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APPROXIMATE SOLUTIONS TO VARIATIONAL INEQUALITIES AND APPLICATIONS

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The aim of the paper is to investigate two concepts of approximate solutions to parametric variational inequalities in topological vector spaces for which the corresponding solution map is closed graph and/or lower semicontinuous (upper and/or lower convergent when considered under perturbations) and to apply the results to the stability of optimization problems with variational inequality constraints.

1. Introduction.

Let (U, τ) be a topological space, (V, σ) be a topological vector space with (V^*, σ^*) as dual space and F be a function from $U \times V$ to V^*. We consider the following parametric variational inequality:

(VI_x) Find $\overline{y} \in \Gamma$ such that $\langle F(x, \overline{y}), \overline{y} - z\rangle \leq 0$ for any $z \in \Gamma$

where Γ is a closed convex subset of V and $\langle \cdot, \cdot \rangle$ denotes the pairing between V and V^*. The pair (σ, σ^*) will be supposed to be such that:

$$\langle u, v\rangle \leq \liminf_{n\to+\infty} \langle u_n, v_n\rangle \tag{1}$$

for any sequence $\{u_n\}_n$ σ-converging to u and any sequence $\{v_n\}_n$ σ^*-converging to v.

If $F(x, \cdot)$ is a monotone hemicontinuous operator it is well known ([11]) that the problem (VI_x) is equivalent to the following one:

$(\widehat{\mathrm{VI}}_x)$ Find $\overline{y} \in \Gamma$ such that : $\langle F(x, z), \overline{y} - z \rangle \leq 0$ for any $z \in \Gamma$.

Let T and $\widehat{T}$ be the multifunctions from U to V defined respectively by the solutions to the problems (VI_x) and $(\widehat{\mathrm{VI}}_x)$ that is:

$$T(x) = \{\overline{y} \in V : \overline{y} \text{ solves } (\mathrm{VI}_x)\}$$
$$\widehat{T}(x) = \{\overline{y} \in V : \overline{y} \text{ solves } (\widehat{\mathrm{VI}}_x)\}.$$

While it is easy to establish conditions under which T or $\widehat{T}$ are sequentially closed graph (see Definition 2.1), in general T and $\widehat{T}$ are not sequentially lower semicontinuous (Definition 2.2 and Remark 2.3). Moreover, if $\{F_n\}_n$ is a sequence of functions from $U \times V$ to V^*, $(\mathrm{VI}_x)_n$ and $(\widehat{\mathrm{VI}}_x)_n$ are the corresponding perturbed variational inequalities and T_n (respectively $\widehat{T}_n$) are the corresponding solution maps, while it is easy to establish conditions under which $\{T_n\}_n$ or $\{\widehat{T}_n\}_n$ are sequentially upperconvergent (Definition 3.2) in general $\{T_n\}_n$ and $\{\widehat{T}_n\}_n$ are not sequentially lowerconvergent (Definition 3.3 and Remark 3.1). However, in dealing with multilevel optimization problems with variational inequality constraints, the sequential lowerconvergence of the sequence of multifunctions defined by the constraints is a crucial property in order to obtain convergence results for the solutions to the perturbed problems. The aim of the paper is to exhibit a concept of approximate solutions for which the corresponding solution map is sequentially lower semicontinuous (lowerconvergent when considered under perturbations) and to apply the results to the stability of optimization problems with variational inequality constraints ([12]).

More precisely in Section 2 we consider two concepts of approximate solutions for (VI_x) and $(\widehat{\mathrm{VI}}_x)$ and we study the continuity properties of the corresponding solution maps. In Section 3 we consider the perturbed problems and we study the properties of upper and lowerconvergence of the sequence of the approximate solution maps. Finally, in Section 4, we apply the previous results to optimization problems with variational inequality constraints under perturbations finding conditions which guarantee convergence of the approximate solutions and /or convergence of the approximate values.

For the sake of simplicity we present the results in a sequential setting but we wish to point out that analogous results can be obtained in a topological setting (by using nets instead of sequences) so during the paper we shall omit the term “sequentially”.

