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ON GENERAL NONLINEAR COMPLEMENTARITY PROBLEMS AND QUASI-EQUILIBRIA

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In this note we compare different approaches for establishing solvability of nonlinear complementarity problems, quasi-variational inequalities, and quasi-equilibrium problems.

1. Introduction and problem setting.

The classical case of a *complementarity problem* consists in finding $\overline{u} \in H$ such that

$$\overline{u} \in K, \; T(\overline{u}) \in K^*, \; \langle \overline{u}, T(\overline{u}) \rangle = 0. \tag{1}$$

Here H is a real Hilbert space, $T : H \to H$ is a given mapping, $K \subset H$ is a closed convex cone, and $K^* := \{z \in H \mid \langle x, z \rangle \geq 0 \quad \forall x \in K\}$ is the polar cone of K. In most applications, $H = \mathbb{R}^n$ and $K = K^* = \mathbb{R}^n_+$. This problem has been studied by many authors, and we refer to the references for further information.

In this paper we are mainly interested in the so-called *quasi-complementarity problem* of finding $\overline{u} \in H$ such that

$$\overline{u} \in C, \; S(\overline{u}) \in K, \; T(\overline{u}) \in K^*, \; \langle S(\overline{u}), T(\overline{u}) \rangle = 0. \tag{2}$$

Here $C \subset H$ is a convex set, and $S : H \to H$ is another mapping. The set C is sometimes introduced artificially to compactify a given problem. Note that problem (2) is symmetric with respect to S, K on one hand and T, K^* on the other hand, since $K^{**} = K$. It seems therefore appropriate to call (2) the symmetric nonlinear complementarity problem (for linear complementarity problems, symmetry has a different meaning).

In fact, whenever possible we formulate the results for a still more general quasi-complementarity problem, namely to find $\overline{u}, \overline{x} \in X$ and $\overline{y} \in X^*$ such that

$$\overline{u} \in C,\ \overline{x} \in K \cap S(\overline{u}),\ \overline{y} \in K^* \cap T(\overline{u}),\ \langle \overline{x}, \overline{y} \rangle = 0. \tag{3}$$

Here X is a real topological vector space, X^* its topological dual (endowed with the weak* topology), $C \subset X$ is a given set, $S : C \rightrightarrows X$ and $T : C \rightrightarrows X^*$ are multivalued mappings ([1]), $K \subset X$ is a closed convex cone, and

$$K^* := \{y \in X^* \mid \langle x, y \rangle \geq 0 \quad \forall x \in K\}$$

its polar cone. If $S = Id$ (the identity mapping), then (3) leads back to a multivalued variant of (1). We consider problem (3) as the pilot problem within this framework.

Problem (1) is a special case of a *variational inequality problem.* In the latter one has to find $\overline{u} \in X$, $\overline{y} \in X^*$ such that

$$\overline{u} \in C,\ \overline{y} \in T(\overline{u}),\ \langle u - \overline{u}, \overline{y} \rangle \geq 0 \quad \forall u \in C. \tag{4}$$

If K is a cone, then one has for arbitrary $\overline{u} \in K$ the equivalence

$$\big(\overline{y} \in K^*,\ \langle \overline{u}, \overline{y} \rangle = 0\big) \iff (\langle u - \overline{u}, \overline{y} \rangle \geq 0 \quad \forall u \in K). \tag{5}$$

So Problem (1) is equivalent with $\overline{u} \in K$, $\langle u - \overline{u}, T(\overline{u}) \rangle \geq 0 \quad \forall u \in K$, and is therefore a special case of (4). The general quasi-complementarity problem (3) on the other hand is more properly related to what is called a *quasi-variational inequality problem.* In the latter one has to determine $\overline{u} \in X$, $\overline{y} \in X^*$ such that

$$\overline{u} \in C,\ \overline{u} \in K(\overline{u}),\ \overline{y} \in T(\overline{u}),\ \langle u - \overline{u}, \overline{y} \rangle \geq 0 \quad \forall u \in K(\overline{u}). \tag{6}$$

Here $C \subset X$, $K : C \rightrightarrows X$, $T : C \rightrightarrows X^*$. With $K(u) \equiv C$, (6) leads back to (4). If we set $K(u) := u - S(u) + K$ in (6), then for every solution $\overline{u}$, $\overline{y}$ of (6) there exists $\overline{x}$ such that $\overline{u}, \overline{x}, \overline{y}$ is a solution of (3); this follows easily by means of (5)

([1]) For multivalued mappings we follow the terminology of Berge [4], except that upper semicontinuity of a mapping $T(\cdot)$ does not imply the compactness of $T(x)$.

(see also the proof of Theorem 4 below). Conversely, if $S(\cdot)$ is single-valued and $\overline{u}, \overline{y}$ solves (3), then $\overline{u}, \overline{y}$ is also a solution of (6) under this choice of $K(\cdot)$.

Problems (4) and (6) can be further generalized to equilibrium problems and quasi-equilibrium problems, respectively. In the *quasi-equilibrium problem* one has to find $\overline{u} \in X$, $\overline{y} \in Y$ such that

$$\text{(7)} \qquad \overline{u} \in C,\ \overline{u} \in K(\overline{u}),\ \overline{y} \in T(\overline{u}),\ f(v, \overline{y}) \geq f(\overline{u}, \overline{y}) \quad \forall v \in K(\overline{u}).$$

Here Y is another topological vector space, $C \subset X$, $K : C \rightrightarrows X$, $T : C \rightrightarrows Y$, $f : C \times Y \to \mathbb{R}$. If $K(u) \equiv C$, then we call (7) an *equilibrium problem.* It is clear that (4) is an equilibrium problem, and (6) is a quasi-equilibrium problem. Quasi-equilibria constitute also an extension of Nash equilibria, which are of fundamental importance in the theory of noncooperative games.

