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# DUAL VARIATIONAL INEQUALITY AND APPLICATIONS TO ASYMMETRIC TRAFFIC EQUILIBRIUM PROBLEM WITH CAPACITY CONSTRAINTS

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This paper deals with a general duality theory developed by means of separation techniques and shows that particular kinds of Dual Variational Inequalities can be obtained as special cases. An important emphasis is put on the study of an asymmetric equilibrium problem with side constraints.

## 1. Introduction.

In this paper, we are concerned with the duality theory for Variational Inequalities with capacity constraints.

We consider the general duality theory presented by M. Castellani - G. Mastroeni in [1] (see also [3] and [6]) and we apply this general theory to the particular case of the Variational Inequality that expresses the traffic equilibrium problem with capacity constraints for which the convex set, that defines the feasible flows, has a specific formulation.
This particular formulation allows us to get an explicit computation of the support function, which plays an important role in the Duality Theory.

Before considering the dual approach derived from the separation theory, we first consider a dual formulation for the Variational Inequalities with capacity

*Key Words*: Variational Inequality, Quasi-Variational Inequality, Separation Theory, Duality.

constraints obtained by means of a particular approach proposed by M. Fukushima - T. Itoh in [2] and we show that actually this formulation is a particular case of the general theory.

However, in both the cases we need that the cost functions have an inverse and, then, the aim of our future works will be to remove this restrictive assumption.

## 2. The M. Fukushima - T. Itoh dual approach.

It is well known that the feasible set of traffic equilibrium problem in terms of path flows can be expressed by means of the convex set

$$K = \left\{ F \in \mathbb{R}^m_+ : \sum_{R_r \in \mathcal{R}_j} F_r = \rho_j \quad j = 1, \dots, l \right\}. \tag{2.1}$$

In $K$ we consider the Variational Inequality

$$H \in K : C(H)(F - H) \geq 0 \quad \forall F \in K \tag{2.2}$$

where

$$C(F) = (C_1(F), \dots, C_m(F))$$

represents the cost function on the paths.
If $W_j$ $j = 1, \dots, \ell$ are O/D pairs, by $\mathcal{R}_j$ we denote the set of paths $R_r$ which connect the O/D $W_j$.

By the equilibrium definition, if $H$ is an equilibrium solution, setting

$$C^j(H) = \min_{R_r \in \mathcal{R}_j} \{C_r(H)\} \tag{2.3}$$

then it results

$$C^j(H) \begin{cases} = C_r(H) & \text{if } H_r > 0 \\ \leq C_r(H) & \text{if } H_r = 0 \end{cases} \qquad R_r \in \mathcal{R}_j, \ j = 1, \dots, l \tag{2.4}$$

that is

$$H_r \begin{cases} = 0 & \text{if } C^j < C_r(H) \\ \geq 0 & \text{if } C^j = C_r(H) \end{cases} \qquad R_r \in \mathcal{R}_j \ , j = 1, \dots, l. \tag{2.5}$$

Setting

$$C(K) = \{\Gamma \in \mathbb{R}^m_+ : \Gamma = C(F), \ F \in K\}$$

let us recall the following result due to M. Fukushima - T. Itoh, whose proof we report for the reader's convenience. We consider an equivalent formulation in terms of path-costs function more useful for our purpose.

**Theorem 2.1.** *Suppose that the cost function $\Gamma : K \to C(K)$ has an inverse* [1]*; set* $\forall \mathscr{C} \in C(K)$

$$K^*(\mathscr{C}) = \Big\{\Gamma \in C(K) : \Gamma_r \geq C^j = \min_{R_r \in \mathscr{R}_j} \{\mathscr{C}_r\}, \quad R_r \in \mathscr{R}_j,\ j = 1, \dots, l\Big\}. \tag{2.6}$$

*Then, if $H \in K$ is a solution to* (2.2), *$\mathscr{C} = C(H)$ is a solution to the Variational Inequality*

$$\sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq 0 \quad \forall \Gamma \in K^*(\mathscr{C}) \tag{2.2$'$}$$

*and viceversa : if $\mathscr{C} \in K^*(\mathscr{C})$ is a solution to* (2.2)$'$, *then $H = C^{-1}(\mathscr{C})$ is a solution to* (2.2).

