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EQUILIBRIUM IN ASYMMETRIC MULTIMODAL TRANSPORT NETWORKS WITH CAPACITY CONSTRAINTS

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The paper presents an equilibrium model for asymmetric multimodal transport networks with capacity constraints. It is shown that these networks may have no equilibrium flow pattern which satisfies the capacity constraints, but additional costs can be imposed on some network links, so that an equilibrium solution that satisfies the capacity constraints always exists. A method of computing both the equilibrium flow pattern and the additional costs is presented, and is used in a simple numerical example.

1. The user equilibrium problem.

Consider an urban area, divided into κ zones, and let $G(N, L)$ be the graph representing the multimodal transport network serving the area, where N is the set of nodes and L is the set of links: m is the number of nodes and a the number of links. Each zone is identified by a centroid, and $w = (i, j)$, $i \neq j$, represents an ordered pair of centroids. Consider a period τ (e.g. the period in the morning, when people go to work) belonging to a sequence of periods during which the probability distribution of the utility function [4] of people living in the urban area is the same. Let:

indicate the transpose of a vector or a matrix

W = the set of ordered w pairs

n = the number of pairs $w \in W$

T_w = the set of transport modes joining the pair $w \in W$

T = $\bigcup_{w \in W} T_w$ the set of all transport modes on the network

u = the number of elements in T

λ_w^t = λ_{ij}^t = the average trip cost expected by users on mode $t \in T_w$ between the pair $(i, j) = w \in W$

N_i = the generation capacity of centroid i, i.e. the number of people that can decide to move from the zone i during τ

d_w^t = d_{ij}^t = the demand on mode t between the pair $(i, j) = w$, that is the average number of people that during τ travel on mode t between w

d_w = $(\ldots d_w^{t'} \ldots)'$ = the vector of demand between the pair $w \in W$ on all modes $t \in T_w$

d = $(\ldots d^{t'} \ldots)'$ = the vector of demand between all $w \in W$ on all modes $t \in T$

d_i = the average number of people that during τ remain in i

$\mathscr{A}_i$ = the measure of the attractiveness of zone i

$\alpha\mathscr{A}_i = a_i$ = the average of utilities that people derive from remaining in i during τ

$\alpha\mathscr{A}_j - \beta\lambda_w^t = a_j - \beta\lambda_w^t$ the average of utilities that people derive from moving from i to j during τ.

We assume that the demand d_{ij}^t is given by a logit model [9]:

$$d_{ij}^t = N_i \frac{e^{a_j - \beta\lambda_{ij}^t}}{e^{a_i} + \sum_{z \neq i} \sum_{t \in T_{iz}} e^{a_z - \beta\lambda_{iz}^t}} \tag{1}$$

and the average number of people that during τ remain in i is:

$$d_i = N_i - \sum_{z \neq i} \sum_{t \in T_{iz}} d_{iz}^t = N_i \frac{e^{a_i}}{e^{a_i} + \sum_{z \neq i} \sum_{t \in T_{iz}} e^{a_z - \beta\lambda_{iz}^t}}. \tag{2}$$

After some manipulations one obtains from (1) and (2):

$$d_{ij}^t = e^{a_j - a_i - \beta\lambda_{ij}^t} \left(N_i - \sum_{z \neq i} \sum_{t \in T_{iz}} d_{iz}^t \right)$$

and

$$\lambda_w^t = \lambda_{ij}^t = \frac{1}{\beta}\left[a_j - a_i - ln\ d_{ij}^t + ln\ (N_i - \sum_{z \neq i} \sum_{t \in T_{iz}} d_{iz}^t)\right] \tag{3}$$

which is the relationship $\lambda_w^t = \lambda_w^t(d)$ between the average trip cost expected by users on mode t between the pair $w \in W$ and the demand vector d. Eq. (3) is called the *inverse* of the *demand function* (1): we take in it $ln(\cdot) = 0$ if $(\cdot) \leq 1$ in order to assure the continuity of λ_{ij}^t.

Let:

S = the set of centroids

P_w^t = the set of paths p joining w on mode t

$P = \bigcup_{\substack{t \in T_w \\ w \in W}} P_w^t$ the set of all p joining all $w \in W$ on all modes $t \in T_w$

M = the number of paths $p \in P$

h_p = the flow on path $p \in P$, that is the average number of users which during τ travel on path p

$h \in \mathbb{R}_+^M$ = the vector of flows on all paths $p \in P$

f_i = the flow on link $i \in L$, that is the average number of users which during τ traverse link i

$f \in \mathbb{R}_+^a$ = the vector of link flows

$f_i = \sum_{p \in P} h_p \delta_{ip}\ \forall i \in L$, where $\delta_{ip} = 1$ if path p traverses link i, 0 otherwise

$d_w^t = \sum_{p \in P} h_p \delta_{wp}^t\ \forall t \in T_w, \forall w \in W$, where $\delta_{wp}^t = 1$ if path p connects the pair w on mode t, 0 otherwise.

The network has v capacity constraints, which are linear functions of link flows [11]:

$$g^r(f) = \sum_{i \in L} \beta_{ri} f_i - H_r \leq 0, \qquad r \in (1, 2, \ldots, v) \tag{4}$$

where H_r are constants.

We can express the capacity constraints as functions of path flows, so that expression (4) becomes:

$$G^r(h) = \sum_{i \in L} \beta_{ri} \sum_{p \in P} h_p \delta_{ip} - H_r \leq 0 \quad \text{for} \quad r \in (1, 2, \ldots, v). \tag{5}$$

The set:

$$\Omega_1 = \{h \in \mathbb{R}^M : h \geq 0,\ G^r(h) \leq 0 \quad \forall r \in (1, 2, \ldots, v)\} \tag{6}$$

is the *supply* set. The set:

$$\Omega_2 = \left\{h \in \mathbb{R}^M : h \geq 0, \quad \sum_{\substack{j \neq z \\ t \in T_{zj}}} \sum_{p \in P^t_{zj}} h_p \leq N_z \quad \forall z \in S\right\} \tag{7}$$

is the *demand* set. Both Ω_1 and Ω_2 are compact polyhedra.