Under additional assumptions Problem (7) can be represented more concisely. Namely, let $\psi : C \times C \to \mathbb{R}$ be defined through

$$\psi(u, v) := \sup_{y \in T(u)} \left(f(v, y) - f(u, y) \right).$$

Then it is obvious, if $\overline{u}, \overline{y}$ solves (7), that $\overline{u}$ is a solution of

$$\text{(8)} \qquad \overline{u} \in C,\ \overline{u} \in K(\overline{u}),\ \psi(\overline{u}, v) \geq 0 \quad \forall v \in K(\overline{u}).$$

Conversely, if $\overline{u}$ solves (8), and if we assume that $K(\overline{u})$ is convex, $T(\overline{u})$ is convex, compact, $\neq \emptyset$, $f(v, y) - f(\overline{u}, y)$ is convex in v, and concave and upper semicontinuous in y, then there exists $\overline{y} \in T(\overline{u})$ such that $\overline{u}, \overline{y}$ solves (7). This is non-trivial (except when T is single-valued), and requires tools from convex analysis; see [5], Lemma 1.

In Section 2 we derive solvability results for (7),(6),(3) from a topological fixed point theorem. In Section 3 we treat the case when T is monotone; here our tool is convex separation in $\mathbb{R}^n$. In Section 4 we treat problem (6) with T single-valued and monotone by means of projection mappings and Banach's fixed point theorem.

2. Existence results.

In this section we obtain results about the existence of solutions of variational inequalities using a topological fixed point theorem. Let us work within the setting of two real, locally convex, separated topological vector spaces X and Y, and let us first investigate the case of a symmetric quasi-equilibrium, in the sense of (9) below.

Theorem 1. *For* $C \subset X$, $D \subset Y$, $S : C \times D \rightrightarrows C$, $T : C \times D \rightrightarrows D$, $f : C \times D \to \mathbb{R}$, $g : C \times D \to \mathbb{R}$ *let the following assumptions hold:*

(i) *C and D are nonempty, compact and convex;*
(ii) *S and T are upper semicontinuous with nonempty, compact, convex values;*
(iii) *f and g are lower semicontinuous; $f(\cdot, y)$ and $g(x, \cdot)$ are quasi-convex;*
(iv) *the functions*

$$F(x, y) := \min\{f(\xi, y) \mid \xi \in S(x, y)\},$$
$$G(x, y) := \min\{g(x, \eta) \mid \eta \in T(x, y)\}$$

are upper semicontinuous on $C \times D$.

Then there exists $(\overline{x}, \overline{y}) \in C \times D$ such that

$$\begin{aligned} &\overline{x} \in S(\overline{x}, \overline{y}), \;\; f(x, \overline{y}) \geq f(\overline{x}, \overline{y}) \quad \forall x \in S(\overline{x}, \overline{y}), \\ &\overline{y} \in T(\overline{x}, \overline{y}), \;\; g(\overline{x}, y) \geq g(\overline{x}, \overline{y}) \quad \forall y \in T(\overline{x}, \overline{y}). \end{aligned} \tag{9}$$

Proof. Define multivalued mappings $A : C \times D \rightrightarrows C$, $B : C \times D \rightrightarrows D$ by

$$A(x, y) := \{\xi \in S(x, y) \mid f(\xi, y) = F(x, y)\},$$
$$B(x, y) := \{\eta \in T(x, y) \mid g(x, \eta) = G(x, y)\}.$$

Then $A(x, y)$ is nonempty, compact and convex for all $(x, y) \in C \times D$. Since S and F are upper semicontinuous and f is lower semicontinuous, the mapping A is upper semicontinuous; this follows from a close examination of the proof of Berge's maximum theorem [4], p. 123. The mapping B has the same properties, and we conclude that the mapping $\phi : C \times D \rightrightarrows C \times D$, defined by $\phi(x, y) := (A \times B)(x, y)$, is upper semicontinuous and has nonempty, compact, convex values. Thus ϕ satisfies the requirements of Ky Fan's theorem [13], p. 109, which implies that there exists a fixed point $(\overline{x}, \overline{y}) \in \phi(\overline{x}, \overline{y})$.

Then $(\overline{x}, \overline{y})$ satisfies (9). □

Remark 1. Let us briefly discuss assumption (iv). The upper semicontinuity of F and G is ensured whenever S and T are l.s.c. and f and g are u.s.c.. This follows from Théorème 1 in [4], p. 122. If $f \equiv 0$ and $g \equiv 0$, then F and G are trivially u.s.c.; Theorem 1 becomes then a pure fixed point result for the mapping $(S \times T)(x, y)$. If $S(x, y) \equiv C$ and f is u.s.c. in the second argument, then F is independent of x and u.s.c. in y, being the infimum of a family of u.s.c. functions.

In somewhat less general form Theorem 1 is well-known in Game Theory; compare [2], p. 282.

