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A REMARK ON INFINITE-DIMENSIONAL VARIATIONAL INEQUALITIES

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An infinite-dimensional Quasi-Variational Inequality is considered, whose domain is expressed as intersection of the level sets of functionals having infinite-dimensional image. A proposal is made to handle it by means of Lagrange multipliers.

Assume we are given the positive integers m and n, the interval $T := [t_0, t_1] \subset \mathbb{R}$ with $t_0 < t_1$, the real Hilbert space Ξ whose elements are $x : T \to \mathbb{R}^n$, the multifunction $X : \Xi \rightrightarrows \Xi$, the functions $A : \Xi \to \Xi$ and

$$\phi_i : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}, \quad i \in \mathcal{I} := \{1, \dots, m\}.$$

Let p be an integer such that $0 \le p \le m$ and consider the sets $\mathcal{I}^0 := \{1, \dots, p\}$, $\mathcal{I}^+ := \{p+1, \dots, m\}$ with the stipulation that $\mathcal{I}^0 = \emptyset$ if $p = 0$ and $\mathcal{I}^+ = \emptyset$ if $p = m$. Let $\tau_0, \tau_1 \in \mathbb{R}$ with $\tau_0 < \tau_1$ be given, and $\forall y \in \Xi$ introduce the set:

$$K(y) := \Big\{ x \in X(y) \cap C^2(T)^n : \\ \int_{\tau_0}^{\tau_1} \phi_i(y(\tau); t, x(t), x'(t))\, d\tau \begin{bmatrix} = 0, & \text{if} & i \in \mathcal{I}^0 \\ \ge 0, & \text{if} & i \in \mathcal{I}^+ \end{bmatrix}, \quad \forall t \in T \Big\},$$

where $x(t) := (x_i(t), i \in \mathcal{I})$, $x'(t) := \left(\frac{dx_i(t)}{dt}, \ i \in \mathcal{I} \right)$, and $C^2(T)^n$ denotes the n times Cartesian product of the set of real-valued functions on T having the first two derivatives continuous.

Consider the following Quasi-Variational Inequality (in short, QVI): to find $y \in K(y)$ such that:

$$\int_{t_0}^{t_1} \langle A(y), x(t) - y(t) \rangle \, dt \geq 0, \quad \forall x \in K(y), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in Ξ.

If we set $\mathcal{C} := O_p \times \mathbb{R}_+^{m-p}$ and consider, $\forall y \in \Xi$, the multifunctions: $\psi_i : \Xi \rightrightarrows \Xi$, with

$$\psi_i(y; x, x') := \left\{ \int_{\tau_0}^{\tau_1} \phi_i(y(\tau); t, x(t), x'(t)) \, d\tau : t \in T \right\} \quad i \in \mathcal{I},$$

where y plays the role of parameter, then we can write

$$K(y) = \left\{ x \in X(y) \cap C^2(T)^n : \psi_i(y; t, x(t), x'(t)) \subseteq \mathcal{C}, \quad \forall t \in T \right\},$$

where $\psi := (\psi_i, i \in \mathcal{I})$; this shows a more general form for $K(y)$, $\mathcal{C}$ being a closed and convex cone with apex at the origin and g a suitable multifunction.

When $X(y)$ and $\psi(y; x, x')$ are independent of y (in this case they will be denoted by X and $\psi(x)$, respectively), then (1) collapses to a Variational Inequality (in short, VI). If, moreover, $K(y) \equiv X(y)$ and, $\forall y$, $K(y)$ is a closed and convex cone with apex at the origin, then (1) collapses to a Quasi-Complementarity System (in a Hilbert space).

We want to give some hints on how to extend to (1) the image space approach defined for constrained extremum problems [4] and for VI having a finite-dimensional image [5], [6].

For each y, ψ_i is a multifunction which sends an element of $X(y)$, i.e. a function $x(t)$, $t \in T$, onto a subset of $\mathbb{R}$, i.e. the image of T through the function of t (x being fixed) $\tilde{\psi}_i(y; t) := \psi_i(y; t, x(t), x'(t))$. Thus, $\forall y$, we consider the multifunction $F : X(y) \rightrightarrows \mathbb{R}^{1+m}$ defined by

$$F(y; x) := \left\{ (u, v) \in \mathbb{R} \times \mathbb{R}^m : \right.$$
$$\left. u = \int_{t_0}^{t_1} \langle A(y), y - x \rangle \, dt; \quad v = \psi(y; t, x, x'), t \in T \right\}.$$

For each y, the image of (1) (see [5], [6]) is the set $F(y; X(y))$ which is a subset of a finite-dimensional space, like it happens when ψ is a function. Thus, the approach adopted in [5], [6], which is heavily based on the finite-dimensionality of the image of the QVI, seems to be useless here. A natural attempt of overcoming the difficulty of handling the infinite-dimensionality of $K(y)$ should be an extension of such an approach. In this order of ideas, instead of accepting the infinite-dimensionality as soon as it appears, we suggest the viewpoint of postponing it as long as possible. This seems possible by introducing a multifunction approach, which has been suggested in [4] for constrained extremum problems, it allows us to circumvent the infinite-dimensionality and to reduce ourselves to handle finite-dimensional sets in order to study the image of (1).