The set $\Omega = \Omega_1 \cap \Omega_2$ is the set of *feasible solutions*: in transport networks, whose study is of actual interest, $\Omega_1 \subset \Omega_2$ and the demand constraints which appear in (7) are not binding in Ω, because a flow vector h which saturates the generation capacity of at least one origin does not satisfy the capacity constraints. The set of feasible link flow vectors is:

$$\Theta = \left\{f \in \mathbb{R}^a_+ : f_i = \sum_{p \in P} h_p \delta_{ip} \quad \forall i \in L,\ h \in \Omega\right\}. \tag{8}$$

Let:

$c_i(f)$ = the average of trip costs perceived by users who travel on link $i \in L$; $c_i(f)$ is continuous function of f in Θ.

$C_p(h)$ = $\sum_{i \in L} c_i(f)\delta_{ip}$ = the average of trip costs perceived by users who travel on path $p \in P$; $C_p(h)$ is continuous function of h in Ω.

$\Lambda^t_w(h)$ = $\lambda^t_w\big(\sum_{p \in P^{t1}_{w_1}} h_p, \ldots, \sum_{p \in P^{tu}_{w_n}} h_p\big)$ = the trip cost expected by users between the pair w on mode t; $\Lambda^t_w(h)$ is continuous function of h in Ω, as a consequence of the hypotheses assumed on Eq. (3).

$C_p(h) - \Lambda^t_w(h)$ = the difference between the perceived cost on path $p \in P^t_w$ and the expected cost between the pair w on mode $t \in T_w$.

$[C(h) - \Lambda(h)] : \Omega \to \mathbb{R}^M$ = the vector function whose components are $C_p(h) - \Lambda^t_w(h)$, $p \in P^t_w$, $\forall t \in T_w$, $\forall w \in W$.

A vector $\overline{h} \in \Omega$ is an *user equilibrium* (UE) *solution* if and only if, for every pair w, every mode $t \in T_w$, and every path $p \in P^t_w$:

$$\begin{aligned} \overline{h}_p > 0 &\Rightarrow C_p(\overline{h}) - \Lambda^t_w(\overline{h}) = 0 \\ \overline{h}_p = 0 &\Rightarrow C_p(\overline{h}) - \Lambda^t_w(\overline{h}) \geq 0. \end{aligned} \tag{9}$$

The traditional approach to the theory of transport network equilibrium [2] is founded on the hypothesis of equivalence between the UE conditions (9) and a variational inequality (VI), which can be written in terms of path flows as follows:

$$\text{(10)} \qquad [C(\bar{h}) - \Lambda(\bar{h})](h - \bar{h}) \geq 0 \quad \text{for all} \quad h \in \Omega.$$

This VI has at least one solution $\bar{h}$ in the compact polyhedron Ω. However it can be shown [5] that this equivalence holds if capacity constraints are written as strict inequalities, whereas if capacity constraints are $\leq$, then the UE problem and the VI problem are not necessarily equivalent, and the solution of the VI problem in the set Ω may not be solution of the UE problem, if any capacity constraint is verified as equality. However additional cost terms can be introduced, which ensure that in all solutions of the VI problem the capacity constraints will in fact be satisfied as strict inequalities. This also ensures that all the solutions of the VI problem in the compact set Ω are solutions of the UE problem, hence the corresponding UE problem has at least one solution (an equilibrium). Thus, introducing appropriate cost terms ensures that the UE problem always has a solution.

The result that additional costs can assure the existence of equilibrium when capacity constraints are active has been obtained also by other authors, using different methods, in the hypothesis of fixed demand (see Eq. (2.3) in [7] and Eqs. (2.4) and (3.10) in [8]).

In this paper the problem of network equilibrium with capacity constraints and elastic demand is studied by an approach different from that used in [5], and a method is presented which provides both the equilibrium solution and the modifications of cost functions that ensure in any case the existence of equilibrium.

2. The solution of equilibrium.

Let

$$s_p(h) = -h_p \leq 0 \qquad \forall p \in P$$

be the nonnegativity constraints and let

$$G^r(h) = \sum_{i \in L} \beta_{ri} \sum_{p \in P} h_p \delta_{ip} - H_r \leq 0 \quad \text{for } r \in (1, 2, \ldots, v)$$

be the capacity constraints. Since the demand constraints are satisfied as strict inequalities in the set Ω of feasible solutions, the latter is defined as follows:

$$\Omega = \{h \in \mathbb{R}^M : s_p(h) \leq 0 \quad \forall p \in P, \quad G^r(h) \leq 0 \quad \text{for } r \in (1, 2, \ldots, v)\}$$

Given a vector $\overline{h} \in \Omega$, the cone D of feasible directions e of Ω at $\overline{h}$ is given by:

$$D = \left\{e : e \neq 0, \quad \overline{h} + \xi e \in \Omega \quad \text{for all } \xi \in (0, \delta) \text{ for some } \delta > 0\right\}.$$

Let $J_s = \left\{p : s_p(\overline{h}) = 0\right\}$ and $J_g = \left\{r : G^r(\overline{h}) = 0\right\}$ be the index sets of constraints $s_p(h) \leq 0$ and $G^r(h) \leq 0$ which are active in $\overline{h}$. Since both $s_p(h)$ and $G^r(h)$ are linear, a direction $e \neq 0$ belongs to D if and only if

$$\begin{aligned} \nabla s_p(\overline{h})' e \leq 0 \quad \forall p \in J_s \\ \nabla G^r(\overline{h})' e \leq 0 \quad \forall r \in J_g. \end{aligned} \tag{11}$$

Theorem. *A vector $\overline{h} \in \Omega$ is a solution of VI* (10) *if and only if there exists a set of multipliers $u_p \geq 0\, \forall p \in J_s$ and $y_r \geq 0\, \forall r \in J_g$ such that:*

$$C(\overline{h}) - \Lambda(\overline{h}) + \sum_{p \in J_s} u_p \nabla s_p(\overline{h}) + \sum_{r \in J_g} y_r \nabla G^r(\overline{h}) = 0\,. \tag{12}$$

Proof. Suppose that $\overline{h}$ is a solution of VI (10). Consider any direction $e \in D$: we can find a $\xi > 0$ such that $\overline{h} + \xi e = h \in \Omega$. Thus $\left[C(\overline{h}) - \Lambda(\overline{h})\right]' \xi e \geq 0$ $\forall e \in D$ for some $\xi > 0$; if we denote by F the set of directions e for which:

$$\left[C(\overline{h}) - \Lambda(\overline{h})\right]' e < 0 \tag{13}$$

we have $D \cap F = \emptyset$. In this case Eq. (12) follows from Farkas' theorem.