2. Approximate solutions for variational inequalities.

For $\varepsilon > 0$ we consider the following problem for any $x \in U$:

$(\mathrm{VI}_x)(\varepsilon)$ Find $\overline{y} \in \Gamma$ such that : $\langle F(x, \overline{y}, \overline{y} - z\rangle \leq \varepsilon$ for any $z \in \Gamma$

$(\widehat{\mathrm{VI}}_x)(\varepsilon)$ Find $\overline{y} \in \Gamma$ such that : $\langle F(x, z), \overline{y} - z\rangle \leq \varepsilon$ for any $z \in \Gamma$

and the corresponding solutions maps defined by:

$$T(x, \varepsilon) = \{\overline{y} \in V : \overline{y} \text{ solves } (\mathrm{VI}_x)(\varepsilon)\}$$
$$\widehat{T}(x, \varepsilon) = \left\{\overline{y} \in V : \overline{y} \text{ solves } (\widehat{\mathrm{VI}}_x)(\varepsilon)\right\}.$$

We have:

$$T(x, 0) = T(x)$$
$$\widehat{T}(x, 0) = \widehat{T}(x).$$

Remark 2.1. Let $x \in U$. If $F(x, \cdot)$ is a monotone operator, that is

$$\langle F(x, y), y - z\rangle \geq \langle F(x, z), y - z\rangle$$

for any $y \in V$ and any $z \in V$, then we have $T(x, \varepsilon) \subset \widehat{T}(x, \varepsilon)$ for any $\varepsilon \geq 0$. However, differently from the case $\varepsilon = 0$, in general the opposite inclusion does not hold, for $\varepsilon > 0$, as it is shown by the following trivial example:
Let $U = V = \mathbb{R}$, $\Gamma = [1/2, 2]$ and $F(x, y) = y^2$.
Then $\widehat{T}(x, \varepsilon) = [1/2, 2\varepsilon + 1/2]$ and $T(x, \varepsilon) = \left[1/2, 1/4 + 1/2\sqrt{1/4 + 4\,\varepsilon}\,\right]$.

Remark 2.2. $T(x) \subset T(x, \varepsilon)$ for any $\varepsilon > 0$ and, for a large class of problems, $T(x)$ is nonempty (see, for example [5], [1]). In the following we will consider the real numbers $\varepsilon \geq 0$ for which $T(x, \varepsilon)$ is nonempty.

Now, when the variational inequality (VI_x) derives from an optimization problem, let us study the connection between the approximate variational inequality solution and the ε-solutions to the optimization problem. For the sake of simplicity we consider non parametric problems.

Let f be a real valued function on V, bounded from below. For $\varepsilon \geq 0$ we denote by $M(\varepsilon)$ the set of ε-solutions to the minimum problem:

$$\text{(P)} \qquad \min_{y \in \Gamma} f(y)$$

that is:

$$M(\varepsilon) = \left\{y_\varepsilon \in \Gamma \text{ such that } f(y_\varepsilon) \leq \inf_{y \in \Gamma} f(y) + \varepsilon\right\}.$$

Proposition 2.1. *Let f be a convex function on V and $f'(y, h)$ be the directional derivative of f at $y \in V$ in the direction $h \in V$. If we consider the variational inequality defined by the problem* (P)*:*

(VI) Find $\overline{y} \in \Gamma$ such that $f'(\overline{y}, \overline{y} - z) \leq 0$ for any $z \in V$

then

$$T(\varepsilon) \subset M(\varepsilon) \subset \widehat{T}(\varepsilon)$$

for any $\varepsilon \geq 0$, where

$$T(\varepsilon) = \left\{\overline{y} \in \Gamma \text{ such that } f'(\overline{y}, \overline{y} - z) \leq \varepsilon \text{ for any } z \in \Gamma\right\}$$

and

$$\widehat{T}(\varepsilon) = \left\{\overline{y} \in \Gamma \text{ such that } f'(z, \overline{y} - z) \leq \varepsilon \text{ for any } z \in \Gamma\right\}.$$

Proof. Let $\overline{y} \in T(\varepsilon)$. From the convexity assumption we have:

$$f(\overline{y}) - f(z) \leq f'(\overline{y}, \overline{y} - z)$$

then: $f(\overline{y}) \leq f(z) + \varepsilon$ for any $z \in \Gamma$.
Let $\overline{y} \in M(\varepsilon)$. From the convexity assumption we have:

$$f'(z, \overline{y} - z) \leq f(\overline{y}) - f(z)$$

for any $z \in \Gamma$, then $f'(z, \overline{y} - z) \leq \varepsilon$ for any $z \in \Gamma$.