Theorem 2. *For $C \subset X$, $D \subset Y$, $S : C \rightrightarrows C$, $T : C \rightrightarrows D$, $f : C \times D \to \mathbb{R}$ let the following assumptions hold:*

(i) *C and D are nonempty, compact and convex;*
(ii) *T is upper semicontinuous with nonempty, compact, convex values;*
(iii) *S is continuous with nonempty, compact, convex values;*
(iv) *$f(\cdot,\cdot)$ is quasiconvex in the first argument, and continuous jointly in both arguments.*

Then there exists $(\overline{x}, \overline{y}) \in C \times D$ such that

$$\overline{x} \in S(\overline{x}),\ \overline{y} \in T(\overline{x}),\ f(x, \overline{y}) \geq f(\overline{x}, \overline{y}) \quad \forall x \in S(\overline{x}). \tag{10}$$

Proof. The conclusion of the theorem is an immediate consequence of Theorem 1, where $g \equiv 0$, together with Remark 1. □

By making use of an additional coercivity condition it is possible to weaken the compactness assumption concerning $S(x)$. The function f, however, must then be convex in the first argument. Let us recall before that, given two subsets A and B of some topological vector space with A convex and $A \subset B$, the core of A relative to B, denoted by $\mathrm{core}_B A$, consists of all elements $a \in A$ such that

$$(a, b] \cap A \neq \emptyset \quad \text{for all} \quad b \in B \setminus A.$$

We note the following: *If $a \in \mathrm{core}_B A$ and $\psi(\cdot)$ is convex, then*

$$\psi(a) \leq \psi(x) \quad \forall x \in A$$

implies

$$\psi(a) \leq \psi(x) \quad \forall x \in B.$$

Indeed, if $b \in B$ and $\psi(b) < \psi(a)$, then from convexity of ψ, $\psi(x) < \psi(a)$ for all $x \in (a, b]$, and therefore also $\psi(x) < \psi(a)$ for some $x \in (a, b] \cap A \neq \emptyset$, a contradiction with the premissa.

Theorem 3. *For $C \subset X$, $D \subset Y$, $T : C \rightrightarrows D$, $S : C \rightrightarrows X$, $f : C \times D \to \mathbb{R}$, $\widetilde{S}(x) := S(x) \cap C$ let the following assumptions hold:*

(i) *C and D are nonempty, compact and convex;*
(ii) *T is upper semicontinuous, and has nonempty, compact, convex values;*
(iii) *$\widetilde{S}$ is continuous, and has nonempty, compact, convex values;*
(iv) *f is convex in the first argument, and continuous jointly in both arguments;*
(v) *for all $x \in C$ with $x \in \widetilde{S}(x)$ and all $y \in T(x)$ there exists $\xi \in \mathrm{core}_{S(x)} \widetilde{S}(x)$ such that $f(\xi, y) \leq f(x, y)$.*

Then (10) *has a solution.*

Proof. By Theorem 2, where we replace $S(\cdot)$ by $\widetilde{S}(\cdot)$, we obtain the existence of $\overline{x} \in \widetilde{S}(\overline{x})$, $\overline{y} \in T(\overline{x})$ such that

$$f(\overline{x}, \overline{y}) \leq f(x, \overline{y}) \quad \forall x \in \widetilde{S}(\overline{x}).$$

Now (v) provides $\overline{\xi} \in \widetilde{S}(\overline{x})$ such that

$$f(\overline{\xi}, \overline{y}) = f(\overline{x}, \overline{y}),$$

hence

$$f(\overline{\xi}, \overline{y}) \leq f(x, \overline{y}) \quad \forall x \in \widetilde{S}(\overline{x}).$$

But since $\overline{\xi} \in \operatorname{core}_{S(\overline{x})} \widetilde{S}(\overline{x})$ and f is convex in the first argument, this inequality even holds true for all $x \in S(\overline{x})$, which implies

$$f(\overline{x}, \overline{y}) \leq f(x, \overline{y}) \quad \forall x \in S(\overline{x}). \qquad \square$$

Note that assumption (v) is trivially satisfied for all $x \in C$ with $x \in \operatorname{core}_{S(x)} \widetilde{S}(x)$, since we may then choose $\xi := x$ independent of y.

Let us now consider the general quasi-complementarity problem, which comes out from (11) below when K is a cone. We assume in the remainder of this section that X is a barrelled (e.g. Banach) space, and that X^* bears the weak* topology.

Theorem 4. *For $K \subset X$, $C \subset X$, $S : C \rightrightarrows X$, $T : C \rightrightarrows X^*$, $K(u) := u - S(u) + K$, $\widetilde{K}(u) := K(u) \cap C$ let the following assumptions hold:*

(i) *C is nonempty, compact and convex;*
(ii) *T is upper semicontinuous and has nonempty, convex, compact values;*
(iii) *$\widetilde{K}$ is continuous and has nonempty, convex, compact values;*
(iv) *for all $u \in C$ with $u \in \widetilde{K}(u)$ and for all $y \in T(u)$ there exists $v \in \operatorname{core}_{K(u)} \widetilde{K}(u)$, such that $\langle v - u, y \rangle \leq 0$.*

Then there exist $\overline{u}, \overline{x} \in X$ and $\overline{y} \in X^*$ such that

$$\text{(11)} \qquad \overline{u} \in C, \ \overline{x} \in K \cap S(\overline{u}), \ \overline{y} \in T(\overline{u}), \ \langle x - \overline{x}, \overline{y} \rangle \geq 0 \quad \forall x \in K.$$

Proof. Due to the fact that T is upper semicontinuous and C is compact, $T(C)$ is also compact [4], p. 116; hence cl conv $T(C)$ is nonempty, convex and – since X is barrelled and X^* bears the weak* topology – again compact. So, setting $Y := X^*$, $D :=$ cl conv $T(C)$, we have found a set D as required in Theorem 3.