Suppose that Eq. (12) holds. By Farkas' theorem we have that any direction e that verifies (11) does not satisfy (13), so that $\left[C(\overline{h}) - \Lambda(\overline{h})\right]' e \geq 0$. Let h be any point belonging to Ω: since Ω is convex, the direction $(h - \overline{h})$ satisfies (11), thus $\overline{h}$ satisfies VI (10) for all $h \in \Omega$. □

We suppose that the gradients $\nabla s_p(\overline{h})$ and $\nabla G^r(\overline{h})$ are independent, so that the multipliers u_p and y_r, given a solution $\overline{h}$ of VI (10), are unique: this hypothesis is in general verified in transport networks. If we let $u_p = 0$ if $s_p(\overline{h}) < 0$ and $y_r = 0$ if $G^r(\overline{h}) < 0$, Eq. (12) becomes:

$$C(\overline{h}) - \Lambda(\overline{h}) + \sum_{p \in P} u_p \nabla s_p(\overline{h}) + \sum_{r=1}^{v} y_r \nabla G^r(\overline{h}) = 0 \tag{14}$$

and we have the following corollary of the previous theorem:

Corollary. *A vector $\overline{h} \in \Omega$ is a solution of VI* (10) *if and only if there exists a set of multipliers $u_p \geq 0\ \forall p \in P$ and $y_r \geq 0\ \forall r \in (1 \dots v)$ which satisfy Eq.* (14) *and the complementarity condition* $\sum_{p \in P} u_p s_p(\overline{h}) + \sum_{r=1}^{v} y_r G^r(\overline{h}) = 0$.

The p_s component of Eq. (14), $p_s \in P_w^t$, is:

$$C_{p_s}(\overline{h}) - \Lambda_w^t(\overline{h}) - u_{p_s} + \sum_{r=1}^{v} y_r \frac{\partial G^r(\overline{h})}{\partial h_{p_s}} = 0 \tag{15}$$

and taking into account the expression (5) of $G^r(h)$, we have:

$$C_{p_s}(\overline{h}) - \Lambda_w^t(\overline{h}) - u_{p_s} + \sum_{i \in L} \sum_{r=1}^{v} \beta_{ri} y_r \delta_{ip_s} = 0\,. \tag{16}$$

If $y_r = 0\ \forall r \in (1, 2, \dots, v)$, Eq. (16) written for all $p_s \in P$ coincides with the definition (9) of UE. As a matter of fact, if $\overline{h}_{p_s} > 0$, $p_s \in P_w^t$, we have $u_{p_s} = 0$ and so $C_{p_s}(\overline{h}) - \Lambda_w^t(\overline{h}) = 0$; whereas if $\overline{h}_{p_s} = 0$, we have $u_{p_s} \geq 0$, and so $C_{p_s}(\overline{h}) - \Lambda_w^t(\overline{h}) \geq 0$. Thus, if capacity constraints are not binding in a solution $\overline{h}$ of VI (10), $\overline{h}$ is an equilibrium solution; and if no solution of VI (10) has the multiplier vector $y = 0$, no equilibrium solution exists. It is worth noting that the same conclusion was obtained in a previous paper using a different approach [5].

Let $\overline{f}_i = \sum_{p \in P} \overline{h}_p \delta_{ip}$; suppose that one increases the link cost function $c_i(f)$ by adding a road price equal to the linear combination of multipliers y_r that appears in Eq. (16). Let

$$c_i^*(f) = c_i(f) + \sum_{r=1}^{v} \beta_{ri} y_r \qquad \forall i \in L \tag{17}$$

be the new cost functions. The modified path cost $C_{p_s}^*(h)$, $p_s \in P_w^t$, is:

$$\begin{aligned} C_{p_s}(h) &= \sum_{i \in L} c_i^*(f) \delta_{ip_s} = \sum_{i \in L} c_i(f) \delta_{ip_s} + \sum_{i \in L} \sum_{r=1}^{v} \beta_{ri} y_r \delta_{ip_s} = \\ &= C_{p_s}(h) + \sum_{i \in L} \sum_{r=1}^{v} \beta_{ri} y_r \delta_{ip_s}\,. \end{aligned} \tag{18}$$

By substituting the expression (18) into Eq. (16), the latter becomes

$$C_{p_s}^*(\overline{h}) - \Lambda_w^t(\overline{h}) - u_{p_s} = 0 \tag{19}$$

and coincides with the definition (9) of UE. Thus, if one adds to link costs $c_i(f)$ $\forall i \in L$ the linear combination of multipliers that appears in Eq. (17), the solution $\overline{h}$ of VI (10) becomes an equilibrium solution.

3. The calculation of the equilibrium solution and of the capacity constraints multipliers.

In this section we propose an iterative procedure which provides both the equilibrium solution and the capacity constraints multipliers, when each capacity constraint is a function of only one link flow:

$$g^i(f) = f_i - H_i \leq 0 \quad \forall i \in L. \tag{20}$$

The proposed procedure is derived from the well-known iterative scheme used for solving the variational inequalities [3].