Now we study the continuity properties of the multifunctions $T(\cdot, \varepsilon)$ and $\widehat{T}(\cdot, \varepsilon)$ defined respectively by $T(x, \varepsilon)$ and $\widehat{T}(x, \varepsilon)$. More precisely we look for conditions ensuring that these multifunctions are closed graph and/or lower semicontinuous in the sense of the following definitions:

Definition 2.1. *A multifunction M from U to V is closed graph at $x_0 \in U$ whenever for any sequence $\{x_n\}_n$ converging to x_0 in U we have:*

$$\limsup_{n \to +\infty} M(x_n) \subset M(x)$$

where

$$\limsup_{n \to +\infty} M(x_n) = \Big\{y \in V : \text{ there exists a sequence } \{y_k\}_k \text{ converging to } y \text{ in } V \text{ such that } y_k \in M(x_{n_k}) \text{ for a subsequence } \{n_k\}\Big\}.$$

Definition 2.2. *A multifunction M from U to V is nearly lower semicontinuous at $x_0 \in U$ ([7]) whenever for any sequence $\{x_n\}_n$ converging to x_0 in U we have:*

$$M(x) \subset \overline{\liminf_{n\to+\infty} M(x_n)}^{\,\text{seq}}$$

where

$$\liminf_{n\to+\infty} A_n = \{y \in V : \text{ there exists a sequence } \{y_n\}_n \text{ converging to } y \text{ in } V \text{ such that } y_n \in A_n \text{ for } n \text{ large}\}$$

and $\overline{A}^{\,\text{seq}}$ is the sequential closure of A.

Remark 2.3. If (U, σ) and (V, τ) are first countable topological spaces the previous definition is equivalent to M is lower semicontinuous ([7])

Definition 2.3. *A multifunction M from U to V is continuous at $x_0 \in U$ if it is lower semicontinuous and closed graph at $x_0 \in U$.*

Now, it is easy to prove the following theorem:

Proposition 2.2. *Let $\varepsilon \geq 0$. The multifunction $T(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ if the function F is continuous on $\{x_0\} \times \Gamma$ and the multifunction $\widehat{T}(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ if the function $F(\cdot, y)$ is continuous at x_0 for any $y \in V$.*

Proof. Let $\{x_n\}_n$ be a sequence converging to x_0 in U and $(y_k)_k$ be a sequence converging to y_0 in V such that $y_k \in T(x_{n_k}, \varepsilon)$. Since Γ is closed and F is continuous on $\{x_0\} \times \Gamma$ it results from (1):

$$\langle F(x_0, y_0), y_0 - z\rangle \leq \liminf_{k\to+\infty} \langle F(x_{n_k}, y_k), y_k - z\rangle \leq \varepsilon$$

and similarly for the second result.

In the case in which V is a Banach space with weak (w) and strong (s) topologies on V and V^* we get:

Corollary 2.1. *Let $\varepsilon \geq 0$. If the function F is continuous from $(\{x_0\} \times \Gamma, \tau \times w)$ to (V^*, s) then the multifunction $T(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ with respect to (τ, w). If F is continuous from $(\{x_0\} \times \Gamma, \tau \times s)$ to (V^*, w) then the multifunction $T(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ with respect to (τ, s).*

An analogous result is obtained for $\widehat{T}(\cdot, \varepsilon)$.

Corollary 2.2. *Let $\varepsilon \geq 0$. If, for any $y \in \Gamma$ the function $F(\cdot, y)$ is continuous at x_0 from (U, τ) to (V^*, s) then $\widehat{T}(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ with respect to (τ, w) and consequently also with respect to (τ, s).*

For what concerning the lower semicontinuity we obtain the following result for $\widehat{T}(\cdot, \varepsilon)$:

Theorem 2.1. *Let $\varepsilon > 0$. If the function F is continuous on $\{x_0\} \times \Gamma$, the pairing $\langle \cdot, \cdot \rangle$ is sequentially continuous and Γ is sequentially compact then the multifunction $\widehat{T}(\cdot, \varepsilon)$ is nearly lower semicontinuous at $x_0 \in U$ (lower semicontinuous if V is a first countable topological space).*

