphi_n(y)), y\rangle \geq -\varepsilon_n$ for all $y \in Y$. Therefore, we get

$$\inf_{y\in Y} \langle A(\varphi_n(y)), y\rangle \geq -\varepsilon_n$$

for each $n \in \mathbb{N}$, hence we have

$$\sup_{\varphi \in C^0(Y,[0,1])} \inf_{y \in Y} \langle A(\varphi(y)), y\rangle \leq 0.$$

By Theorem 1 our claim follows. □

Theorem 4. *Let E be a real Banach space, with $\dim(E) > 1$, $Y = \{y \in E: \|y\| = 1\}$, $A: [0, 1] \to E^*$ a weakly-star continuous operator which is twice differentiable in $]0, 1[$, $\{\sigma_n\}$ a sequence of positive real numbers, with $\{\sigma_n\} \to 0$. Let*

$$\Gamma = \{(t, y) \in]0, 1[\times Y: \langle A'(t), y\rangle = 0 \textit{ and } \langle A''(t), y\rangle \leq 0\}.$$

Assume that:

(i) *for each $y \in Y$ one has $\{t \in [0, 1]: \langle A(t), y\rangle \geq 0\} \neq \emptyset$;*
(ii) $\{\langle A(t), y\rangle: (t, y) \in \Gamma\} \cap (\bigcup_n \{-\sigma_n\}) = \emptyset$;
(iii) *for each $y \in Y$ and each $t_0 \in \{0, 1\}$ at least one of the following assertions holds:*
   (a) *t_0 is not a local maximum for the function $t \to \langle A(t), y\rangle$;*
   (b) $\langle A(t_0), y\rangle \notin \bigcup_n \{-\sigma_n\}$.

Then there exists $t^* \in [0, 1]$ such that $A(t^*) = 0_{E^*}$.

Proof. Let $y \in Y$ and $n \in \mathbb{N}$ be fixed. If $t \in]0, 1[$ is a local maximum for the function $t \to \langle A(t), y\rangle$, then $(t, y) \in \Gamma$, hence, by (ii), $\langle A(t), y\rangle \neq -\sigma_n$. Therefore, taking into account (iii), there is not a point $\tilde{t} \in [0, 1]$ such that $\tilde{t}$ is a local maximum for $t \to \langle A(t), y\rangle$ and $\langle A(\tilde{t}), y\rangle = -\sigma_n$. Now, let

$$t_1 = \inf\{t \in [0, 1]: \langle A(t), y\rangle \geq -\sigma_n\},$$
$$t_2 = \inf\{t \in [0, 1]: \langle A(t), y\rangle > -\sigma_n\}.$$

We note that by (i) we have

$$\emptyset \neq \{t \in [0, 1]: \langle A(t), y\rangle > -\sigma_n\} \subseteq \{t \in [0, 1]: \langle A(t), y\rangle \geq -\sigma_n\}.$$

Of course we have $t_1 \leq t_2$. Assume that $t_1 < t_2$. Of course, we have $\langle A(t_1), y\rangle \geq -\sigma_n$. If $\langle A(t_1), y\rangle > -\sigma_n$, then $t_1 \in \{t \in [0, 1]: \langle A(t), y\rangle > -\sigma_n\}$, hence $t_1 \geq t_2$, against our assumption $t_1 < t_2$. Therefore, $\langle A(t_1), y\rangle = -\sigma_n$. Of course, for each $t \in [t_1, t_2[$ we have $\langle A(t), y\rangle \leq -\sigma_n$. Thus, if $t_1 > 0$, since we have $\langle A(t), y\rangle < -\sigma_n$ for all $t \in [0, t_1[$, the point t_1 is a local maximum of the function $t \to \langle A(t), y\rangle$ such that $\langle A(t_1), y\rangle = -\sigma_n$, and this is a contradiction by the first part of the proof. If $t_1 = 0$, since $\langle A(t), y\rangle \leq -\sigma_n$ for each $t \in [t_1, t_2[$ and $A\langle(t_1), y\rangle = -\sigma_n$, again $t_1 = 0$ is a local maximum of the function

$t \to \langle A(t), y\rangle$ such that $\langle A(t_1), y\rangle = -\sigma_n$, another contradiction. Therefore $t_1 = t_2$. By Theorem 3 our claim follows. □

We remark that assumptions (ii) and (iii) of Theorem 4 are satisfied, for instance, if for each $\sigma > 0$ one has

$$]-\sigma, 0[\, \not\subseteq \{\langle A(t), y\rangle : (t, y) \in \Gamma\}$$

and for each $y \in Y$ and $t_0 \in \{0, 1\}$ at least one of the following conditions hold:
(a) the point t_0 is not a local maximum of $t \to \langle A(t), y\rangle$;
(b) $\langle A(t_0), y\rangle \geq 0$.

It is not difficult to construct examples of operators A satisfying the assumptions of Theorem 4. Another application of Theorem 1 is the following.

Theorem 5. *Let* $A: [0, 1] \to \mathbb{R}^n (n \geq 2)$ *be a continuous operator,* $Y = \{y \in \mathbb{R}^n : \|y\| = 1\}$. *Assume that for each* $\varepsilon > 0$ *there exists* $L_\varepsilon > 0$ *such that, for each finite set* $\{y_1, \ldots, y_k\} \subseteq Y$, *there exists a set* $\{t_1, \ldots, t_k\} \subseteq [0, 1]$ *such that*

$$(1) \qquad \begin{cases} \langle A(t_i), y_i\rangle \geq -\varepsilon & \text{for each } i = 1, \ldots, k \\ |t_i - t_j| \leq L_\varepsilon \|y_i - y_j\| & \text{for all } i, j = 1, \ldots, k. \end{cases}$$

Then there exists $t^* \in [0, 1]$ *such that* $A(t^*) = 0_{\mathbb{R}^n}$.

Proof. Fix $\varepsilon > 0$. Let Σ be the space of all functions $\varphi: Y \to [0, 1]$ that are Lipschitzian with constant L_ε, endowed with the uniform convergence topology. By the Ascoli-Arzelà theorem the space Σ is compact. For each $y \in Y$, let $F_y = \{\varphi \in \Sigma : \langle A(\varphi(y)), y\rangle \geq -\varepsilon\}$. The continuity of A implies that each set F_y is closed. Now, let $D = \{y_1, \ldots, y_k\}$ be any finite subset of Y, and let $t_1, \ldots, t_k \in [0, 1]$ satisfying (1). Let $g: D \to [0, 1]$ be defined by setting $g(y_i) = t_i$, for each $i \in \{1, \ldots, k\}$. By (1) we have

$$|g(y_i) - g(y_j)| \leq L_\varepsilon \|y_i - y_j\| \quad \text{for all} \quad i, j \in \{1, \ldots, k\},$$

hence g is Lipschitzian on D with constant L_ε. By a classical extension result, there exists a function $\psi: Y \to [0, 1]$ which is Lipschitzian on Y with the same constant L_ε such that $\psi|_D = g$. Therefore, $\psi \in \bigcap_{i=1}^k F_{y_i}$ and the family $\{F_y\}_{y \in Y}$ has the finite intersection property. Since Σ is compact, there exists $\tilde{\varphi} \in \bigcap_{y \in Y} F_y$. That is, $\tilde{\varphi} \in C^0(Y, [0, 1])$ and

$$\inf_{y \in Y} \langle A(\tilde{\varphi}(y)), y\rangle \geq -\varepsilon.$$

By Theorem 1 our claim follows. □

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