During each step of the procedure a nonlinear programming problem is solved, and its solution is the starting point of the successive step. At the beginning of the k-step a vector $h^k \in \Omega$ is known, and f^k is the corresponding link flow vector. During the k-step we consider the following symmetric link cost functions, in which the cost c_i^k on link i is a function only of the flow f_i on this link:

$$c_i^k(f_i) = c_i(f_1^k, \dots f_{i-1}^k, f_i, f_{i+1}^k, \dots, f_a^k) \qquad \forall i \in L \tag{21}$$

and the following symmetric inverse demand function, in which the trip cost $\lambda_w^{t,k}$ on mode t between the pair w is a function only of the demand d_w^t:

$$\lambda_w^{t,k}(d_w^t) = \lambda_w^t(d_1^{t_1,k}, \dots, d_{w-1}^{t,k}, d_w^t, d_{w+1}^{t,k}, \dots, d_n^{t_u,k}) \qquad \forall t \in T_w, \ \forall w \in W. \tag{22}$$

We write:

$$\overline{C}_p(h^k, h) = \sum_{i \in L} c_i^k(f_i)\delta_{ip} = \tag{23}$$
$$= \sum_{i \in L} c_i\Big(\dots \sum_{p \in P} h_p^k \delta_{i-1,p}, \sum_{p \in P} h_p \delta_{ip}, \sum_{p \in P} h_p^k \delta_{i+1,p} \dots\Big)\delta_{ip} \quad \forall p \in P$$

$$\overline{\Lambda}_w^t(h^k, h) = \lambda_w^{t,k}\Big(\sum_{p \in P} h_p \delta_{wp}^t\Big) = \tag{24}$$
$$= \lambda_w^t\Big(\dots \sum_{p \in P} h_p^k \delta_{w-1,p}^t, \sum_{p \in P} h_p \delta_{wp}^t, \sum_{p \in P} h_p^k \delta_{w+1,p}^t \dots\Big) \quad \forall t \in T_w, \ \forall w \in W$$

$[\overline{C}(h^k, h) - \overline{\Lambda}(h^k, h)] : \Omega \to \mathbb{R}^M$ = the vector function whose components are $\overline{C}_p(h^k, h) - \overline{\Lambda}^t_w(h^k, h)$ $p \in P^t_w$, $\forall t \in T_w$ $\forall w \in W$.

During the k-step we solve the following problem. Given the compact polyhedron:

$$\Omega = \Big\{h \in \mathbb{R}^M : h_p \geq 0 \quad \forall p \in P, \quad \sum_{p \in P} h_p \delta_{ip} - H_i \leq 0 \quad \forall i \in L, \quad \sum_{\substack{j \neq z \\ t \in T_{zj}}} \sum_{p \in P^t_{zj}} h_p \leq N_z \quad \forall z \in S\Big\} \tag{25}$$

find a solution $h^{k+1} \in \Omega$ of the following VI:

$$\left(\overline{C}(h^k, h^{k+1}) - \overline{\Lambda}(h^k, h^{k+1})\right)'(h - h^{k+1}) \geq 0 \quad \text{for } h \in \Omega \tag{26}$$

and the corresponding multipliers $y_i^{k+1} \forall i \in L$ of capacity constraints.

It easy to verify that the operator $\overline{C}(h^k, h) - \overline{\lambda}(h^k, h)$ of VI (26) is the gradient of the following function $\Omega \to \mathbb{R}$:

$$R(h^k, h) = \sum_{i \in L} \int_0^{\sum_{p \in P} h_p \delta_{ip}} c_i(f_1^k, \dots, f_{i-1}^k, x, f_{i+1}^k, \dots, f_a^k)\, dx - \tag{27}$$

$$- \sum_{\substack{t \in T_w \\ w \in W}} \int_0^{\sum_{p \in P} h_p \delta^t_{wp}} \lambda^t_w(d_1^{1,k}, \dots, d_{w-1}^{t,k}, y, d_{w+1}^{t,k}, \dots, d_n^{u,k})\, dy$$

where, as usual:

$$f_j^k = \sum_{p \in P} h_p^k \delta_{jp} \quad \text{and} \quad d_w^{t,k} = \sum_{p \in P} h_p^k \delta^t_{wp} .$$

As a matter of fact the p_s component of the gradient $\nabla R(h^k, h)$ is:

$$\frac{\partial R(h^k, h)}{\partial h_{p_s}} = \sum_{i \in L} c_i\Big(f_1^k, \dots, f_{i-1}^k, \sum_{p \in P} h_p \delta_{ip}, f_{i+1}^k, \dots, f_a^k\Big)\delta_{ip_s} - \tag{28}$$

$$- \sum_{\substack{t \in T_w \\ w \in W}} \lambda^t_w\Big(d_1^{1,k}, \dots, d_{w-1}^{t,k}, \sum_{p \in P} h_p \delta^t_{wp}, d_{w+1}^{t,k}, \dots, d_n^{u,k}\Big)\delta^t_{wp_s}$$

and taking into account Eq. (23) and Eq. (24), and that $\delta_{wp_s} = 1$ if $p_s \in P_w^t$, 0 otherwise, we have:

$$\frac{\partial R(h^k, h)}{\partial h_{p_s}} = \overline{C}_{p_s}(h^k, h) - \overline{\Lambda}_w^t(h^k, h) \tag{29}$$

where the pair w is connected on mode t by path p_s.

Thus a solution h^{k+1} and the constraint multipliers of the following nonlinear programming problem:

$$\min[R(h^k, h) : h \in \Omega] \tag{30}$$

are solution and multipliers of VI (26).

We denote by $s_p(h_p) = -h_p \leq 0 \; \forall p \in P$ and

$$G^i(h) = \sum_{p \in P} h_p \delta_{ip} - H_i \leq 0 \quad \forall i \in L$$

the nonnegativity and capacity constraints. Taking into account that the demand constraints are not active in Ω, the Kuhn and Tucker (KT) condition of problem (30) in h^{k+1} is:

$$\nabla R(h^k, h^{k+1}) + \sum_{p \in P} u_p^{k+1} \nabla s_p(h^{k+1}) + \sum_{i \in L} y_i^{k+1} \nabla G^i(h^{k+1}) = 0 \tag{31}$$

where u_p^{k+1} and y_i^{k+1} are the Lagrange multipliers of the nonnegativity and of the capacity constraints, which are unique as a consequence of the supposed independence of $\nabla s_p(h^{k+1})$ and $\nabla G^i(h^{k+1})$.