From Theorem 3, where $S(\cdot)$ is replaced by $K(\cdot)$, we obtain $\overline{u} \in X$, $\overline{y} \in X^*$ satisfying

$$\overline{u} \in C,\ \overline{u} \in K(\overline{u}),\ \overline{y} \in T(\overline{u}),\ \langle u - \overline{u}, \overline{y}\rangle \geq 0 \quad \forall u \in K(\overline{u}).$$

In view of the definition of $K(u)$ there exists $\overline{x} \in S(\overline{u})$, such that

$$\overline{u} \in \overline{u} - \overline{x} + K,\ \langle u - \overline{u}, \overline{y}\rangle \geq 0 \quad \forall u \in \overline{u} - \overline{x} + K.$$

It follows that $\overline{x} \in K \cap S(\overline{u})$ and $\langle x - \overline{x}, \overline{y}\rangle \geq 0 \quad \forall x \in K$. Altogether we have obtained a solution of (11). $\square$

In particular, if S is the identity mapping and $C \subset K$, we have $K(u) \equiv K$, and $\widetilde{K}(u) \equiv C$. Thus we obtain the following

Corollary 1. *Let the following assumptions hold:*

(i) $K \subset X$ *is convex;*
(ii) $T : K \rightrightarrows X^*$ *is upper semicontinuous and has nonempty, convex, compact values;*
(iii) *there exists* $C \subset K$ *convex, compact, nonempty such that for all* $u \in C \setminus \operatorname{core}_K C$ *there exists* $v \in \operatorname{core}_K C$ *satisfying* $\sup_{y \in T(u)} \langle v - u, y\rangle \leq 0$.

Then there exist $\overline{u} \in C$, $\overline{y} \in T(\overline{u})$ *such that* $\langle v - \overline{u}, \overline{y}\rangle \geq 0 \quad \forall v \in K$.

Note that assumption (iii) is trivially fulfilled if K is compact. Indeed, we simply set $C := K$ and note that $K \setminus \operatorname{core}_K K = \emptyset$.

Let us mention that Theorem 4 and Corollary 1 remain valid if X is a normed space and X^* bears the norm topology; X^* is then complete, and so the compactness of $T(C)$ implies again the compactness of $\operatorname{cl}\operatorname{conv} T(C)$.

3. The monotone case.

Working with monotone mappings makes it possible to weaken the continuity assumptions concerning T. Moreover in this case we can use separation results instead of fixed point results. Recall that a multivalued mapping $T : X \rightrightarrows X^*$ is called *monotone* iff

$$(12) \qquad \langle x_1 - x_2, \xi_1 - \xi_2\rangle \geq 0 \quad \forall (x_1, \xi_1), (x_2, \xi_2) \in \operatorname{graph} T.$$

In the proof of the next theorem we shall use the following lemma named after Fan-Glicksberg-Hoffman.

Lemma. *If Γ is a convex set and $f_i : \Gamma \to \mathbb{R}$ $(i = 1, \ldots, n)$ are convex functions such that* $\max_i f_i(x) \geq 0 \, \forall x \in \Gamma$, *then there exist real numbers $u_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_i u_i = 1$ such that $\sum_i u_i f_i(x) \geq 0 \ \ \forall x \in \Gamma$.*

The lemma is an immediate consequence of the separation theorem for convex sets in $\mathbb{R}^n$, for if $\max_i f_i(x) \geq 0$ on Γ, then $0 \in \mathbb{R}^n$ is not an interior point of the convex set $D := \{z \in \mathbb{R}^n \mid x \in \Gamma, \ z_i \geq f_i(x) \ (i = 1, \ldots, n)\}$, hence can be weakly separated from D by a linear functional $\langle u, \cdot \rangle$ with $u \neq 0$ [31].

Theorem 5. *Let X, Y be real topological vector spaces. For $C \subset K \subset X$, $T : C \rightrightarrows Y$, $f : K \times C \times Y \to \mathbb{R}$ let the following assumptions hold:*

(i) *K is convex; C is compact, convex, nonempty;*

(ii) *T is upper semicontinuous on $[\overline{x}, x]$ for all $\overline{x}, x \in C$; T has compact, convex, nonempty values;*

(iii) *for all $\overline{x} \in C$, $x \in C$, $y \in Y$,*
$f(\cdot, \overline{x}, y)$ is convex and lower semicontinuous,
$f(x, \overline{x}, \cdot)$ is concave,
$f(x, \cdot, \cdot)$ is upper semicontinuous on $[\overline{x}, x] \times Y$;

(iv) *$f(x, x, y) = 0$ for all $(x, y) \in$* graph T; *f and T are jointly monotone in the sense that $f(x_2, x_1, y_1) + f(x_1, x_2, y_2) \leq 0 \quad \forall (x_1, y_1), (x_2, y_2) \in$* graph T;

(v) *for all $x \in C$ and all $y \in T(x)$ there exists $\xi \in \mathrm{core}_K C$ such that $f(\xi, x, y) \leq 0$.*

Then there exists $\overline{x} \in C$, $\overline{y} \in T(\overline{x})$ such that $f(x, \overline{x}, \overline{y}) \geq 0$ for all $x \in K$.

