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SOME RELATIONS BETWEEN DUALITY THEORY FOR EXTREMUM PROBLEMS AND VARIATIONAL INEQUALITIES

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After revisiting the well-known relationships with the minimax theory, some duality results for constrained extremum problems are related to variational inequalities. In particular, the connections with saddle point conditions and gap functions associated to the variational inequality are analysed.

1. Introduction.

Consider the variational inequality:find $y^* \in K \subseteq \mathbb{R}^n$ such that

$$\text{(VI)} \qquad \langle F(y^*), x - y^* \rangle \geq 0 \qquad \forall x \in K$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$. From the simple remark that if the point $(y^*, x^*) \in K \times K$ is a saddle point for the function $\phi(y, x) := \langle F(y), y - x \rangle$ on $K \times K$ that is

$$(1.1) \quad \langle F(y^*), y^* - x \rangle \leq \langle F(y^*), y^* - x^* \rangle \leq \langle F(y), y - x^* \rangle \quad \forall (y, x) \in K \times K$$

then y^* is a solution for (VI), we will consider (Section 2) the problem suggested by the second inequality of (1.1), the problem of finding $x^* \in K$ such that

$$\text{(VI}^*) \qquad \langle F(y), y - x^* \rangle \geq 0 \qquad \forall y \in K.$$

It is known that if F is continuous and pseudomonotone then $y^* \in K$ solves (VI) if and only if $x^* = y^*$ solves (VI*).

In Section 3 we will consider the generalized complementary problem:

(GCP) find $y \in K$ such that $F(y) \in K^*$ and $\langle F(y), y \rangle = 0$

where K is assumed to be a closed convex cone and K^* is the positive polar of K.

We will show that, if (GCP) has a solution, in the particular case in which K is a closed convex cone containing the origin of $\mathbb{R}^n$ and the function $\phi(y, x)$ is convex (or invex in the differentiable case) with respect to y, $\forall x \in K$, then a solution x^* of (VI*) is a solution of the dual of the constrained extremum problem:

$$\min \langle F(y), y \rangle \quad \text{s.t. } F(y) \in K^*, \ \ y \in K$$

when its extreme value is equal to zero; in this case the previous problem is equivalent to (GCP). Since with the above convexity hypothesis (GCP) is also equivalent to (VI), we obtain a duality relation between (VI) and (VI*) equivalent to the saddle point condition expressed by (1.1).

Section 4 will be devoted to the analysis of the gap function approach for the solution of a variational and quasi-variational inequality, pointing out some connection with the minimax formulation for a (VI) considered in Section 2.

2. Variational inequalities and minimax theory.

Let us recall some notations that will be used in what follows: $\operatorname{int} M$ will denote the interior of the set $M \subseteq \mathbb{R}^n$, if C is a cone

$$C^* := \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \quad \forall x \in C \}$$

will denote the positive polar of C, $a \underset{C}{\geq} b$ iff $a - b \in C$. Moreover we will say that the mapping F is pseudo-monotone on K if:

$$\langle F(y), x - y \rangle \geq 0 \quad \text{implies} \quad \langle F(x), x - y \rangle \geq 0 \quad \forall x, y \in K.$$

In this Section we will point out some relationships between minimax theory and variational inequalities. This will be useful since it will allow to show some connections with duality that is deeply related to the minimax theory. Let us recall the following result due to K. Fan [4] as reported in [1]:

Theorem 2.1. *Let E be a topological vector space, let K be a nonempty compact set in E and let ϕ be a real-valued function on $K \times K$. Suppose that*

(a) $\phi(y, y) \leq 0 \quad \forall y \in K$;
(b) $\forall$ *fixed* $y \in K$, *the map* $x \to \phi(x, y)$ *is quasiconcave on* K;
(c) *for every fixed* $x \in K$, *the map* $y \to \phi(x, y)$ *is lower semicontinuous on* K.

Then there exists a vector $y^* \in K$ *such that* $\phi(x, y^*) \leq 0 \quad \forall x \in K$.