Consider the following new link cost functions $\forall i \in L$:

$$\begin{aligned} \hat{c}_i(f_i) &= c_i^k(f_i) \quad \text{if } f_i \leq H_i \\ \hat{c}_i(f_i) &= c_i^k(f_i) + \frac{1}{\varepsilon}(f_i - H_i) \quad \varepsilon > 0 \quad \text{if } f_i > H_i \end{aligned} \tag{32}$$

and denote by $\hat{R}(h^k, h)$ the function (27) when the cost functions (32) are used. We have:

$$\hat{R}(h^k, h) = R(h^k, h) + \alpha(h) \tag{33}$$

where

$$\alpha(h) = \sum_{i \in L} \eta_i \int_{H_i}^{\sum_{p \in P} h_p \delta_{ip}} (x - H_i) \frac{1}{\varepsilon} \, dx = \sum_{i \in L} \eta_i \frac{\left[\sum_{p \in P} h_p \delta_{ip} - H_i\right]^2}{2\varepsilon} \tag{34}$$

with $\eta_i = 0$ if $\sum_{p\in P} h_p\delta_{ip} - H_i \le 0$ and $\eta_i = 1$ if $\sum_{p\in P} h_p\delta_{ip} - H_i > 0$, is a *differentiable penalty function.*

In order to calculate both the vectors h^{k+1} and y^{k+1}, we solve the following problem of nonlinear programming:

(35)

$$\min\Big[\hat{R}(h^k, h) : h \in \mathbb{R}^M,\ \sum_{\substack{j\neq z\\ i\in T_{zj}}} \sum_{p\in P'_{zj}} h_p \le N_z\ \forall z\in S,\ s_p(h_p) \le 0\ \forall p \in P\Big].$$

It can be shown (see [1], pp. 366-368) that when $\varepsilon \to 0$ the solution of problem (35) tends to the solution h^{k+1} of problem (30). Let h_ε be the solution of problem (35) for a particular value of ε. If ε is sufficiently small, since $h_\varepsilon \to h^{k+1}$ as $\varepsilon \to 0$, we have that the demand constraints, which are not active in h^{k+1}, are also not active in h_ε; and that $\sum_{p\in P} h_{p\varepsilon}\delta_{ip} - H_i < 0$ for all capacity constraints for which $\sum_{p\in P} h_p^{k+1}\delta_{ip} - H_i < 0$. Thus the KT condition of problem (35) in h_ε for ε sufficiently small can be written:

$$\nabla R(h^k, h_\varepsilon) + \sum_{i\in I} y_{i\varepsilon}\nabla\Big(\sum_{p\in P} h_{p\varepsilon}\delta_{ip} - H_i\Big) + \sum_{p\in P} u_{p\varepsilon}\nabla s_p(h_\varepsilon) = 0 \tag{36}$$

where I is the set of links whose capacity constraints are active in h^{k+1}, $u_{p\varepsilon}\ \forall p \in P$ are the Lagrangian multipliers of the nonnegativity constraints $s_p(h) \le 0$, and

$$y_{i\varepsilon} = \frac{\sum_{p\in P} h_{p\varepsilon}\delta_{ip} - H_i}{\varepsilon}. \tag{37}$$

By noting that $G^i(h) = \sum_{p\in P} h_p\delta_{ip} - H_i$, Eq. (36) can be written:

$$\nabla R(h^k, h_\varepsilon) + \sum_{i\in I} y_{i\varepsilon}\nabla G^i(h_\varepsilon) + \sum_{p\in P} u_{p\varepsilon}\nabla s_p(h_\varepsilon) = 0. \tag{38}$$

We observe that R, $G^i\ \forall i \in L$ and $s_p\ \forall p \in P$ are continuously differentiable and that $h_\varepsilon \to h^{k+1}$ as $\varepsilon \to 0$. Since there exist unique Lagrange multipliers $y_i^{k+1} \ge 0\ \forall i \in I$, $y_i^{k+1} = 0\ \forall i \notin I$ and $u_p^{k+1} \ge 0\ \forall p \in P$ which solve Eq. (31), by comparing Eq. (31) with Eq. (38) as $\varepsilon \to 0$ we have that $\lim_{\varepsilon\to 0} y_{i\varepsilon} = y_i^{k+1}$ $\forall i \in I$ and $\lim_{\varepsilon\to 0} u_{p\varepsilon} = u_p^{k+1}\ \forall p \in P$.

Thus the solution of problem (35) for $\varepsilon \to 0$ gives the solution of problem (30), and at the same time the values of Lagrange multipliers y_i^{k+1}. The solution of problem (35) can be made arbitrarily close to the solution of problem (30) by choosing ε sufficiently small. However high computational difficulties can arise if we use a very small ε value. For this reason the popular approach to the solution of problems that use penalty functions employs a sequence of decreasing parameters. With each new value of ε a problem (35) is solved, starting with the solution corresponding to the previously chosen parameter value.

Suppose that the sequence $\{h^k\}$ obtained in the successive steps of the calculation procedure is convergent and $\lim_{k\to\infty} h^k = \overline{h}$.