Proof. Let $(x_i, y_i) \in \mathrm{graph}\, T$ $(i = 1, \ldots, n)$ be chosen arbitrarily. The lower semicontinuous function $\max_i f(\cdot, x_i, y_i)$ assumes its minimum on the compact set $\Sigma := \mathrm{conv}\{x_1, \ldots, x_n\}$ in some point $\xi \in \Sigma$, say. Let $m := \max_i f(\xi, x_i, y_i)$. Then $\max_i f(x, x_i, y_i) \geq m \ \forall x \in \Sigma$. By the lemma of Fan-Glicksberg-Hoffman follows the existence of $u_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_i u_i = 1$ and $\sum_i u_i f(x, x_i, y_i) \geq m \quad \forall x \in \Sigma$. Let $\widetilde{x} := \sum_i u_i x_i$. Then $\widetilde{x} \in \Sigma$, and therefore

$$m \leq \sum_i u_i f(\widetilde{x}, x_i, y_i) \leq \sum_i \sum_j u_i u_j f(x_j, x_i, y_i) =$$
$$= \frac{1}{2} \sum_{i,j} u_i u_j \left(f(x_j, x_i, y_i) + f(x_i, x_j, y_j) \right) \leq 0.$$

From $\max_i f(\xi, x_i, y_i) = m \leq 0$ follows then

$$f(\xi, x_i, y_i) \leq 0 \quad (i = 1, \ldots, n).$$

Let $S(x, y) := \{\xi \in C \mid f(\xi, x, y) \leq 0\}$, $(x, y) \in \operatorname{graph} T$. We have just proved that any finite collection of the closed sets $S(x, y)$ has nonempty intersection. Since C is compact, the entire family has nonempty intersection. So there exists $\overline{x} \in C$ satisfying

$$f(\overline{x}, x, y) \leq 0 \,\forall (x, y) \in \operatorname{graph} T.$$

Let $x \in C$ be arbitrary and set $x(t) := \overline{x} + t(x - \overline{x})$ where $0 \leq t \leq 1$. We have just proved that $f(\overline{x}, x(t), y) \leq 0$ for all $y \in T(x(t))$. Therefore for all $y \in T(x(t))$ we have

$$0 = f(x(t), x(t), y) \leq tf(x, x(t), y) + (1 - t)f(\overline{x}, x(t), y) \leq tf(x, x(t), y).$$

Hence $f(x, x(t), y) \geq 0$ for all $y \in T(x(t))$, provided $t > 0$. This means that the function $\varphi(t) := \max\{f(x, x(t), y) | y \in T(x(t))\}$ $(0 \leq t \leq 1)$ satisfies $\varphi(t) \geq 0$ for $0 < t \leq 1$. Since T is u.s.c. on $[\overline{x}, x]$ and $f(x, \cdot, \cdot)$ is u.s.c. on $[\overline{x}, x] \times Y$ it follows from Berge's maximum theorem [4] that φ is u.s.c. on $[0, 1]$. Therefore we obtain $\varphi(0) \geq 0$, i.e., $\max\{f(x, \overline{x}, y) | y \in T(\overline{x})\} \geq 0$. So for every $x \in C$ there exists $y \in T(\overline{x})$ such that $f(x, \overline{x}, y) \geq 0$. Assume now, for contradiction, that the family of closed sets $F(x, \varepsilon) := \{y \in T(\overline{x}) \mid f(x, \overline{x}, y) \geq -\varepsilon\}$, where $x \in C$ and $\varepsilon > 0$, has empty intersection on $T(\overline{x})$. Then, since $T(\overline{x})$ is compact, there exist $x_1, \ldots, x_n \in C$ and $\varepsilon_1 > 0, \ldots, \varepsilon_n > 0$ such that $\bigcap_{i=1}^n F(x_i, \varepsilon_i) = \emptyset$. With $\overline{\varepsilon} := \min_i \varepsilon_i$ this implies that $\min_i f(x_i, \overline{x}, y) \leq -\overline{\varepsilon} \,\forall y \in T(\overline{x})$. By the lemma of Fan-Glicksberg-Hoffman we obtain $u_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_i u_i = 1$ such that $\sum_i u_i f(x_i, \overline{x}, y) \leq -\overline{\varepsilon} \,\forall y \in T(\overline{x})$. Let $\widetilde{x} := \sum_i u_i x_i$. Then $\widetilde{x} \in C$ and $f(\widetilde{x}, \overline{x}, y) \leq -\overline{\varepsilon}$ for all $y \in T(\overline{x})$, in contradiction to what has already been proved. Hence the family of sets $F(x, \varepsilon)$ has nonempty intersection. This means the existence of $\overline{y} \in T(\overline{x})$ such that

$$f(x, \overline{x}, \overline{y}) \geq 0 \quad \forall x \in C.$$

By assumption there exists then $\xi \in \operatorname{core}_K C$ with $f(\xi, \overline{x}, \overline{y}) = 0$, hence $f(\xi, \overline{x}, \overline{y}) \leq f(x, \overline{x}, \overline{y}) \,\forall x \in C$. Since $\xi \in \operatorname{core}_K C$, and $f(\cdot, \overline{x}, \overline{y})$ is convex, the latter inequality holds even true for all $x \in K$. Hence we have

$$f(x, \overline{x}, \overline{y}) \geq 0 \quad \forall x \in K. \qquad \square$$

Remark 2. We note that under the assumptions of Theorem 5 for every $\overline{x} \in C$ the following statements are equivalent:

(a) $\quad (\exists \overline{y} \in T(\overline{x}))\,(\forall x \in C)\, f(x, \overline{x}, \overline{y}) \geq 0;$

(b) $\quad (\forall x \in C)\,(\forall y \in T(x))\, f(\overline{x}, x, y) \leq 0.$

Indeed, the implication (b)$\Longrightarrow$(a) has been shown in the proof of Theorem 5, and the implication (a)$\Longrightarrow$(b) follows from the joint monotonicity of f and T.

From now on let X be locally convex and separated.