An immediate corollary of the previous theorem is the well known result on the existence of a solution of (VI) due to Hartman and Stampacchia [7]:

Corollary 2.1. *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous on the convex and compact set* $K \subseteq \mathbb{R}^n$. *Then there exists* $y^* \in K$ *such that*

$$\langle F(y^*), x - y^* \rangle \geq 0 \quad \forall x \in K.$$

Proof. Let $\phi(x, y) := \langle F(y), y - x \rangle$; the hypothesis of Theorem 2.1 are fulfilled and therefore the thesis follows.

Consider now the problem of finding $x^* \in K$ such that

$$\text{(VI*)} \qquad \langle F(y), y - x^* \rangle \geq 0 \qquad \forall y \in K.$$

It is known [9] that if F is a continuous pseudomonotone mapping then $y^* \in K$ solves (VI) if and only if $x^* = y^*$ solves (VI*).
Still from Theorem 2.1 we obtain the following result:

Proposition 2.1. *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous on the convex and compact set* $K \subseteq \mathbb{R}^n$ *and* $\forall x \in K$, *the map* $y \to \langle F(y), y - x \rangle$ *quasi-convex on* K. *Then both problems* (VI) *and* (VI*) *admit a solution.*

Proof. Let $\phi(x, y) := \langle F(y), y - x \rangle$; (VI) admits a solution for the previous corollary, moreover the function $-\phi(x, y)$ satisfies the hypothesis of Theorem 2.1 and therefore there exists x^* such that $-\phi(x^*, y) \leq 0$ from which thesis follows.

The next proposition provides a minimax formulation for the problems (VI) and (VI*):

Proposition 2.2. *Let* $\phi(x, y) := \langle F(y), y - x \rangle$;

i) (VI) *admits a solution if and only if*

$$\text{(2.1)} \qquad \inf_{y \in K} \sup_{x \in K} \phi(x, y) = 0$$

and the infimum is attained;

ii) (VI*) *admits a solution if and only if*

$$\sup_{x\in K} \inf_{y\in K} \phi(x, y) = 0$$

and the supremum is attained;

3i) $y^* \in K$ *and* $x^* \in K$ *are the respective solutions of the problems* (VI) *and* (VI*) *if and only if* (x^*, y^*) *is a saddle point for* $\phi(x, y)$ *on* $K \times K$.

Proof. i) See [8].

ii) Let $h(x) := \inf_{y\in K} \langle F(y), y - x\rangle$. It is immediate that $h(x) \le 0\ \forall x \in K$ and that $x^* \in K$ is a solution for (VI*) if and only if $h(x^*) = 0$ which is equivalent to the fact that

$$0 = h(x^*) = \max_{x\in K} h(x) = \max_{x\in K} \inf_{y\in K} \phi(x, y).$$

3i) Since y^* and x^* are solutions of the problems (VI) and (VI*), it will necessarily be $\langle F(y^*), y^* - x^*\rangle = 0$ because x^* and $y^* \in K$ and the two inequalities given by (VI) and (VI*) must hold.
Therefore, from i) and ii), it follows that the point (x^*, y^*) is a saddle-point for $\phi(x, y)$ on $K \times K$.
Viceversa let $(x^*, y^*) \in K \times K$ a saddle point for ϕ on $K \times K$, that is:

$$\langle F(y^*), y^* - x\rangle \le \langle F(y^*), y^* - x^*\rangle \le \langle F(y), y - x^*\rangle \quad \forall (y, x) \in K \times K.$$

Evaluating the previous inequalities at the point (x^*, y^*) we obtain

$$\langle F(y^*), y^* - x^*\rangle = 0$$

from which the thesis follows since the first inequality states that y^* is a solution for (VI), while the second that x^* is a solution for (VI*).

3. Complementary problems and variational inequalities:duality connections.

Following the results of the last section we will give a dual interpretation of the problem (VI*) with respect to a complementary problem.