Proof. We start by proving that the following multifunction is lower semicontinuous at $x_0 \in U$:

$$S(\cdot, \varepsilon) : x \to \{y \in \Gamma \text{ such that} : \langle F(x, z), y - z \rangle < \varepsilon \text{ for any } z \in \Gamma\}.$$

If $\{x_n\}_n$ is a sequence converging to x_0 on U and we consider $y_0 \in S(x_0, \varepsilon)$ then we have that $y_0 \in S(x_n, \varepsilon)$ for n large. Indeed, assuming that $y_0 \notin S(x_{n_k}, \varepsilon)$ for a subsequence $\{n_k\}_k$, there exists a sequence $\{z_k\}_k$, with $z_k \in \Gamma$, such that $\langle F(x_{n_k}, z_k), y_0 - z_k \rangle \leq \varepsilon$ for any k. From the compactness of Γ there exists a subsequence of $\{z_k\}_k$ converging to a point $z_0 \in \Gamma$ and for such z_0 we have $\langle F(x_0, z_0), y_0 - z_0 \rangle \geq \varepsilon$ in contradiction with the assumption $y_0 \in S(x_0, \varepsilon)$.

Now let us prove that:

$$\widehat{T}(x, \varepsilon) \subset \overline{(S(x, \varepsilon))}^{\text{seq}} \tag{2}$$

for any $x \in X$. Let $y_0 \in \widehat{T}(x, \varepsilon)$, $y_0' \in S(x, \varepsilon)$ and $y_n' = (1 - \alpha_n) y_0 + \alpha_n y_0'$ where $\alpha_n \in [0, 1]$ for any n and $\alpha_n \to 0$.
The sequence $\{y_n'\}_n$ is convergent to y_0 and we can prove that $y_n' \in S(x, \varepsilon)$ for any n. In fact

$$\begin{aligned} \langle F(x, z), y_n' - z \rangle &= \langle F(x, z), (1 - \alpha_n)(y_0 - z) + \alpha_n (y_0' - z) \rangle = \\ &= (1 - \alpha_n) \langle F(x, z), (y_0 - z) \rangle + \alpha_n \langle F(x, z), (y_0' - z) \rangle < \\ &< (1 - \alpha_n)\varepsilon + \alpha_n \varepsilon = \varepsilon \end{aligned}$$

for any n. Then, from relation (2) and lower semicontinuity of $S(\cdot, \varepsilon)$ at x_0 we infer that:

$$\widehat{T}(x, \varepsilon) \subset \overline{(S(x, \varepsilon))}^{\text{seq}} \subset \overline{\liminf_{n \to +\infty} (S(x_n, \varepsilon))}^{\text{seq}} \subset \overline{\liminf_{n \to +\infty} (\widehat{T}(x_n, \varepsilon))}^{\text{seq}}.$$

When (V, σ) is first countable the sequential closure coincides with the topological closure and the lim inf is always a closed subset of V.

When we deal with Banach spaces we obtain:

Proposition 2.3. *Let $\varepsilon > 0$. If the function F is continuous from $(\{x_0\} \times \Gamma, \tau \times w)$ to (V^*, s) and Γ is weakly sequentially compact then $\widehat{T}(\cdot, \varepsilon)$ is continuous with respect to (τ, s) at $x_0 \in U$.*

Proof. From Corollary 2.2 we have that $\widehat{T}(\cdot, \varepsilon)$ is closed graph at $x_0 \in U$ with respect to (τ, s), while lower semicontinuity can be proved arguing as in Theorem 2.1.

3. Approximate solutions for variational Inequalities under perturbations.

Given $\{F_n\}_n$ a sequence of functions from $U \times V$ to V^*, for $\varepsilon \geq 0$ we define the following perturbed problems:

$(\mathrm{VI}_x)_n(\varepsilon)$ Find $\overline{y} \in \Gamma$ such that : $\langle F_n(x, \overline{y}), \overline{y} - z \rangle \leq \varepsilon$ for any $z \in \Gamma$

$(\widehat{\mathrm{VI}}_x)_n(\varepsilon)$ Find $\overline{y} \in \Gamma$ such that : $\langle F_n(x, z), \overline{y} - z \rangle \leq \varepsilon$ for any $z \in \Gamma$

and the corresponding solution maps:

$$T_n(x, \varepsilon) = \{\overline{y} \in V : \overline{y} \text{ solves } (\mathrm{VI}_x)_n(\varepsilon)\}$$
$$\widehat{T}_n(x, \varepsilon) = \left\{\overline{y} \in V : \overline{y} \text{ solves } (\widehat{\mathrm{VI}}_x)_n(\varepsilon)\right\}.$$

First let us recall some convergence properties for functions and multifunctions.