*Proof.* Let us show that if $H \in K$ is a solution to (2.2), then $\mathscr{C} = C(H)$ is a solution to (2.2)$'$.
Let us remark that from (2.4), it results:

$$H_r\,(C_r(F) - C_r(H)) \geq 0 \quad R_r \in \mathscr{R}_j,\ j = 1, \dots, l.$$

In fact, if $C_r(H) = C^j$, for every $\Gamma = C(F)$ belonging to (2.6), it follows

$$\Gamma_r = C_r(F) \geq C^j$$

and moreover, from (2.5)

$$H_r \geq 0,$$

so that the product is non-negative.
On the contrary, if $C_r(H) > C^j$, from (2.5) it follows

$$H_r = 0.$$

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[1] Notice, i.e., that if $C$ satisfies the hypothesis of strong monotonicity

$$\langle C(F_1) - C(F_2), F_1 - F_2\rangle \geq \alpha \|F_1 - F_2\|,\ \alpha > 0,$$

then $C$ has an inverse. In the sequel, we will consider the following notation: $\mathscr{C} = C(H)$.

Summing over all paths, we obtain

$$\sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} H_r \left(C_r(F) - C_r(H)\right) \geq 0 \quad \forall F \in K. \tag{2.7}$$

In virtue of the assumption that the cost function $\Gamma = C(F)$ has an inverse $F = C^{-1}(\Gamma)$ with $F : C(K) \to K$, (2.7) can be written as

$$\sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq 0 \quad \forall \Gamma \in K^*(\mathscr{C}).$$

Viceversa. Let $\mathscr{C} \in K^*(\mathscr{C})$ be a solution to (2.2)′; let us show that $H = C^{-1}(\mathscr{C})$ is a solution to (2.2).
First, we notice that $H \in K$ because $\mathscr{C} \in C(K)$.
If we write (2.2)′ as

$$\sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} \left[ - C_r^{-1}(\mathscr{C}) \right] (\Gamma_r - \mathscr{C}_r) \leq 0$$

and take into account that from (2.6)

$$\Gamma_r \geq C^j \quad R_r \in \mathcal{R}_j, \ j = 1, \ldots, l$$

by a theorem of the alternative, there exist $\lambda_r \geq 0$, $R_r \in \mathcal{R}_j$, $j = 1, \ldots, l$ such that

$$\begin{cases} \lambda_r = C_r^{-1}(\mathscr{C}) & \text{if } \mathscr{C}_r = C^j \\ \lambda_r = 0 & \text{if } \mathscr{C}_r > C^j. \end{cases} \tag{2.8}$$

Since we know that $H_r = C_r^{-1}(\mathscr{C})$, that is $\mathscr{C}_r = C_r(H_r)$, (2.8) coincides with (2.5).

**Remark 2.1.** $C^j$ are the Lagrangean variables associated to the equations

$$\sum_{R_r \in \mathcal{R}_j} F_r = \rho_j \quad j = 1, \ldots, l.$$

They correspond to the Lagrangean multipliers found by [5] in the standard case.

**Remark 2.2.** Problem (2.2)′ could be more convenient from a computational point of view in virtue of the simpler formulation of the constraints set.

**Remark 2.3.** Notice that, even if we start from a model with fixed demand, that is the convex set $K$ doesn't depend on the solution $H$ by $\rho = \rho(H)$, the dual problem is always expressed by a Quasi-Variational Inequality, since the convex $K^*$ depends on the solution $\mathscr{C}$.