Theorem 6. *For $C \subset K \subset X$, $T : C \rightrightarrows Y$, $S : C \rightrightarrows K$, $f : K \times C \times Y \to \mathbb{R}$, $\widetilde{S}(x) := S(x) \cap C$ let assumptions* (i)-(iv) *of Theorem 5 hold. Assume furthermore that:*

(v) *$\widetilde{S}$ is upper semicontinuous and has compact, convex, nonempty values;*
(vi) *the function $H(u, \overline{x}) := \sup\{f(\overline{x}, x, y) | x \in \widetilde{S}(u), y \in T(x)\}$ is lower semicontinuous on $C \times C$;*
(vii) *for all $x \in C$ with $x \in \widetilde{S}(x)$ and all $y \in T(x)$ there exists $\xi \in \mathrm{core}_{S(x)}\widetilde{S}(x)$ such that $f(\xi, x, y) \leq 0$.*

Then there exists $(\overline{u}, \overline{y}) \in C \times Y$ such that

$$\overline{u} \in S(\overline{u}), \ \overline{y} \in T(\overline{u}), \ f(x, \overline{u}, \overline{y}) \geq 0 \quad \forall x \in S(\overline{u}).$$

Proof. For every $u \in C$ there exists $\overline{x} \in C$, $\overline{y} \in T(\overline{x})$ such that

$$\overline{x} \in \widetilde{S}(u), \ f(x, \overline{x}, \overline{y}) \geq 0 \quad \forall x \in \widetilde{S}(u). \tag{13}$$

This follows from Theorem 5, where we replace K and C by $\widetilde{S}(u)$. Let $Q(\cdot) : C \rightrightarrows C$ denote the mapping which assigns to each $u \in C$ the set of all $\overline{x} \in C$ such that (13) is satisfied with some $\overline{y} \in T(\overline{x})$. Then $Q(u) \neq \emptyset$ for all $u \in C$. By Remark 2, $Q(u)$ has the equivalent representation

$$Q(u) = \{\overline{x} \in \widetilde{S}(u) \mid f(\overline{x}, x, y) \leq 0 \quad \forall x \in \widetilde{S}(u), \ \forall y \in T(x)\}.$$

From this follows that $Q(u)$ is compact and convex, and

$$Q(u) = \{\overline{x} \in \widetilde{S}(u) \mid H(u, \overline{x}) \leq 0\}.$$

Since $\widetilde{S}(\cdot)$ is upper semicontinuous and H is lower semicontinuous, we conclude that $Q(\cdot)$ is upper semicontinuous. Hence Q has a fixed point $\overline{u} \in Q(\overline{u})$.

Then $\overline{u} \in \widetilde{S}(\overline{u})$, and from (13) there exists $\overline{y} \in T(\overline{u})$ such that $f(x, \overline{u}, \overline{y}) \geq 0$ for all $x \in \widetilde{S}(\overline{u})$. Assumption (vii) permits to extend the latter inequality to all $x \in S(\overline{u})$. □

Remark 3. If $S(u) \equiv C$, then assumption (vi) is automatically satisfied: $H(u, \overline{x})$ is then independent of u and is lower semicontinuous in $\overline{x}$, since it is the supremum of a family of functions which are lower semicontinuous in $\overline{x}$.

We choose now $Y := X^*$, and we assume that X^* bears the weak* topology. Then the function $\langle\cdot,\cdot\rangle : X\times X^* \to \mathbb{R}$ is continuous in each argument separately, and jointly continuous in both arguments on $L\times X^*$, where $L \subset X$ is an arbitrary line. So we may choose $f(\xi, x, y) := \langle \xi - x, y\rangle$ in Theorem 6, and the requested continuity assumptions on f are satisfied. Moreover, if T is monotone, then f and T are jointly monotone in the sense of Theorem 5. Recall that $T : C \rightrightarrows X^*$ is said to be weakly hemicontinuous iff, for every line segment $L \subset C$, T is upper semicontinuous from L into X^*, when X^* is endowed with the weak* topology. Then the next result follows from Theorem 6 in the same way as Theorem 4 followed from Theorem 3.

Theorem 7. *For $K \subset X$, $C \subset X$, $S : C \rightrightarrows X$, $T : C \rightrightarrows X^*$, $K(u) := u - S(u) + K$, $\widetilde{K}(u) := K(u) \cap C$ let the following assumptions hold:*

(i) *C is compact, convex, nonempty;*

(ii) *T is monotone, weakly hemicontinuous, and has compact, convex, nonempty values;*

(iii) *$\widetilde{K}$ is upper semicontinuous and has compact, convex, nonempty values; the function $H(u,\overline{x}) := \sup\{\langle \overline{x} - x, y\rangle \mid x \in \widetilde{K}(u),\ y \in T(x)\}$ is lower semicontinuous on $C \times C$;*

(iv) *for all $u \in C$ with $u \in \widetilde{K}(u)$ and for all $y \in T(u)$ there exists $v \in \operatorname{core}_{K(u)}\widetilde{K}(u)$ such that $\langle v - u, y\rangle \le 0$.*

Then there exist $\overline{u}, \overline{x} \in X$ and $\overline{y} \in X^*$ such that

$$\overline{u} \in C,\ \overline{x} \in K \cap S(\overline{u}),\ \overline{y} \in T(\overline{u}),\ \langle x - \overline{x}, \overline{y}\rangle \ge 0 \quad \forall x \in K.$$

For comparison with Corollary 1 we note the following special case of both Theorem 5 and Theorem 7.

Corollary 2. *Let the following assumptions hold:*

(i) *$K \subset X$ is convex;*

(ii) *$T : K \rightrightarrows X^*$ is monotone, weakly hemicontinuous, and has compact, convex, nonempty values;*

(iii) *there exists $C \subset K$ compact, convex, nonempty such that for all $u \in C \setminus \operatorname{core}_K C$ there exists $v \in \operatorname{core}_K C$ satisfying $\sup\limits_{y\in T(u)} \langle v - u, y\rangle \le 0$.*

Then there exist $\overline{u} \in C$, $\overline{y} \in T(\overline{u})$ such that $\langle v - \overline{u}, \overline{y}\rangle \ge 0 \quad \forall v \in K$.