Definition 3.1. *A sequence $\{f_n\}_n$ of functions from U to V is continuously convergent to f if, for any $x_0 \in U$ and for any sequence $\{x_n\}_n$ converging to x_0, we have:*

$$\lim_{n \to +\infty} f_n(x_n) = f(x_0).$$

Definition 3.2 ([6]). *A sequence $\{M_n\}_n$ of multifunctions from U to V is upper-convergent to a multifunction M if*

$$\limsup_{n \to +\infty} M_n(x_n) \subset M(x_0)$$

for any $x_0 \in U$ and for any sequence $\{x_n\}_n$ converging to x_0 in U.

Definition 3.3 ([7]). *A sequence $\{M_n\}_n$ of multifunctions from U to V is nearly convergent to a multifunction M if*

$$M(x_0) \subset \overline{\liminf_{n \to +\infty} \left(M_n(x_n)\right)}^{\,\mathrm{seq}}$$

for any $x_0 \in U$ and for any sequence $\{x_n\}_n$ converging to x_0 in U.

Remark 3.1. In first countable spaces $\{M_n\}_n$ nearly lowerconvergent to M on U is equivalent to $\{M_n\}_n$ lowerconvergent to M on U ([7]), that is:

$$M(x_0) \subset \liminf_{n\to+\infty} M_n(x_n)$$

for any $x_0 \in U$ and for any sequence $\{x_n\}_n$ converging to x_0 in U.

Proposition 3.1. *Let $\varepsilon \geq 0$. The sequence $\{T_n(\cdot,\varepsilon)\}_n$ (resp. $\{\widehat{T}_n(\cdot,\varepsilon)\}_n$) is upperconvergent to $T(\cdot,\varepsilon)$ (resp. $\widehat{T}(\cdot,\varepsilon)$) on U if the sequence $\{F_n\}_n$ is continuously convergent on $U \times V$ (resp. the sequence $\{F_n(\cdot, y)\}_n$ is continuously convergent to $F(\cdot, y)$ on U for any $y \in V$).*

Proof. Let $x_0 \in U$, $\{x_n\}_n$ be a sequence converging to x_0 in U and $\{y_k\}_k$ be a sequence converging to y_0 such that $y_k \in T_{n_k}(x_{n_k},\varepsilon)$ (resp. $y_k \in \widehat{T}_{n_k}(x_{n_k},\varepsilon)$). Since the sequence $\{F_n\}_n$ is continuously convergent to F on $U \times V$ (resp. the sequence $F_n(\cdot, y)$ is continuous convergent to $F(\cdot, y)$ on U for any $y \in V$) it results for any $z \in \Gamma$:

$$\langle F(x_0, y_0), y_0 - z\rangle \leq \liminf_{k\to+\infty} \langle F_{n_k}(x_{n_k}, y_k), y_k - z\rangle \leq \varepsilon$$

(resp.

$$\langle F(x_0, z), y_0 - z\rangle \leq \liminf_{k\to+\infty} \langle F_{n_k}(x_{n_k}, z), y_k - z\rangle \leq \varepsilon).$$

In the case in which V is Banach space with weak (w) and strong (s) topologies on V and V^* we get:

Corollary 3.1. *Let $\varepsilon \geq 0$. If the sequence $\{F_n\}_n$ is continuously convergent from $(U\times V, \tau\times w)$ to (V^*, s) then the sequence $\{T_n(\cdot,\varepsilon)\}_n$ is upperconvergent to $T(\cdot,\varepsilon)$ on U with respect to (τ, w).*
If the sequence $\{F_n\}_n$ is continuously convergent from $(U\times V, \tau\times s)$ to (V^*, w) then the sequence $\{T_n(\cdot,\varepsilon)\}_n$ is upperconvergent to $T(\cdot,\varepsilon)$ on U with respect to (τ, s).