Since $\overline{C}(h^k, h^{k+1})$ and $\overline{\Lambda}(h^k, h^{k+1})$ are continuous, we have:

$$
\begin{aligned}
&\lim_{k\to\infty} \overline{C}(h^k, h^{k+1}) = \overline{C}(\overline{h}, \overline{h}) = C(\overline{h}) \\
&\lim_{k\to\infty} \overline{\Lambda}(h^k, h^{k+1}) = \overline{\Lambda}(\overline{h}, \overline{h}) = \Lambda(\overline{h})
\end{aligned} \tag{39}
$$

and from VI (26):

$$
\begin{aligned}
\lim_{k\to\infty} &\left[\overline{C}(h^k, h^{k+1}) - \overline{\Lambda}(h^k, h^{k+1})\right](h - h^{k+1}) = \\
&= \left[C(\overline{h}) - \Lambda(\overline{h})\right](h - \overline{h}) \geq 0
\end{aligned} \tag{40}
$$

so that $\overline{h}$ is a solution of VI (10).

We observe that $\lim_{k\to\infty} \nabla R(h^k, h^{k+1}) = C(\overline{h}) - \Lambda(\overline{h})$ and that both G^i $\forall i \in L$ and s_p $\forall p \in P$ are continuously differentiable. Since there exists unique multipliers that, given $\overline{h}$, solve Eq. (14), by comparing Eq. (14) with Eq. (31) as $k \to \infty$, we have:

$$
u_p = \lim_{k\to\infty} u_p^k \quad \forall p \in P \qquad y_i = \lim_{k\to\infty} y_i^k \quad \forall i \in L\,. \tag{41}
$$

Thus the equilibrium flow vector and the corresponding additional costs $y_i \forall i \in L$ can be obtained by solving a succession of nonlinear programming problems (35). In general the solution of problems (35) is computed in terms of vectors of link flows f and of demand d instead of path flows, in order to avoid the numeration of paths. If we consider that $\sum_{p\in P} h_p\delta_{ip} = f_i$ and $\sum_{p\in P} h_p\delta^t_{wp} = d^t_w$, Eq. (27) can be written as a function of f and d:

$$
R(h^k, h) = S(f, d) = \sum_{i\in L} \int_0^{f_i} c_i(f_1^k, \ldots, f_{i-1}^k, x, f_{i+1}^k, \ldots f_a^k)\, dx -
$$

$$-\sum_{\substack{t\in T_w\\ w\in W}} \int_0^{d_w^t} \lambda_w^t(d_1^k, \ldots, d_{w-1}^{t,k}, y, d_{w+1}^{t,k}, \ldots, d_n^{u,k})\, dy$$

and Eq. (34) can be written as a function of f:

$$\alpha(h) = \gamma(f) = \sum_{i\in L} \eta_i \, \frac{f_i - H_i}{\varepsilon} \, .$$

If we use the variables f and d instead of h, and denote $\hat{S}(f, d) = S(f, d) + \gamma(f)$, the problem (35) is transformed into the following:

$$\min \left[\hat{S}(f, d) : f_i = \sum_{p\in P} h_p \delta_{ip} \quad \forall i \in L, \tag{42}$$

$$d_w^t = \sum_{p\in P} h_p \delta_{wp} \quad \forall t \in T_w \, \forall w \in W, \quad h_p \geq 0 \quad \forall p \in P, \quad \sum_{\substack{t\in T_{zj}\\ j\neq z}} d_{zj}^t \leq N_z \right]$$

which can be solved by the usual algorithms of traffic assignment to networks with elastic demand (see [12], Chapter 6): in such a way we can calculate both the vectors f^{k+1} and d^{k+1}, and the multipliers $y_i^{k+1} = \lim_{\varepsilon\to 0} \frac{f_i^{k+1} - H_i}{\varepsilon} \; \forall i \in L$. It is worth noting that $\hat{S}(f, d)$ is strictly convex, so that the solution (f^{k+1}, d^{k+1}) of problem (42) and the multipliers y_i^{k+1} $\forall i \in L$ are unique. However the uniqueness of f^k, d^k does not ensure the uniqueness of the sequence $\{f^k, d^k\}$, because the latter depends on its starting point. If we use the variables (f, d), VI (10) is transformed into the following [2]:

$$c(\overline{f})'(f - \overline{f}) - \lambda(\overline{d})'(d - \overline{d}) \geq 0 \, . \tag{43}$$

The convergence point of the various sequences $\{f^k, d^k\}$ obtained using different starting points is unique if the solution of VI (43) is unique; in this case also the corresponding multipliers $y_i \forall i \in L$ are unique. The uniqueness of the solution of VI (43) depends on characteristics of its operator $c(f) - \lambda(d)$ [6]; however the uniqueness of the solutions of VI (43) is verified in most actual urban transport networks [10].

It is worth noting that, even if VI (10) has various solutions, only that solution whose capacity constraint multipliers are used as additional costs becomes the equilibrium solution. Thus the imposition of these additional costs ensures the uniqueness of the equilibrium solution in any case.

The iterative procedure can be stopped when the distance $\|f^{k+1} - f^k\|$ is less than a fraction ξ of the length $\|f^k\|$ of f^k:

