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OPTIMIZATION PROBLEMS WITH SIDE CONSTRAINTS AND GENERALIZED EQUILIBRIUM PRINCIPLES

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In this paper we show how the "Wardrop Equilibrium Principle" must be modified when capacities for the paths of a network are introduced. Moreover we show that the equilibrium condition can be still expressed in terms of a Variational inequality.

1. Introduction.

Variational inequalities describe equilibria in network flow problems in the general setting of non standard and asymmetric cost functions.

Recently it was remarked by F. Giannessi [2] and P. Ferrari [1] that the models usually considered do not take into account the "capacity" of the paths of network and, consequently, the same "Wardrop Equilibrium principle" is unrealistic.

Then it seems to be very reasonable to introduce capacities (i.e. a particular kind of side constraints) for the paths and to carry out researches to find

i) How the "Wardrop Equilibrium principle" must be modified

ii) and if the modified principle can be still expressed in terms of Variational Inequalities.

Moreover T. Larsson and M. Patriksson [3] present a standard side constrained traffic equilibrium problem that leads to an optimization formulation and has optimality conditions that correspond to a generalization of Wardrop Equilibrium Principle.

T. Larsson and M. Patriksson also observe that, under the additional hypothesis that flow relationships modelled trough the introduction of asymmetric travel cost functions are better represented by a set of side constraints, the asymmetric model is equivalent to a symmetric one with a travel cost function containing unknown multipliers.

The approach above is appreciable, but its effectiveness must be verified.

The aim of this paper is to show how the "Wardrop Equilibrium Principle" must be modified when capacities are introduced in the asymmetric equilibrium model and that it can be equivalently expressed in terms of Variational Inequalities.

Moreover we provide a comparison of the results obtained in the general case and those obtained in the standard one, and, as a consequence, an appropriate meaning of Lagrange multipliers will arise.

2. Generalized user's equilibrium principle.

Let us consider a traffic network (N, L, W) where (see A. Maugeri [4] for more details):

$L = (P_1, P_2, \dots, P_q)$ is the set of nodes,

$N = \{a_1, a_2, \dots, a_n\}$ is the set of links,

$W = \{w_1, w_2, \dots, w_\ell\}$ is the set of Origin Destination pairs

and let us denote by:

$R_r \quad r = 1, \dots, m$ the paths connecting the O/D pairs

$\mathscr{R}_j = \{R_r\colon R_r$ connects the O/D pair $w_j\}$

$$\mathscr{R} = \bigcup_{j=1}^{\ell} \mathscr{R}_j$$

$F' = (F_1, F_2, \dots, F_m)$ the path-flow vector
(the apex denotes the transposition);

$C'(F) = (C_1(F), C_2(F), \dots, C_m(F))$ the path-cost vector;

$\rho' = (\rho_1, \rho_2, \dots, \rho_\ell)$ the vector travel demand between the O/D pairs w_j;

$$\varphi = \{\varphi_{jr}\}_{\substack{j=1,\dots,\ell \\ r=1,\dots,m}} \qquad \varphi_{jr} = \begin{cases} 1 & \text{if } R_r \in \mathscr{R}_j \\ 0 & \text{otherwise} \end{cases}$$

(φ is the pair-path incidence matrix);

$$\Delta = \{\delta_{ir}\}_{\substack{i=1,\dots,n \\ r=1,\dots,m}} \qquad \delta_{ir} = \begin{cases} 1 & \text{if } a_i \in R_r \\ 0 & \text{otherwise} \end{cases}$$

(Δ is the link-flow incidence matrix);

$f = (f_1, f_2, \dots, f_n)$ link-flow vector;

$c(f) = (c_1(f), c_2(f), \dots, c_n(f))$ the link-cost vector;

it results: $f = \Delta F$, $c(F) = \Delta' c(\Delta F)$.

Let us assume that there exist m positive numbers Γ_r, $r = 1, 2, \dots, m$, that represent the capacities on path-flow:

$$F_r \le \Gamma_r \quad r = 1, 2, \dots, m. \tag{2.1}$$

If we denote by γ_i, $i = 1, 2, \dots, n$, the capacities, of the links a_i, $i = 1, 2, \dots, n$, then it results:

$$\Delta\Gamma \le \gamma. \tag{2.2}$$

Now setting

$$K = \Big\{F \in R^m : 0 \le F_r \le \Gamma_r,\ r = 1, 2, \dots, m,$$
$$\sum_{r=1}^{m} \varphi_{jr} F_r = \rho_j,\ j = 1, 2, \dots, \ell\Big\}$$

and assuming that

$$\sum_{r=1}^{m} \varphi_{jr}\, \Gamma_r \ge \rho_j \quad j = 1, 2, \dots, \ell \tag{2.3}$$