An analogous result is obtained for $\widehat{T}_n(\cdot,\varepsilon)$:

Corollary 3.2. *Let $\varepsilon \geq 0$. If, for any $y \in V$, the sequence $\{F_n(\cdot, y)\}_n$ is continuously convergent, from (U,τ) to (V^*, s) then the sequence $\{\widehat{T}_n(\cdot,\varepsilon)\}_n$ is upperconvergent to $\widehat{T}(\cdot,\varepsilon)$ on U with respect to (τ, w).*

Theorem 3.1. *Let $\varepsilon > 0$. Let the pairing $\langle \cdot, \cdot \rangle$ be sequentially continuous and Γ be sequentially compact. The sequence $\{\widehat{T}_n(\cdot, \varepsilon)\}_n$ is nearly lowerconvergent (lowerconvergent if the space V is a first countable topological space) to $\widehat{T}(\cdot, \varepsilon)$ on U if the sequence $\{F_n\}_n$ is continuously convergent to F on $U \times V$.*

Proof. Let $\varepsilon > 0$. We start by proving that the following sequence of multifunctions is nearly lowerconvergent on U to $S(\cdot, \varepsilon)$ (as defined in the proof of Theorem 2.1):

$$S_n(\cdot, \varepsilon) : x \to \{y \in \Gamma \text{ such that} : \langle F_n(x, z), y - z \rangle < \varepsilon \text{ for any } z \in \Gamma\}.$$

If $\{x_n\}_n$ is a sequence converging to x_0 on U and we consider $y_0 \in S(x_0, \varepsilon)$ then, for n large, we have that $y_0 \in S_n(x_n, \varepsilon)$.
Indeed, assume that $y_0 \notin S_{n_k}(x_{n_k}, \varepsilon)$ for a subsequence $\{n_k\}_k$ then there exists a sequence $\{z_k\}_k$, with $z_k \in \Gamma$, such that

$$\langle F_{n_k}(x_{n_k}, z_k), y_0 - z_k \rangle \geq \varepsilon$$

for any k. From the compactness of Γ there exists a subsequence of $\{z_k\}_k$ converging to a point $z_0 \in \Gamma$ and for such z_0 we have

$$\langle F(x_0, z_0), y_0 - z_0 \rangle \geq \varepsilon$$

in contradiction with the assumption $y_0 \in S(x_0, \varepsilon)$.
Therefore, from (2), we have:

$$\widehat{T}(x, \varepsilon) \subset \overline{(S(x, \varepsilon))}^{\text{seq}} \subset \overline{\liminf_{n \to +\infty} (S_n(x_n, \varepsilon))}^{\text{seq}} \subset \overline{\liminf_{n \to +\infty} (\widehat{T}_n(x_n, \varepsilon))}^{\text{seq}}.$$

Arguing as in Proposition 2.3 we have:

Proposition 3.2. *Let $\varepsilon > 0$. If the sequence $\{F_n\}_n$ is continuously convergent from $(U \times V, \tau \times w)$ to (V^*, s) and Γ is weakly sequentially compact then the sequence $\{\widehat{T}_n(\cdot, \varepsilon)\}_n$ is lowerconvergent to $\widehat{T}(\cdot, \varepsilon)$ with respect to (τ, s).*

Combining Corollary 3.2 and Proposition 3.2 we obtain:

Corollary 3.3. *Let $\varepsilon > 0$. If Γ is weakly sequentially compact and the sequence $\{F_n\}_n$ is continuously convergent from $(U \times V, \tau \times w)$ to (V^*, s) then, for any $x_0 \in U$ and for any sequence $\{x_n\}_n$ converging to x_0, the sequence $\{\widehat{T}_n(x_n, \varepsilon)\}_n$ is Mosco convergent to $\widehat{T}(x_0, \varepsilon)$, that is:*

$$w\text{-}\limsup_{n \to +\infty} \widehat{T}_n(x_n, \varepsilon) \subset \widehat{T}(x_0, \varepsilon) \subset s\text{-}\liminf_{n \to +\infty} \widehat{T}_n(x_n, \varepsilon).$$

4. Applications.

Let f and f_n, for any $n \in \mathbb{N}$, be extended real valued functions on $U \times V$ and consider the following problems:

(P)
Find $\overline{x} \in U$ such that :

$$\inf_{y \in T(\overline{x})} f(\overline{x}, y) = \inf_{x \in U} \inf_{y \in T(x)} f(x, y) = \alpha,$$

where $T(x)$ is defined as in Section 1

and

$(\mathrm{P})_n$
Find $\overline{x}_n \in U$ such that :

$$\inf_{y \in T_n(\overline{x}_n)} f_n(\overline{x}_n, y) = \inf_{x \in U} \inf_{y \in T_n(x)} f_n(x, y) = \alpha_n ,$$

where $T_n(x) = T_n(x, 0)$ is defined as in Section 3.