First of all we will recall some properties concerning saddle points of the lagrangean function associated to a constrained extremum problem of the following kind:

$$\text{(P)} \qquad \min f(x) \quad \text{s.t.} \quad x \in X,\ g(x) \in K$$

where $K \subseteq \mathbb{R}^m$ is a closed convex cone, $X \subseteq \mathbb{R}^n$ and $g : X \to \mathbb{R}^m$.
Let $L(x, \lambda) : X \times K^* \to \mathbb{R}$, $L(x, \lambda) := f(x) - \langle \lambda, g(x) \rangle$, the lagrangean function associated to (P).
Let us recall the following well-known results:

Proposition 3.1. *If* (x^*, λ^*) *is a saddle point for* $L(x, \lambda)$ *on* $X \times K^*$, *that is*

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*) \quad \forall x \in X, \ \forall \lambda \in K^*,$$

then λ^* *is a solution of the lagrangean dual of the problem* (P) *defined by:*

$$\text{(D)} \qquad \sup_{\lambda \in K^*} \inf_{x \in X} L(x, \lambda).$$

Proposition 3.2. *Let* $L(x, \lambda)$ *be convex (or invex in the differentiable case) with respect to* x, $\forall \lambda \in K^*$ *and let* $x' \in \mathbb{R}^n$ *such that* $g(x') \in \operatorname{int} K$. *Then*

i) x^* *is a solution for* (P) *if and only if there exists* $\lambda^* \in K^*$ *such that* (x^*, λ^*) *is a saddle point for* L *on* $X \times K^*$.
ii) *if* (P) *has a solution then* (D) *has the same optimal value of* (P).

For a proof of the previous propositions see for example [5].
Consider the generalized complementary problem:

$$\text{(GCP)} \qquad \text{find } y \in K \text{ such that } F(y) \in K^* \text{ and } \langle F(y), y \rangle = 0$$

where $K \subseteq \mathbb{R}^n$ is a closed convex cone, containing the origin of $\mathbb{R}^n$, and K^* is the positive polar of K.
It is known that [9], [3]:

Remark 3.1 ([6]). y^* solves (GCP) if and only if y^* solves (VI) with feasible set K and F defined as for (GCP).

Proposition 3.3 ([3]). *A vector* $y^* \in K$ *solves* (GCP) *if and only if* y^* *is a solution of the problem:*

$$(3.1) \qquad \min \langle F(y), y \rangle \quad \text{s.t.} \quad F(y) \in K^*, \ y \in K$$

and $\langle F(y^*), y^* \rangle = 0$.

The previous proposition allows us to express the complementary problem (GCP) in terms of an optimization problem of which we can consider the lagrangean dual. We will see that, if (GCP) has a solution, then the dual problem of (3.1) is equivalent to the problem (VI*).

Remark 3.2. We observe that the lagrangean function associated to (3.1) defined by $L : K \times K \to \mathbb{R}$ (we recall that $(K^*)^* = K$ since $0 \in K$) $L(y, \lambda) := \langle F(y), y \rangle - \langle \lambda, F(y) \rangle$ coincides with the function $\phi(x, y) := \langle F(y), y - x \rangle$ defined in Proposition 2.2.

Proposition 3.4. *Suppose that the function* $L(y, \lambda) := \langle F(y), y - \lambda \rangle$ *is convex with respect to* y, $\forall \lambda \in K$, *that there exists* $y' \in K$ *such that* $F(y') \in \operatorname{int} K^*$ *and that the problem* (GCP) *has a solution.*

Then (VI*) *admits a solution and* x^* *solves* (VI*) *if and only if* $\lambda^* = x^*$ *is a solution of the lagrangean dual of* (3.1).

Proof. Let $y^* \in K$ a solution of (GCP) and, from Remark 3.1, of (VI). For Proposition 3.3 y^* is a solution of problem (3.1) and, for Proposition 3.2 i), this is equivalent to the fact that there exists $\lambda' \in K$ such that (y^*, λ') is a saddle point of the lagrangean function associated to (3.1), that is the function $L : K \times K \to \mathbb{R}$ $L(y, \lambda) := \langle F(y), y - \lambda \rangle$ (see Remark 3.2).
Applying Proposition 2.2 3i) we obtain that is equivalent to the fact that λ' is a solution for (VI*) and the proposition is proven.