## 3. The dual model with capacity constraints on the paths.

Let us suppose, now, that on the paths there are capacity constraints i.e.

$$K = \left\{ F \in \mathbb{R}^m_+ : F_r \leq \Phi_r \; r = 1, \dots, m, \; \sum_{R_r \in \mathcal{R}_j} F_r = \rho_j, \; j = 1, \dots, l \right\}. \tag{3.1}$$

In this case, if $H$ is an equilibrium solution according with the generalized Wardrop principle (see [4]), setting

$$C^j(H) = \max_{r \in B_j} C_r(H) \tag{3.2}$$

where

$$B_j = \{r : R_r \in \mathcal{R}_j, H_r > 0\}, \tag{3.3}$$

$$L^j_r(H) = C^j(H) - C_r(H) \quad r \in B_j, \tag{3.4}$$

$$\widetilde{C}_r(H) = \begin{cases} C_r(H) + L^j_r(H) = C^j(H) & \text{if } r \in B_j \\ C_r(H) & \text{if } r \notin B_j, \; R_r \in \mathcal{R}_j \end{cases} \tag{3.5}$$

by the equilibrium definition, it follows

$$H_r \begin{cases} = 0 & \text{if } C^j(H) < \widetilde{C}_r(H) \\ \geq 0 & \text{if } C^j(H) = \widetilde{C}_r(H). \end{cases} \tag{3.6}$$

For a generic $F \in K$, let us set

$$C^j(F) = \max_{r \in B_j} C_r(F)$$

and

$$L^j_r(F) = C^j(F) - C_r(F) \quad r \in B_j$$

$$\widetilde{C}_r(F) = \begin{cases} C_r(F) + L^j_r(F) = C^j(F) & \text{if } r \in B_j = \{r : R_r \in \mathcal{R}_j \; , F_r > 0\} \\ C_r(F) & \text{if } r \notin B_j, \; R_r \in \mathcal{R}_j. \end{cases}$$

Define

$$\widetilde{C}(K) = \{\Gamma \in \mathbb{R}^m_+ : \Gamma = \widetilde{C}(F), \; F \in K\}$$

and assume that the function $\widetilde{C}$ has an inverse defined from $\widetilde{C}(K)$ to $K$; we have the following

**Theorem 3.1.** *Let us set* $\forall \mathscr{C} \in C(K)$

$$K^*(\mathscr{C}) = \left\{\Gamma \in \widetilde{C}(K) : \Gamma_r \geq C^j = \min_{R_r \in \mathscr{R}_j} \{\mathscr{C}_r\},\ R_r \in \mathscr{R}_j,\ j = 1, \ldots, l\right\}. \tag{3.7}$$

*Then, if* $H \in K$ *is a solution to* (2.2), $\mathscr{C} = \widetilde{C}(H)$ *is a solution to the Variational Inequality*

$$\sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} \widetilde{C}_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq 0 \quad \forall \Gamma \in K^*(\mathscr{C}) \tag{3.8}$$

*and viceversa.*

*Proof.* Let us show that if $H \in K$ is a solution to (2.2), then $\mathscr{C} = \widetilde{C}(H)$ is a solution to (3.8).
Remark that from (3.5) and (3.6), we derive

$$H_r \left(\widetilde{C}_r(F) - \widetilde{C}_r(H)\right) \geq 0 \quad R_r \in \mathscr{R}_j,\ j = 1, \ldots, l.$$

In fact, if $\widetilde{C}_r(H) = C^j(H)$, for every $\Gamma = \widetilde{C}(F) \in K^*(\mathscr{C})$, it follows

$$\Gamma_r = \widetilde{C}_r(F) \geq C^j(H)$$

and moreover from (3.6), it follows

$$H_r \geq 0,$$

so that the product is non-negative.
On the contrary, if $\widetilde{C}_r(H) > C^j(H)$, from (3.6) it follows

$$H_r = 0.$$

Summing over all paths, we obtain:

$$\sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} H_r \left(\widetilde{C}(F) - \widetilde{C}_r(H)\right) \geq 0 \quad \forall F \in K. \tag{3.9}$$