Problem (P), which arises, for example, in shape optimization, in economic modelling and in design transportation networks, has been investigated from different points of view: necessary conditions, sensitivity analysis and algorithms (see [12] and [4] for references). Nevertheless, no results have been given in a general framework on the connections between (P_n) and (P) and, particularly, on the convergence of the solutions to (P_n) and of the value α_n of (P_n). Unfortunately, even for nice perturbations of F and f these properties are not always satisfied, as shown by the following example:

Example 4.1. Let $U = V = [0, 1]$, $f(x, y) = f_n(x, y) = x - y + 1$, $F(x, y) = 0$, $F_n(x, y) = 1/n$, $\Gamma = [0, 1]$. It is easy to verify that $\alpha = 0$, $\alpha_n = 1$ then the convergence of the values is not obtained.

So, in line with what done in [10] for the weak Stackelberg problem and [8] for the strong one, we will use the concept of approximate solution for variational inequalities defined by $(\widehat{\mathrm{VI}}_x)(\varepsilon)$ in Section 2 in order to obtain not only existence but also convergence results under perturbations on the data and more precisely, for $\varepsilon > 0$, we consider the following problems:

$\mathrm{P}(\varepsilon)$
Find $\overline{x}_\varepsilon \in U$ such that :

$$\inf_{y \in \widehat{T}(\overline{x}_\varepsilon, \varepsilon)} f(\overline{x}_\varepsilon, y) = \inf_{x \in U} \inf_{y \in \widehat{T}(x, \varepsilon)} f(x, y) = \widehat{\alpha}(\varepsilon)$$

where $\widehat{T}(x, \varepsilon)$ is defined as in Section 2 and

$$\widehat{\mathrm{P}}_n(\varepsilon) \qquad \begin{array}{l} \text{Find } \overline{x}_{\varepsilon,n} \in U \text{ such that :} \\ \inf_{y\in\widehat{T}_n(\overline{x}_{\varepsilon,n},\varepsilon)} f_n(\overline{x}_{\varepsilon,n}, y) = \inf_{x\in U} \inf_{y\in\widehat{T}_n(x,\varepsilon)} f_n(x, y) = \widehat{\alpha}_n(\varepsilon) \\ \text{where } \widehat{T}_n(x, \varepsilon) \text{ is defined as in Section 3.} \end{array}$$

Let $\widehat{M}(\varepsilon)$ and $\widehat{M}_n(\varepsilon)$ be the set of solutions respectively of $\widehat{\mathrm{P}}(\varepsilon)$ and $\widehat{\mathrm{P}}_n(\varepsilon)$. Then we have:

Theorem 4.1. *Let $\varepsilon > 0$. Let U be a sequentially compact topological space. If the pairing is continuous and Γ is sequentially compact, under the following assumptions:*

1) *for any $y \in V$, the sequence $\{F_n\}_n$ continuously converges to F,*
2) *for any $(x, y) \in U \times V$, any sequence $\{x_n\}_n$ converging to x in U and any sequence $\{y_n\}_n$ converging to y in V we have:*

$$f(x, y) \leq \liminf_{n\to+\infty} f_n(x_n, y_n),$$

3) *for any $(x, y) \in U \times V$, there exists a sequence $\{\overline{x}_n\}_n$ converging to x in U such that for any sequence $\{y_n\}_n$ converging to y in V we have:*

$$f(x, y) \geq \limsup_{n\to+\infty} f_n(\overline{x}_n, y_n),$$

we have:

i) $\limsup_{n\to+\infty} \widehat{M}_n(\varepsilon) \subset \widehat{M}(\varepsilon),$

ii) $\lim_{n\to+\infty} \widehat{\alpha}_n(\varepsilon) = \widehat{\alpha}(\varepsilon).$