$$\|f^{k+1} - f^k\| < \xi \, \|f^k\|$$

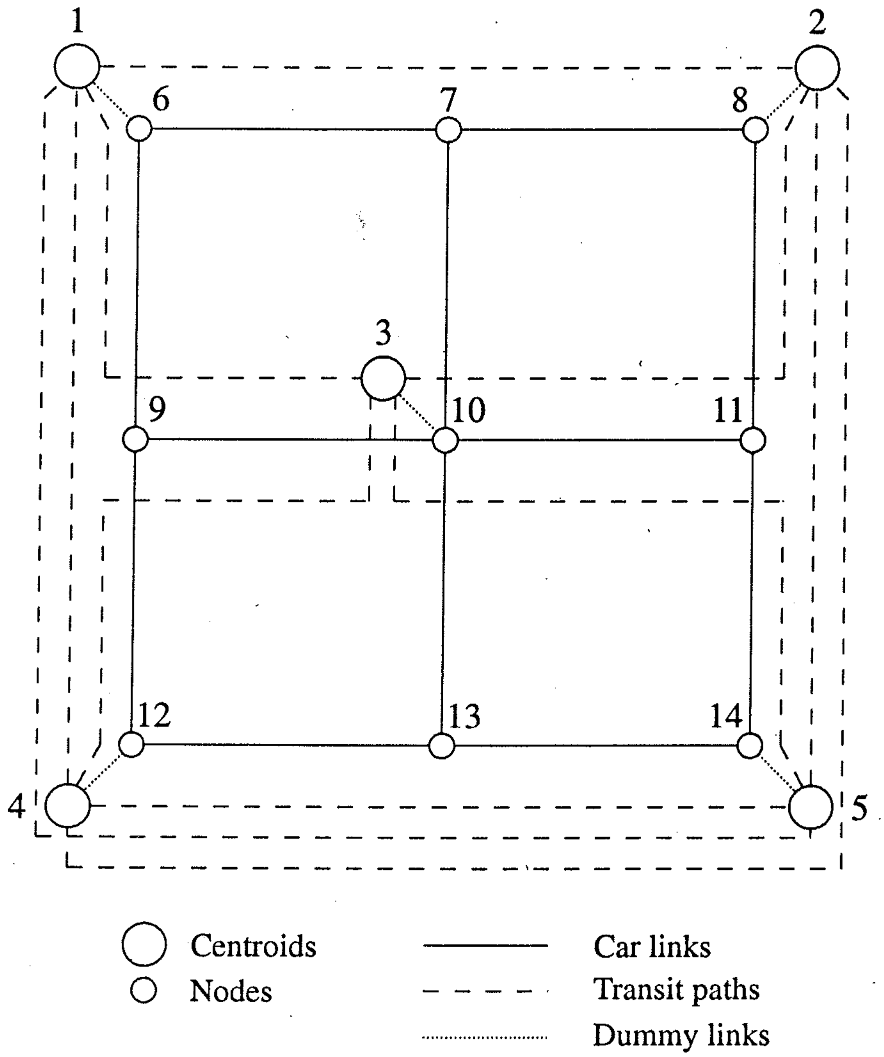

Fig. 1. Bimodal network considered in the computational example

If ξ is sufficiently small, the vectors f^{k+1} and y^{k+1} so obtained are good estimates of $\overline{f}$ and y.

4. A computational example.

The method illustrated in the previous Section has been applied to the small network reported in Fig. 1, which is travelled by two transport modes: car and transit. Nodes $1 \ldots 5$ are centroids: O/D flows depart from them and arrive at them, but cannot go through them. Only one transit path joins each pair of centroids; every centroid is connected by a dummy link to car links. Some roads are travelled by transit along with cars, so that the transport cost of transit depends on car flows, whereas it is supposed that the car link costs do not depend on transit flows.

Transport cost coincides with journey time. The time on dummy links is 0.1 minutes; the time of every car link depends only on the flow f_i on it and is given by:

$$t_i = a_i + b_i \left[\frac{f_i}{1000} \right]^4 \tag{44}$$

where parameters a_i and b_i are reported in Tab. 1 for every car link i.

Link	a_i	b_i	EC_i
6 - 7	6.00	7.26	1200
6 - 9	4.00	2.94	4000
7 - 6	6.00	7.26	1200
7 - 8	8.00	7.04	1300
7 - 10	6.00	2.94	1200
8 - 7	8.00	7.04	1300
8 - 11	7.00	2.10	1300
9 - 6	4.00	2.94	4000
9 - 10	9.00	3.42	1300
9 - 12	6.00	1.44	1300
10 - 7	6.00	2.94	1200
10 - 9	9.00	3.42	1300
10 - 11	10.00	3.00	1000
10 - 13	7.00	1.33	1400
11 - 8	7.00	2.10	1300
11 - 10	10.00	3.00	1000
11 - 14	8.00	1.92	1400
12 - 9	6.00	1.44	1300
12 - 13	9.00	5.94	1200
13 - 10	7.00	1.33	1400
13 - 12	9.00	5.94	1200
13 - 14	8.00	2.40	1200
14 - 11	8.00	1.92	1400
14 - 13	8.00	2.40	1200

Tab. 1. Parameters of cost functions and capacity values EC_i (Pph) on car links of the network in Fig. 1

The journey time on transit is independent of transit flows, and is given by the line-haul time and of the access and waiting time t_a which is equal to 10 minutes for all pairs w. The line-haul time t_w^b for each pair w is given by the sum of the line-haul time when flows on car links are zero, which is reported in Tab. 2, and of congestion time $b_i \left[\frac{f_i}{1000}\right]^4$ due to car flows.

O/D	1	2	3	4	5
1	–	29	12	28	27
2	29	–	17	32	32
3	12	17	–	16	15
4	28	32	16	–	31
5	27	32	15	31	–

Tab. 2. Transit line-haul times (min) between the O/D of the network in Fig. 1 when flows on car links are zero

Flows on both car and transit links are measured in passengers per hour (Pph). Capacity constraints express the condition that flow f_i on every car link has to be less that the EC_i value reported in Tab. 1

Since the transit costs depend on flows on car links, the equilibrium flows on network and the additional costs were obtained by the iterative procedure shown in the previous Section, solving a succession of nonlinear programming problems. At the k step the function of demand for car travel $d_w^{c,k}$ is obtained from Eq. (1), which in the case under examination becomes:

$$d_w^{c,k} = d_w \frac{\exp\left[-\beta t_w^c\right]}{\exp\left[-\beta t_w^c\right] + \exp\left[-\gamma t_a - \beta t_w^{b,k}\right]} \tag{45}$$

where t_w^c is the journey time by car for pair w, $t_w^{b,k}$ is the transit line-haul time depending on flows on car links obtained in the $k-1$ step, and coefficients β and γ are equal to 0.1 and 0.15 respectively.