The following scheme summarizes the relations considered in the last two sections.
By means of a saddle point condition for the function $\langle F(y), y - x \rangle$ we have introduced the problem (VI*):

$$\text{(VI)} \xleftrightarrow{\text{Saddle point condition}} \text{(VI}^*\text{)}$$

In the hypothesis that K is a closed convex cone with $0 \in K$ we have the following equivalence ($\Longleftrightarrow$):

$$\text{(VI)} \Longleftrightarrow \text{(GCP)} \Longleftrightarrow \text{(P)}$$

| Duality |

$$\text{(VI}^*\text{)} \Longleftrightarrow \text{(P}^*\text{)}$$

where (P) is problem (3.1) and (P*) its lagrangean dual; two problem are equivalent ($\Longleftrightarrow$) if their solutions coincide.

4. Variational inequalities and gap functions.

Consider the variational inequality (VI) with feasible set $K := \{x \in X : g(x) \underset{C}{\geq} 0\}$ where $g : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ and $C \subset \mathbb{R}^m$ is a closed convex cone. We observe that $y^* \in K$ is a solution for (VI) iff the constrained extremum problem:

$$\text{P}(y^*) \qquad \max_{x \in X} \langle F(y^*), y^* - x \rangle$$

admits a solution such that $p(y^*) = 0$, where

$$p(y) := \max_{x \in K} \langle F(y), y - x \rangle. \tag{4.1}$$

Definition 4.1 ([9]). *Given a variational inequality* (VI), $p : \mathbb{R}^n \to \mathbb{R}$ *is a gap function for* (VI) *iff:*

i) $p(y) \geq 0 \quad \forall y \in K;$
ii) $p(y) = 0$ *if and only if* y *is a solution for* (VI).

Remark 4.1. From the previous definition it is immediate that (VI) is equivalent to the problem of finding the global minimum of a gap function on the set K.

Proposition 4.1. *The function* p *defined by* (4.1) *is a gap function.*

Proof. We have to prove i) and ii) of Definition 4.1.
i) is immediate since $x^* = y$ is a feasible point for $P(y)$ $\forall y \in K$.
ii) follows from i) of Proposition 2.2.

Remark 4.2. We observe that the gap function formulation of (VI), with gap function defined by (4.1), is equivalent to the minimax formulation given by (2.1).

The gap function given by (4.1), which was introduced by Auslender [2], is the marginal function of the parametric problem P(y) by which we express, in an equivalent form, the variational inequality (VI). Therefore it is possible to define the gap function $p(y)$ considering a problem Q(y) equivalent to P(y), where the equivalence must be interpreted in the sense that Q(y) and P(y) have the same marginal functions.

To illustrate this scheme consider a function $\gamma : \mathbb{R}^m \to \mathbb{R}^m$ with the following property:

$$\gamma(z) \underset{C}{\geq} 0 \iff z \underset{C}{\geq} 0 \quad \forall z \in g(X) := \left\{v \in \mathbb{R}^m : v = g(x),\ x \in X\right\}.$$

It is immediate that the parametric problem

$$\mathrm{Q}(y) \qquad \max_x \langle F(y), y-x\rangle \quad \text{s.t.} \quad \gamma(g(x)) \underset{C}{\geq} 0,\ x \in X$$

is equivalent to the problem P(y), since the respective feasible regions and objective functions coincide.

Remark 4.3. The advantage of considering problem Q(y) lies in the fact that the function $\gamma(g(x))$ may have better properties than $g(x)$, for example the convexity. In this way the construction of the gap function $p(y)$ may be simplified.