Proof. From Proposition 3.1 and Theorem 3.1 we infer that the sequence $\{\widehat{T}_n(\cdot, \varepsilon)\}_n$ is lower and upperconvergent to $\widehat{T}(\cdot, \varepsilon)$. So, from Proposition 4.3.1 in [6] we can deduce that the sequence of marginal function $\{\widehat{\omega}_n(\cdot, \varepsilon)\}_n$ (defined by:

$$\widehat{\omega}_n(x, \varepsilon) = \inf_{y\in\widehat{T}_n(x,\varepsilon)} f_n(x, y)$$

for any $x \in U$) is epiconvergent to the marginal function $\widehat{\omega}(\cdot, \varepsilon)$ (defined by:

$$\widehat{\omega}(x, \varepsilon) = \inf_{y\in\widehat{T}(x,\varepsilon)} f(x, y)$$

for any $x \in u$). Then, from a classical result ([2], [3] or Proposition 2.3.1 in [6]), we obtain i) and ii).

Similarly we have:

Corollary 4.1. *Let $\varepsilon > 0$. If the assumptions of Corollary 3.3 and the following are satisfied:*

1) *for any $(x, y) \in U \times V$, any sequence $\{x_n\}_n$ converging to x in U and any sequence $\{y_n\}_n$ weakly converging to y in V we have:*

$$f(x, y) \leq \liminf_{n\to+\infty} f_n(x_n, y_n),$$

2) *for any $(x, y) \in U \times V$, there exists a sequence $\{\overline{x}_n\}_n$ converging to x in U such that for any sequence $\{y_n\}_n$ strongly converging to y in V we have:*

$$f(x, y) \geq \limsup_{n\to+\infty} f_n(\overline{x}_n, y_n);$$

then:

i) $w\text{-}\limsup_{n\to+\infty} \widehat{M}_n(\varepsilon) \subset \widehat{M}(\varepsilon)$,

ii) $\lim_{n\to+\infty} \widehat{\alpha}_n(\varepsilon) = \widehat{\alpha}(\varepsilon)$.

REFERENCES

[1] C. Baiocchi - A. Capelo, *Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libera,* Volume 1. Quaderni dell'UMI. Pitagora, Bologna, 1978.

[2] G. Buttazzo - G. Dal Maso, *Γ-convergence and Optimal Control Problems,* Jour. of Optimazion Theory and Appl., 17 (1982), pp. 385-407.

[3] E. De Giorgi - T. Franzoni, *Su un tipo di convergenza variazionale,* Atti Accademia Nazionale dei Lincei, Scienze Fisiche Matematiche e Naturali, 58 (1975), pp. 842-850.

[4] P.T. Harker - J.S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems; a survey of theory, algorithms and applications,* Mathematical Programming, 48 (1990), pp. 161-220.

[5] D. Kinderlehrer - G. Stampacchia, *An introduction to variational inequalities and their applications,* Academic Press, New-York, 1980.

[6] M.B. Lignola - J. Morgan, *Convergence of marginal functions with dependent constraints,* Optimization, 23 (1992), pp 189-213.

[7] M.B. Lignola - J. Morgan, *Existence and approximation results for MinSup problems,* Proceedings of the International Conference on Operation Research 90 in Vienna, G. Feichtinger - W. Buhler - F.J. Radermacher - P. Stahly editors., Springer Verlag, 1992, pp. 157-164.

[8] M.B. Lignola - J. Morgan, *Regularized bilevel programming problem,* Preprint no 22-1993, Pubblicazione del Dipartimento di Matematica e Applicazioni dell'Università di Napoli.

[9] M.B. Lignola - J. Morgan, *Convergences for variational inequalities and Generalized inequalities,* Preprint no 6-1994, Pubblicazione del Dipartimento di Matematica e Applicazioni dell'Università di Napoli.

[10] P. Loridan - J. Morgan, *New results on approximate solutions in two level optimization,* Optimization, 20 (1989), pp. 819-836.

[11] G. Minty, *Monotone (non linear) operator in Hilbert space,* Duke Math. Journal, 29 (1962), pp. 341-346.

[12] J.V. Outrata, *An optimization problem with variational inequality constraints,* SIOPT 3, n.2, (1994).

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