Each nonlinear programming problem under examination is equivalent to that obtained if one considers only the car network with elastic demand given by (45): flow on the transit link between the pair w is $d_w^b = d_w - d_w^c$. In this case the demand d can be assigned to the bimodal network by the usual Frank-Wolfe algorithm, if one attributes to each transit path j the following dummy cost function (see [12], pp. 155-157):

$$t_j^k = \frac{1}{\beta}\left[ln \frac{d_w^b}{d_w - d_w^b} + \beta t_w^{b,k} + \gamma t_a\right]. \tag{46}$$

A cost function $\hat{t}_i(f_i)$ modified following (32) was attributed to every car link:

$$\hat{t}_i = a_i + b_i \left[\frac{f_i}{1000}\right]^4 \quad \text{if } f_i \leq EC_i$$

(47)

$$\hat{t}_i = a_i + b_i \left[\frac{f_i}{1000}\right]^4 + \frac{1}{\varepsilon}(f_i - EC_i),\ \varepsilon > 0 \quad \text{if } f_i > EC_i\ .$$

The Frank-Wolfe assignment procedure was repeated 20 times in each k step, starting with $\varepsilon = 100$ and taking in each successive assignment half the ε value of the preceding one. The equilibrium flow vector obtained in each assignment was assumed as starting point in the successive one.

k	2	3	4	5	6	7	8	9	10
ξ	0.0385	0.0302	0.0104	0.0035	0.0030	0.0038	0.0036	0.0041	0.0031
η	0.3695	0.0902	0.1098	0.0710	0.0950	0.1033	0.0479	0.0581	0.0354

Tab. 3. Values of the convergence gaps ξ and η at various steps k of the equilibrium computation for the network in Fig. 1

Ten steps of nonlinear programming problems were solved, in order to verify the convergence of the sequence $\{f^k\}$ and of the sequence $\{y^k\}$ of the capacity constraints multipliers. The values of $\xi = \dfrac{\|f^{k+1} - f^k\|}{\|f^k\|}$ and of $\eta = \dfrac{\|y^{k+1} - y^k\|}{\|y^k\|}$ obtained for different $k+1$ values when $\varepsilon = 0.00019$, are reported in Tab. 3. It can be noted that the convergence is very quick; the fluctuations of the gap η are essentially due to the difficulties in computing the exact values of $y_{i\varepsilon}$ when ε is very close to zero. The flows $f_{i\varepsilon}$ on car links and the estimates $y_{i\varepsilon}$ of multipliers obtained for different values of ε during the step $k = 6$ are reported in Tab. 4. It can be noted the progressive approach of link flows $f_{i\varepsilon}$ to capacities EC_i as ε approaches 0.

Link i	$\varepsilon = 100.0$		$\varepsilon = 6.25$		$\varepsilon = 0.195$		$\varepsilon = 0.0061$		$\varepsilon = 0.00019$	
	$f_{i\varepsilon}$	$y_{i\varepsilon}$	$f_{i\varepsilon}$	$y_{i\varepsilon}$	$f_{i\varepsilon}$	$y_{i\varepsilon}$	$f_{i\varepsilon}$	$y_{i\varepsilon}$	$f_{i\varepsilon}$	$y_{i\varepsilon}$
6-7	1374.02	1.740	1260.80	9.728	1202.66	13.610	1200.08	12.498	1200.00	13.586
6-9	1659.48	0.000	1613.24	0.000	1614.92	0.000	1614.51	0.000	1614.51	0.000
7-6	1348.18	1.482	1245.83	7.334	1201.62	8.284	1200.05	8.020	1200.00	6.798
7-8	1357.75	0.577	1323.87	3.819	1300.85	4.333	1300.03	4.617	1300.00	5.127
7-10	1159.90	0.000	1106.95	0.000	1045.91	0.000	1043.52	0.000	1043.44	0.000
8-7	1443.86	1.439	1367.17	10.747	1303.05	15.648	1300.10	16.030	1300.00	13.925
8-11	1826.75	5.267	1440.95	22.552	1305.39	27.611	1300.17	27.222	1300.00	26.617
9-6	1602.68	0.000	1562.32	0.000	1551.36	0.000	1551.99	0.000	1551.97	0.000
9-10	1367.76	0.678	1308.94	1.430	1289.61	0.000	1289.60	0.000	1289.61	0.000
9-12	1812.95	5.130	1414.94	18.390	1304.48	22.927	1300.12	20.340	1300.00	19.532
10-7	1047.93	0.000	1048.68	0.000	1042.66	0.000	1043.42	0.000	1043.44	0.000
10-9	1245.17	0.000	1227.30	0.000	1225.57	0.000	1227.06	0.000	1227.07	0.000
10-11	997.83	0.000	968.96	0.000	987.05	0.000	986.67	0.000	986.67	0.000
10-13	1168.31	0.000	1119.71	0.000	1114.66	0.000	1113.75	0.000	1113.71	0.000
11-8	1708.67	4.087	1398.87	15.819	1304.04	15.555	1300.10	16.026	1300.00	16.291
11-10	1078.85	0.789	1011.73	1.879	989.75	0.000	986.74	0.000	986.67	0.000
11-14	1645.01	2.450	1455.23	8.837	1402.17	11.131	1400.07	11.529	1400.00	9.779
12-9	1878.73	5.787	1445.65	23.305	1304.96	25.384	1300.15	24.962	1300.00	24.743
12-13	1459.43	2.594	1304.85	16.777	1204.62	23.632	1200.13	22.103	1200.00	21.581
13-10	1247.20	0.000	1158.64	0.000	1117.04	0.000	1113.79	0.000	1113.71	0.000
13-12	1390.41	1.904	1267.89	10.862	1202.75	14.088	1200.10	16.527	1200.00	14.132
13-14	1592.26	3.923	1295.01	15.202	1203.07	15.725	1200.10	16.197	1200.00	14.772
14-11	1607.95	2.080	1455.92	8.947	1402.51	12.873	1400.07	12.138	1400.00	10.599
14-13	1602.13	4.021	1296.98	15.518	1203.58	18.350	1200.11	17.574	1200.00	20.048

Tab. 4. Flows $f_{i\varepsilon}$ (Pph) and estimates $y_{i\varepsilon}$ of the capacity constraints multipliers for the network in Fig. 1 corresponding to various values of parameter ε at the 6th step of the equilibrium computation

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