Remark 4. Set $\psi(u,v) := \sup\limits_{y\in T(u)} \langle v - u, y\rangle$. Then $\psi(u,u) = 0 \quad \forall u \in K$, and $\psi(u,\cdot)$ is convex and lower semicontinuous. If X is a reflexive Banach space provided with the weak topology (which preserves the requested continuity properties of $\langle\cdot,\cdot\rangle$) and K is closed, then each of the following conditions (a)–(e) (voidly satisfied if K is bounded) is sufficient for assumption (iii) in Corollary 2.

(a) *There exists $a \in K$ and $R > 0$ such that $\psi(u,a) \leq 0$ for all $u \in K$ with $\|u-a\| = R$.*

Indeed, if we set $C := \{u \in K \mid \|u-a\| \leq R\}$, then C is weakly compact, and $a \in \mathrm{core}_K C$. Moreover $u \in C \setminus \mathrm{core}_K C \Longrightarrow \|u-a\| = R \Longrightarrow \psi(u,a) \leq 0$. We can choose $v := a$ independent of u to obtain $\psi(u,a) \leq 0$ for all $u \in C \setminus \mathrm{core}_K C$.

(b) *There exists $a \in K$ such that $\psi(u,a) \to -\infty$ if $\|u-a\| \to +\infty$, $u \in K$.*

Indeed, (b) implies (a).

(c) *There exists $a \in K$ such that $(\psi(u,a) + \psi(a,u))/\|u-a\| \to -\infty$ if $\|u-a\| \to +\infty$, $u \in K$.*

Indeed, $\alpha := \min\{\psi(a,u) \mid u \in K, \ \|u-a\| \leq 1\}$ is finite, and from convexity of $\psi(a,\cdot)$ and $\psi(a,a) = 0$ follows $\psi(a,u) \geq \alpha\|u-a\|$ for all $u \in K$ with $\|u-a\| \geq 1$. Hence, for $\|u-a\| \to +\infty$,

$$\psi(u,a)/\|u-a\| \leq (\psi(u,a) + \psi(a,u))/\|u-a\| - \alpha \to -\infty,$$

and therefore $\psi(u,a) \to -\infty$. So (c) implies (b).

(d) *There exists $a \in K$ and $\gamma > 0$ such that $\alpha := \inf_{u \in K} \psi(a,u) > -\infty$ and $(\psi(u,a) + \psi(a,u))/\|u-a\| \leq -\gamma$ for all $u \in K$ with $\|u-a\| \geq 1$.*

Indeed, (d) implies (b), since for $\|u-a\| \to +\infty$,

$$\psi(u,a) \leq -\gamma\|u-a\| - \alpha \to -\infty.$$

(e) *There exists $a \in K$ such that $\psi(a,u) \to +\infty$ if $\|u-a\| \to +\infty$, $u \in K$.*

Indeed, the monotonicity of $T(\cdot)$ implies that $\psi(u,a) + \psi(a,u) \leq 0$ for all $u \in K$, and therefore $\psi(u,a) \leq -\psi(a,u) \to -\infty$ if $\|u-a\| \to +\infty$. Hence (e) implies (b).

If K is a cone (as in the complementarity problem), then the condition $\inf_{u \in K} \psi(a,u) > -\infty$ occurring in (d) above is equivalent with $K^* \cap T(a) \neq \emptyset$. If K is a cone and $X = \mathbb{R}^n$, then the condition $\psi(a,u) \to +\infty$ occurring in (e) is fulfilled if $(\mathrm{int}\, K^*) \cap T(a) \neq \emptyset$.

Condition (b) is satisfied in particular if there exists $a \in K$, $b \in X^*$ and $\alpha > 0$ such that

$$\langle u-a, y-b\rangle \geq \alpha\|u-a\|^2 \quad \text{for all} \quad (u,y) \in \mathrm{graph}\, T. \tag{14}$$

In fact, (14) implies that

$$\psi(u,a) = \sup_{y\in T(u)} \langle a-u, y\rangle \le \langle u-a, -b\rangle - \alpha\|u-a\|^2 \le \|b\|\,\|u-a\| - \alpha\|u-a\|^2,$$

and thus (b) is fulfilled.

For the following result we assume that $X = H = X^*$ is a Hilbert space.

Theorem 8. *Let $K \subset H$ be nonempty, closed and convex, let $T : K \rightrightarrows H$ be weakly hemicontinuous with nonempty, convex, compact values, and assume that*

$$\langle x_1 - x_2, y_1 - y_2\rangle \ge p(\|x_1 - x_2\|)\cdot\|x_1 - x_2\|^2 \quad \forall (x_i, y_i) \in \operatorname{graph} T$$

$(i = 1, 2)$, where $p(s) > 0$ for $s > 0$ and $p(\cdot)$ is non-increasing.
Then there exist $\overline{x} \in K$ (unique) and $\overline{y} \in T(\overline{x})$ such that $\langle x-\overline{x}, \overline{y}\rangle \ge 0 \quad \forall x \in K$.