The next proposition provides another equivalent formulation of the gap function $p(y)$ by means of the lagrangean dual of the problem Q(y):

Proposition 4.2. *Assume that the function $-\gamma(g(x))$ is convex and there exists a point $x^* \in X$ such that $\gamma(g(x^*)) \in \operatorname{int} C$. Then*

$$p(y) = \min_{\lambda \in C^*} \max_{x \in X} \left[\langle F(y), y-x\rangle + \langle \lambda, \gamma(g(x))\rangle\right]. \tag{4.2}$$

Proof. In the hypothesis of the proposition Q(y) is a regular convex problem $\forall y \in K$; therefore the lagrangean dual of Q(y), which is the problem expressed by the right term of (4.2), has the same optimal value of Q(y) $\forall y \in K$.

Consider now the case of a quasi-variational inequality (QVI) defined in the following way:

$$\text{find } y \in K(y) \text{ s. t. } \langle F(y), x-y\rangle \geq 0 \quad \forall x \in K(y) := \left\{x \in X : g(x,y) \underset{C}{\geq} 0\right\},$$

where $g : X \times X \to \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$.

For a quasi-variational inequality the definition of the gap functions is slightly modified. Let $Y := \left\{y \in X : y \in K(y)\right\}$.

Definition 4.2 ([6]). *Given a quasi-variational inequality* (QVI), *$p : Y \to \mathbb{R}$ is a gap function for* (QVI) *if and only if:*

i) *$p(y) \geq 0\ \forall y \in Y$:*
ii) *$p(y) = 0$ if and only if y is a solution for* (QVI).

As in the case of a variational inequality a gap function for (QVI) may be obtained considering the marginal function of the parametric problem:

$$\mathrm{QP}(y) \qquad \max_x \langle F(y), y-x\rangle \quad \text{s.t.} \quad x \in K(y) := \left\{x \in X : g(x,y) \underset{C}{\geq} 0\right\};$$

we observe that

$$p(y) := \max_{x \in K(y)} \langle F(y), y - x \rangle \tag{4.3}$$

is a gap function for (QVI).

In the quasi-variational case, considering the lagrangean dual of QP(y), it is possible to formulate a gap function for (QVI) non necessarily equal to $p(y)$ defined by (4.3).

Consider the optimal function $\Psi(y)$ of the lagrangean dual of QP(y),

$$\Psi(y) := \min_{\lambda \in C^*} \max_{x \in X} \left[\langle F(y), y - x \rangle + \langle \lambda, g(x, y) \rangle \right].$$

We have seen that, in the case of a simple variational inequality with regular constraints, the optimal function of P(y) coincides with the one of its lagrangean dual $\forall\, y \in K$. In the present case in general we have that $p(y) \neq \Psi(y)$. If we ensure that $p(y^*) = \Psi(y^*)$ whenever $p(y^*) = 0$ we can show that $\Psi(y)$ is a gap function for QVI as stated in the following theorem.
Let $S := \{ y \in Y \mid p(y) = 0 \}$.

Theorem 4.1. *Let* $L(x; \lambda, y) := \langle F(y), y - x \rangle + \langle \lambda, g(x, y) \rangle$ *be convex (or invex in the differentiable case) with respect to* x, $\forall\, \lambda \in C^*$, $\forall\, y^* \in S$, *and assume that,* $\forall\, y^* \in S$, *there exists* $x' \in X$ *such that* $g(x', y^*) \in \operatorname{int} C$. *Then*

$$\Psi(y) : \min_{\lambda \in C^*} \max_{x \in X} L(x, \lambda; y)$$

is a gap function for (QVI).

Proof. We have to prove that $\psi(y)$ fulfils i) and ii) of Definition 4.2. Since $\Psi(y)$ is the optimal value of the lagrangean dual of QP(y), for the weak duality theorem, we have $\Psi(y) \geq p(y) \geq 0\ \forall\, y \in Y$, that is i).
From ii) of Proposition 3.2 we obtain that $\psi(y) = 0\ \forall\, y \in S$; therefore ii) of Definition 4.2 holds and the theorem is proven.

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