we may give the following

Generalized user's equilibrium principle. *A vector $H \in K$ will be an equilibrium vector (from the user's point of view) when for every $\mathscr{R}_j$ and for every $R_r, R_s \in \mathscr{R}_j$ if it results*

$$C_r(H) > C_s(H) \quad \textit{and} \quad H_s < \Gamma_s$$

then

$$H_r = 0$$

and if

$$C_r(H) > C_s(H) \quad \textit{and} \quad H_s = \Gamma_s$$

then

$$H_r \ge 0.$$

Remark 2.1. Let H be an equilibrium vector according with the principle above. Let us set:

$$\begin{aligned} A_j =& \{r : 1 \le r \le m,\ \varphi_{jr} = 1\} \quad j = 1, \dots, \ell \\ B_j =& \{r \in A_j : H_r > 0\} \quad j = 1, \dots, \ell \\ D_j =& A_j \setminus B_j. \end{aligned}$$

and let us denote

$$C^j(H) = \max_{r \in B_j} C_r(H).$$

It is easy to show that:

If $r \in B_j$ *and* $C^r(H) < C^j(H)$, *then* $H_r = \Gamma_r$.

Remark 2.2. Let H be an equilibrium solution and let us set:

$$L_r^j(H) = C^j(H) - C_r(H) \quad r \in B_j \tag{2.4}$$

and

$$\widetilde{C}_r(H) = \begin{cases} C_r(H) + L_r^j(H) = C^j(H) & \text{if } r \in B_j \\ C_r(H) & \text{if } r \in D_j. \end{cases} \tag{2.5}$$

Then it is easy to show that the "Generalized Equilibrium Principle" can be expressed in the following way:

$H \in K$ *is an equilibrium vector if* $\forall \mathscr{R}_j$, $\forall R_r, R_s \in \mathscr{R}_j$ *if it results*

$$\widetilde{C}_r(H) > \widetilde{C}_s(H) \tag{2.6}$$

then

$$H_r = 0.$$

Proof. Let H be an equilibrium solution and let us assume that the inequality

$$\widetilde{C}_r(H) > \widetilde{C}_s(H)$$

holds. Then both the indexes r, s can not belong to B_j; moreover it is also impossible that it results $r \in B_j$, $s \in D_j$ because, in this situation, H_s must be greater than zero. Conversely if $r \in D_j$ and $s \in B_j$ the estimate above becomes

$$C_r(H) > C^j(H)$$

and

$$H_r = 0.$$

Now let us show

Theorem 2.1. *$H \in K$ is an equilibrium solution if and only if*

$$C(H)(F-H) \geq 0 \quad \forall F \in K. \tag{2.7}$$

Proof. Let H be an equilibrium solution and let us observe that

$$C(H)(F-H) = \sum_{j=1}^{\ell} \sum_{r \in A_j} C_r(H)(F_r - H_r) =$$

$$= \sum_{j=1}^{\ell} \left\{ \sum_{r \in B_j} C_r(H)(F_r - H_r) + \sum_{r \in D_j} C_r(H)(F_r - H_r) \right\} =$$

$$= \sum_{j=1}^{\ell} \left\{ \sum_{\substack{r \in B_j \\ L_r(H)>0}} C_r(H)(F_r - H_r) + \sum_{\substack{r \in B_j \\ L_r(H)=0}} C_r(H)(F_r - H_r) + \sum_{r \in D_j} C_r(H) F_r \right\} \geq$$

$$\geq \sum_{j=1}^{\ell} \left\{ \sum_{\substack{r \in B_j \\ L_r^j(H)>0}} \left(C^j(H) - L_r^j(H)\right)(F_r - H_r) + \sum_{\substack{r \in B_j \\ L_r^j(H)=0}} C^j(H)(F_r - H_r) + \right.$$

$$\left. + \sum_{r \in D_j} C^j(H) F_r \right\} = \sum_{j=1}^{\ell} C^j(H) \sum_{r \in Aj} (F_r - H_r) -$$

$$- \sum_{j=1}^{\ell} L_r^j(H) \sum_{\substack{r \in B_j \\ L_r^j(H)>0}} (F_r - H_r) = - \sum_{j=1}^{\ell} L_r^j(H) \sum_{r \in B_j} (F_r - \Gamma_r) \geq 0,$$

having taken into account the result of Remark 2.1.