By the assumption that the cost function $\Gamma = \widetilde{C}(F)$ defined from $K$ to $\widetilde{C}(K)$ has an inverse $F = \widetilde{C}^{-1}(\Gamma)$ defined from $\widetilde{C}(K)$ to $K$, (3.9) can be written as

$$\sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} \widetilde{C}^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq 0 \quad \forall \Gamma \in K^*(\mathscr{C}).$$

Viceversa. Let $\mathscr{C} \in K^*(\mathscr{C})$ be a solution to (3.8) and let us show that $H = \widetilde{C}^{-1}(\mathscr{C})$ is a solution to (2.2).
First, we remark that $H \in K$ because $\mathscr{C} \in \widetilde{C}(K)$.
If we write (3.8) as

$$\sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} \left[ -\widetilde{C}_r^{-1}(\mathscr{C}) \right] (\Gamma_r - \mathscr{C}_r) \leq 0$$

and take into account that from (3.7)

$$\Gamma_r \geq C^j \quad R_r \in \mathscr{R}_j, \; j = 1, \ldots, l$$

by a theorem of the alternative, there exist $\lambda_r \geq 0$, $R_r \in \mathscr{R}_j$, $j = 1, \ldots, l$ such that

$$\text{(3.10)} \qquad \begin{cases} \lambda_r = \widetilde{C}_r^{-1}(\mathscr{C}) & \text{if } \mathscr{C}_r = C^j \\ \lambda_r = 0 & \text{if } \mathscr{C}_r > C^j. \end{cases}$$

Since we known that $H_r = \widetilde{C}_r^{-1}(\mathscr{C})$, that is $\mathscr{C}_r = \widetilde{C}_r(H)$, (3.10) coincides with (3.6).

## 4. Dual Variational Inequality in terms of separation theory.

Consider the cost-path function $C(F)$:

$$C : K \to R_+^m$$

where

$$K = \left\{ F \in R_+^m : \sum_{R_r \in \mathscr{R}_j} F_r = \rho_j \quad j = 1, \ldots, l \right\}.$$

Assume that

$$C : K \to C(K)$$

has an inverse

$$\Gamma = C(F) \Longleftrightarrow F = C^{-1}(\Gamma).$$

Since the cost function $C(F)$ is not defined in the set $-K$ as required in the duality theory obtained by means of separation theory, let us consider the extension of $C(F)$ to $-K$ obtained setting

$$C(-F) = -C(F) \quad \forall F \in K$$

and assuming the reasonable condition

$$C(0) = 0.$$

The extension

$$C : K \cup \{-K\} \to C(K) \cup \{-C(K)\}$$

has still an inverse

$$C^{-1}(\Gamma) : C(K) \cup \{-C(K)\} \to K \cup \{-K\}$$

and preserves the eventual continuity of $C$.
Let

$$I_K(F) = \begin{cases} 0 & \text{if } F \in K \\ +\infty & \text{if } F \notin K \end{cases}$$

be the indicator function of the convex set $K$, then the Dual Variational Inequality associated to the Variational Inequality

$$H \in K \quad C(H)(F - H) \geq 0 \quad \forall F \in K \tag{4.1}$$

by means of the separation theory is:

$$\mathscr{C} \in C(K) \quad \sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} \left( - C_r^{-1}(-\mathscr{C})\right)(\Gamma_r - \mathscr{C}_r) = \tag{4.2}$$

$$= \sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq I_K^*(\mathscr{C}) - I_K^*(\Gamma)$$

where

$$I_K^*(\Gamma) = \sup_{\mathbb{R}^m} (\langle \Gamma, F \rangle - I_K(F)) = \sup_K \langle \Gamma, F \rangle.$$