We start with the obvious remark that $y \in X(y)$ is a solution of (1) iff the system (in the unknown x):

$$\int_{t_0}^{t_1} \langle A(y), y - x \rangle \, dt > 0; \quad \psi(y; t, x, x') \subseteq C; \; x \in X(y) \tag{2}$$

is impossible. For each y, consider the functions $\omega_i : X(y) \times T \times X(y) \to \mathbb{R}$, $i \in \mathcal{I}$, and denote by $\Omega(y)$ the set vectors $\omega(y) := (\omega_i(y),\ i \in \mathcal{I})$, whose elements are not identically zero on T and such that $\omega_i(y) \geq 0$, $i \in \mathcal{I}^+$. Consider the vector function $\Phi : X(y) \times \mathbb{R}^{1+m} \times \Omega(y) \to \mathbb{R}^{1+m}$, defined, $\forall x \in X(y)$, by:

$$\Phi(y; F(y; x); \omega(y)) :=$$
$$\left(\int_{t_0}^{t_1} \langle A(y), y - x \rangle \, dt \,,\ \int_{t_0}^{t_i} \omega_i(y; t, x) \psi_i(y; t, x, x') \, dt,\ i \in \mathcal{I} \right);$$

and set

$$\mathcal{H} := \left\{ (u, v_1, \dots, v_m) \in \mathbb{R}^{1+m} : u > 0;\ v_i = 0,\ i \in \mathcal{I}^0;\ v_i \geq 0;\ i \in \mathcal{I}^+ \right\}.$$

We ask Φ to be such that, $\forall y \in X(y)$,

$$F(y; x) \subseteq \mathcal{H} \iff \Phi(y; F(y; x); \omega(y)) \in \mathcal{H},\ \forall \omega(y) \in \Omega(y). \tag{3}$$

When this happens, Φ can be called generalized selection function (in short, GSF) and ω selection multiplier (in short, SM); indeed, Φ selects an element from the set $F(y; x)$. The relation (3) holds under assumptions like those which make valid the so-called Fundamental Lemma of Calculus of Variations; it would be useful to find general conditions on (1) under which (3) holds.

The main purpose here consists in suggesting a way for extending to (1) the separation scheme in the image space introduced in [5], [6]. Such a scheme cannot be applied to (2): y being a solution of (1), $F(y; X(y))$ and $\mathcal{H}$ are not necessarily disjoint (unlike what happens in [5], [6] where ψ is a function), so that they cannot be separated, even nonlinearly. When (3) holds, $\forall y$, the set $F(y; X(y))$ can be equivalently (in the sense of preserving the impossibility of (2)) replaced with an element (of its convex hull), which plays the role of "representative" of the set, namely Φ; this justifies the connection with the theory of selections for multifunctions.

Thus, assuming that (3) holds, we can introduce the selected image of (1) as the set:

$$\mathcal{K}(y; \omega) := \Phi(y; F(y; X(y)); \omega(y)), \quad y \in X(y), \ \omega(y) \in \Omega(y).$$

Thus, obviously, y is a solution of (1) iff there exists a SM $\omega(y)$, such that

$$\mathcal{H} \cap \mathcal{K}(y; \omega(y)) = \emptyset. \tag{4}$$

Now, a separation scheme can be set up to prove (4) as a consequence of separations of $\mathcal{H}$ and $\mathcal{K}$. Hence, it is conceivable to try to transfer here all the topics, which have been discussed in [5], [6] for the case where ψ is a function (as the definition of gap functions, duality, penalty, etc.), even if this does not seem trivial.

Recently, it has been shown that a VI or a QVI can be useful in the study of equilibrium flows in a network (see [2], [3]). The VI and QVI, which have been studied in this context, are independent of time. They do not necessarily reflect a static situation, in the sense that they may mirror the average behaviour of flows in a given time interval. In spite of this simplification, such models have represented a great improvement with respect to the optimization ones. Due to this fact it would be important to introduce explicitly the time in such models. Among several advantages, this should contribute to achieve a better definition of equilibrium flows in a network; indeed a solution of a static VI or QVI cannot represent in a fully satisfactory way the real notion of equilibrium flows in a network, whose nature is essentially dynamic. The above remarks might suggest a way for studying dynamic models.

At least in the field of equilibrium flows on a network it would be useful to introduce a time-lag in the models. For instance, instead of the functional which appears in (1), it might be interesting to consider one of type:

$$\int_{t_0}^{t_1} \langle A(y(t-\Delta)), x(t) - y(t-\Delta) \rangle \, dt, \tag{5}$$

where Δ may or may not dependent on t, and represents a delay with which the users of a network react to what happens. The extrema of integration in (5) may also contain a time-lag not necessarily equal to Δ.

Another generalization of (1) is that where the operator A depends also on the gradient of y; in this case (1) is replaced with

$$\int_{t_0}^{t_1} \langle A(y, y'), x - y\rangle \, dt. \tag{6}$$

If there exists a function f such that

$$A(y, y') = f'_y - \frac{d}{dt} f'_{y'},$$

and $K(y)$ is independent of y and is an open subset of Ξ, then (1) is VI which leads to the classic Euler equation.

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