Conversely let H a solution to (2.7) and suppose that there exist $w_j \in W$, $R_r, R_s \in \mathbb{R}_j$ such that

$$C_r(H) > C_s(H) \quad \text{and} \quad H_s < \Gamma_s.$$

Let us show that $H_r = 0$. To this and let us suppose $H_r > 0$ and let us consider the vector F, the components of which are such that

$$F_h = \begin{cases} H_h & \text{if } h \neq r, s \\ H_r - [\Gamma_s - H_s] & \text{if } r = h \quad \text{and} \quad H_r > \Gamma_s - H_s \\ \Gamma_s & \text{if } h = s. \end{cases}$$

Then $F \in K$ and we have the contradiction:

$$C(H)(F-H) = C_r(H)(H_r - (\Gamma_s - H_s) - H_r) + C_s(H)(\Gamma_s - H_s) =$$
$$= \left(C_s(H) - C_r(H)\right)(\Gamma_s - H_s) < 0.$$

Analogously if $0 < H_r < \Gamma_s - H_s$ it is enough to consider the vector F the components of which are:

$$F_h = \begin{cases} H_h & \text{if } h \neq r, s \\ 0 & \text{if } h = r \\ H_r + H_s & \text{if } h = s \end{cases}$$

and to repeat the same calculation.

3. A comparison with a standard model with side constraints.

T. Larsson and M. Patriksson consider the standard model, i.e. a model with separable travel link cost functions (we use our notations):

$$c_i = c_i(f_i) \quad i = 1, \dots, n \tag{3.1}$$

and recalls that a solution to Wardrop conditions can be found by solving the convex network optimization problem

$$\min T(f) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \int_0^{f_i} c_i(s)\, ds \tag{3.2}$$

subject to:

$$\sum_{r=1}^{m} \varphi_{jr} F_r = \rho_j \quad j = 1, \dots, \ell \tag{3.3}$$

$$F_r \geq 0 \quad r = 1, \dots, m \tag{3.4}$$

$$f_i = \sum_{r=1}^{m} \delta_{ir} F_r \quad i = 1, \dots, n. \tag{3.5}$$

To capture supplementary traffic flow relationship they introduce side constraints:

$$g_k(f) \leq 0 \quad k \in \mathscr{C} \tag{3.6}$$

where

$$g_k : \mathbb{R}^n_+ \to \mathbb{R} \quad k \in \mathscr{C}$$

are convex and continuously differentiable functions and $\mathscr{C}$ consists of the index set of the links, nodes, paths or O/D pairs, or any combination of subsets of them and he shows the following

Theorem 3.1. *Let C^j, $j = 1, \ldots, \ell$, and β_k, $k \in \mathscr{C}$, the Lagrange multipliers for the constraints* (3.3) *and* (3.6), *respectively. If (h, H) is a solution to the problem* (3.1) - (3.6) *then*

$$H_r > 0 \Longrightarrow \widetilde{C}_r(H) = C^j \quad j = 1, \ldots, \ell \tag{3.7}$$

$$H_r = 0 \Longrightarrow \widetilde{C}_r(H) \geq C^j \quad j = 1, \ldots, \ell \tag{3.8}$$

where

$$\widetilde{C}_r(H) = C_r(H) + \sum_{i=1}^{n} \delta_{ir} \sum_{k \in \mathscr{C}} \mathscr{B}_k \frac{\partial g_k(f)}{\partial f_i}. \tag{3.9}$$

From this result, the authors deduce that the optimality conditions of the problem (3.1)-(3.6) give rise to a Wardrop equilibrium principle in terms of *generalized path travel costs*.
The proof is obtained by considering the stationary point conditions for the Lagrangean function

$$L(f, \beta) = \sum_{i=1}^{n} \int_0^{f_i} c_i(s)\, ds + \sum_{k \in \mathscr{C}} \beta_k g_k(f) -$$
$$- \sum_{r=1}^{m} \lambda_r F_r - \sum_{i=1}^{n} \Big(f_i - \sum_{r=1}^{m} \delta_{ir} F_r\Big)\mu_i - \sum_{i=1}^{\ell} \Big(\sum_{r=1}^{m} \varphi_{jr} F_r - \rho_j\Big) C^j$$

where λ_r, μ_i are the Lagrange multipliers for the constraints (3.4) and (3.5) respectively.
In the case of simple upper bounds on the link flows

$$f_i - \gamma_i \leq 0 \quad i = 1, \ldots, n$$

(3.9) reduces to

$$\widetilde{C}_r(H) = C_r(H) + \sum_{i=1}^{n} \delta_{ir} \beta_i. \tag{3.10}$$

From a comparison with (2.5) we obtain (the optimal multipliers are not necessary unique):

$$\sum_{i=1}^{n} \delta_{i_k} \beta_i = L_r(H) \quad \text{for} \quad r \in B_j$$

$$\sum_{i=1}^{n} \delta_{ir} \beta_i = 0 \quad \text{for} \quad r \in D_j$$

and it is possible to obtain an interpretation of the optimal Lagrange multipliers. In fact

$$\sum_{i=1}^{n} \delta_{ik}\beta_i$$

can be considered as the optimal Lagrange multipliers with respect to the path and, then, we may conclude that they represent an *Additional equilibrium cost on saturated paths.*

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