$I_K^*(\Gamma)$ is also known as the support function of $K$. We have to compute $I_K^*(\Gamma)$ with $\Gamma \geq 0$, $\Gamma \in C(K)$. We have a P. L. problem

$$\begin{cases} \sup \langle \Gamma, F \rangle \\ F_r \geq 0 \\ \sum\limits_{R_r \in \mathscr{R}_j} F_r = \rho_j \ j = 1, \ldots, l \end{cases} \tag{4.3}$$

where in the coefficient matrix of the system

$$\sum_{R_r \in \mathscr{R}_j} F_r = \rho_j \quad j = 1, \ldots, l$$

there are only canonical vectors.
In virtue of this remark, it is easy to show that, setting

$$\Gamma_{\nu_j} = \max_{R_r \in \mathcal{R}_j} \Gamma_r \quad j = 1, \dots, l,$$

it results

$$\sup\langle \Gamma, F \rangle = \sum_{j=1}^{l} \Gamma_{\nu_j} \rho_j$$

and, then, the problem (2.2) becomes

$$\mathscr{C} \in C(K) \quad \sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq \sum_{j=1}^{l} \left( \mathscr{C}_{\mu_j - \Gamma_{\nu_j}} \right) \rho_j \quad \forall \Gamma \in C(K)$$

where

$$\mathscr{C}_{\mu_j} = \max_{R_r \in \mathcal{R}_j} \mathscr{C}_r \quad j = 1, \dots, l.$$

Then, M. Fukushima - T. Itoh dual can be obtained by choosing the vectors $\Gamma \in C(K)$ such that

$$\Gamma_{\nu_j} \leq \mathscr{C}_{\mu_j} \quad j = 1, \dots, l.$$

Let us consider, now, the model with capacity constraints on the paths.

$$K = \left\{ F \in \mathbb{R}^m_+ : \sum_{R_r \in \mathcal{R}_j} F_r = \rho_j, \ F_r \leq \Phi_r \ R_r \in \mathcal{R}_j, \ j = 1, \dots, l \right\}.$$

Let us suppose that

$$\sum_{R_r \in \mathcal{R}_j} \Phi_r \geq \rho_j \quad j = 1, \dots, l.$$

Then, assuming that the function $C : K \to C(K)$ fulfils the conditions above about the extension, we get the Dual Variational Inequality

$$(4.2)' \qquad \mathscr{C} \in C(K) : \sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq I_K^*(\mathscr{C}) - I_K^*(\Gamma).$$

Also in this case we are able to compute $I_K^*(\Gamma)$. We have to solve the problem

$$\begin{cases} \sup\langle \Gamma, F \rangle \\ F_r \geq 0 \quad R_r \in \mathcal{R}_j, & j = 1, \dots, l \\ \sum_{R_r \in \mathcal{R}_j} F_r = \rho_j & j = 1, \dots, l \\ F_r \leq \Phi_r & r = 1, \dots, m. \end{cases}$$

Let us dispose $\Gamma_r : R_r \in \mathscr{R}_j$ in decreasing order:

$$\Gamma_{r_1} \geq \Gamma_{r_2} \geq \ldots \geq \Gamma_{re_j}$$

and define $r_{\nu_j}$ the index such that it results

$$\Phi_{r_1} + \Phi_{r_2} + \cdots + \Phi_{r_{\nu_{j-1}}} \leq \rho_j$$
$$\Phi_{r_1}, + \Phi_{r_2} + \cdots + \Phi_{r_{\nu_{j-1}}} + \mathcal{E}_{r_{\nu_j}} = \rho_j, \quad \mathcal{E}_{r_{\nu_j}} \geq 0.$$

Then, we have:

$$\sup\langle \Gamma, F\rangle = \sum_{j=1}^{l} \sum_{r_i \leq r_{\nu_{j-1}}} \Gamma_r \Phi_{r_i} + \Gamma_{r_{\nu_j}} \mathcal{E}_{r_{\nu_j}}$$

and, so, (4.2)′ becomes

$$\mathscr{C} \in C(K) : \sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} C_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq$$
$$\geq \sum_{j=1}^{l} \Big[ \sum_{s_i \leq s_{\nu_{j-1}}} \mathscr{C}_{s_i} \Phi_{s_i} - \sum_{r_i \leq r_{\nu_{j-1}}} \Gamma_{r_i} \Phi_{r_i} \Big] + \mathscr{C}_{s_{\mu_j}} \mathcal{E}_{S_{\mu_j}} - \Gamma_{r_{\nu_j}} \mathcal{E}_{r_{\nu_j}}.$$

It may be convenient to look for the dual problem by using

$$\widetilde{C}_r(F) = \begin{cases} C_r(F) + L_r^j(F) = C^j(F) & \text{if } r \in B_j = \{r : R_r \in \mathscr{R}_j, F_r > 0\} \\ C_r(F) & \text{otherwise} \end{cases}$$

where

$$C^j(F) = \max_{r \in B_j} C_r(F)$$

and

$$L_r^j(F) = C^j(F) - C_r(F) \quad r \in B_j,$$

(see [4] for more details).
In this case, the Variational Inequality (4.1) can be expressed as

$$H \in K : \widetilde{C}(H)(F - H) \geq 0 \quad \forall F \in K.$$

If we assume that

$$\widetilde{C} : K \to \widetilde{C}(K)$$

fulfils the same conditions of $C$ about the extension, being the expression of the function $I_K^*(\Gamma)$ the same,we obtain:

(4.2)″
$$\mathscr{C} \in \widetilde{C}(K) : \sum_{j=1}^{l} \sum_{R_r \in \mathscr{R}_j} \widetilde{C}_r^{-1}(\mathscr{C})(\Gamma_r - \mathscr{C}_r) \geq$$
$$\geq \sum_{j=1}^{l} \Big[ \sum_{s_i \leq s_{\nu_{j-1}}} \mathscr{C}_{s_i} \Phi_{s_i} - \sum_{r_i \leq r_{\nu_{j-1}}} \Gamma_{r_i} \Phi_{r_i} \Big] + \mathscr{C}_{s_{\mu_j}} \mathcal{E}_{s_{\mu_j}} - \Gamma_{r_{\nu_j}} \mathcal{E}_{r_{\nu_j}}.$$

To obtain again M. Fukushima - T. Itoh dual, we need to restrict to the vectors

$$\Gamma \in \widetilde{C}(K)$$

such that

$$\sum_{r_i \leq r_{\nu_{j-1}}} \Gamma_{r_i} \Phi_{r_i} + \Gamma_{r_{\nu_j}} \mathcal{E}_{r_{\nu_j}} \leq \sum_{s_i \leq \mu_{j-1}} \mathscr{C}_{s_i} \Phi_{s_i} + \mathscr{C}_{s_{\mu_j}} \mathcal{E}_{s_{\mu_j}} \text{ (2)}.$$

**Remark 4.1.** The Lagrangean function associated to the Variational Inequality, in our case, is

$$\langle C(F), F \rangle + I_K(F) + I_K^* (C(F)) = \Lambda_F (F, F, C(F))$$

and $H \in K$ is a solution to the Variational Inequality if $(H, \mathscr{C} = C(H))$ is solution to

$$\max_{\Gamma \in C(K)} \min_{F \in K} \Big\{ \langle \Gamma, F \rangle + I_K(F) + I_K^*(\Gamma) \Big\} =$$
$$= -\langle C(H), H \rangle + I_K(H) + I_K^*(\mathscr{C}).$$

(2) If the assumption of strong monotonicity holds, the solution of the Variational Inequality restricted to a convex subset of the convex $K^*$ doesn't change.

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