Proof. Let $u \in H$ be arbitrary, and set $\psi_u(x) := T(x) + x - u$. Then

$$\langle x_1 - x_2, \eta_1 - \eta_2\rangle \ge \|x_1 - x_2\|^2 \quad \forall (x_i, \eta_i) \in \operatorname{graph}\psi_u, \ (i = 1, 2).$$

Hence $\psi_u(\cdot)$ is monotone and satisfies (14). So from Corollary 2 for any $u \in H$ there exists a solution to

$$\overline{x}_u \in K, \ \overline{\eta}_u \in \psi_u(\overline{x}_u), \ \langle x - \overline{x}_u, \overline{\eta}_u\rangle \ge 0 \quad \forall x \in K.$$

$\overline{x}_u$ is easily seen to be unique. Let $\chi : H \to K$ be the mapping which assigns to each $u \in H$ this $\overline{x}_u \in K$. For $u_i \in H$ $(i = 1, 2)$ set $\overline{x}_i := \chi(u_i)$, with the corresponding elements $\overline{\eta}_i \in \psi_{u_i}(\overline{x}_i)$, where $\overline{\eta}_i = \overline{y}_i + \overline{x}_i - u_i$, $\overline{y}_i \in T(\overline{x}_i)$. Then for $i = 1, 2$ we have

$$\langle x - \overline{x}_i, \overline{\eta}_i\rangle \ge 0 \quad \forall x \in K,$$

in particular $\langle \overline{x}_2 - \overline{x}_1, \overline{\eta}_1\rangle \ge 0$ and $\langle \overline{x}_1 - \overline{x}_2, \overline{\eta}_2\rangle \ge 0$. Addition yields

$$\begin{aligned} 0 &\le \langle \overline{x}_1 - \overline{x}_2, \overline{\eta}_2 - \overline{\eta}_1\rangle \\ &= \langle \overline{x}_1 - \overline{x}_2, \overline{y}_2 - \overline{y}_1\rangle + \langle \overline{x}_1 - \overline{x}_2, \overline{x}_2 - \overline{x}_1\rangle + \langle \overline{x}_1 - \overline{x}_2, u_1 - u_2\rangle \\ &\le [-p(\|\overline{x}_1 - \overline{x}_2\|)\cdot\|\overline{x}_1 - \overline{x}_2\| - \|\overline{x}_1 - \overline{x}_2\| + \|u_1 - u_2\|]\cdot\|\overline{x}_1 - \overline{x}_2\|. \end{aligned}$$

With $q(\cdot) := 1/(1 + p(\cdot))$ it follows that $q(\cdot) \le 1$ and

$$\|\overline{x}_1 - \overline{x}_2\| \le q(\|\overline{x}_1 - \overline{x}_2\|)\cdot\|u_1 - u_2\| \le \|u_1 - u_2\|.$$

Since $q(\cdot)$ is non-decreasing, we obtain altogether

$$\|\chi(u_1) - \chi(u_2)\| = \|\overline{x}_1 - \overline{x}_2\| \leq q(\|u_1 - u_2\|) \cdot \|u_1 - u_2\|,$$

where $q(s) < 1$ for $s > 0$. Hence, according to [6], χ has a unique fixed point $\overline{u} = \chi(\overline{u})$. This implies that $\overline{u} \in K$ and that there exists $\overline{\eta} \in \psi_{\overline{u}}(\overline{u}) = T(\overline{u})$ such that $\langle x - \overline{u}, \overline{\eta}\rangle \geq 0 \quad \forall x \in K$. $\square$

If K is a cone, then the previous two results give conditions for the existence of a solution to the multivalued complementarity problem

$$\overline{x} \in K, \ \overline{y} \in K^* \cap T(\overline{x}), \ \langle \overline{x}, \overline{y}\rangle = 0. \tag{15}$$

In fact these results can even be applied to the quasi-complementarity problem

$$\overline{u} \in C, \ \overline{x} \in K \cap S(\overline{u}), \ \overline{y} \in K^* \cap T(\overline{u}), \ \langle \overline{x}, \overline{y}\rangle = 0, \tag{16}$$

where S and T are defined on C. We assume that $S(C) \supset K$, and we define $\Gamma : K \rightrightarrows X^*$ through $\Gamma := TS^{-1}$. Then (16) is equivalent with

$$\overline{x} \in K, \ \overline{y} \in K^* \cap \Gamma(\overline{x}), \ \langle \overline{x}, \overline{y}\rangle = 0,$$

i.e., with the multivalued complementarity problem (15). So we obtain conditions on TS^{-1} which ensure the existence of a solution to (16). In this setting Theorem 8 should be compared with [20], Theorem 2.

4. Projection methods.

Let now $H = H^*$ be a Hilbert space. Given a nonempty closed convex subset $A \subset H$ we shall use the projection mapping $P_A(\cdot) : H \to A$ which assigns to each $x \in H$ the nearest point in A. The following characterization is well known [37], p. 239: For any $x \in H$,

$$y = P_A(x) \iff (y \in A \text{ and } \langle \xi - y, y - x\rangle \geq 0 \, \forall \xi \in A)\,. \tag{17}$$

Let $K : H \rightrightarrows H$ be a multifunction with nonempty closed convex values, and let $T : H \to H$ be a given mapping. We are looking for a solution of the quasivariational inequality

$$\overline{u} \in K(\overline{u}), \ \langle x - \overline{u}, T(\overline{u})\rangle \geq 0 \quad \forall x \in K(\overline{u}). \tag{18}$$

Since the validity of (18) is not affected by multiplication of T with a positive scalar, and in view of (17), $\overline{u}$ is a solution of (18) if, and only if,

$$\overline{u} = P_{K(\overline{u})}(\overline{u} - \tau \cdot T(\overline{u})) \quad \text{for some} \quad \tau > 0.$$

Hence (18) is equivalent to finding a fixed point, for some $\tau > 0$, of the mapping

$$\Phi_\tau(u) := P_{K(u)}(u - \tau \cdot T(u)) : H \to H. \tag{19}$$

Theorem 9. *Let the following assumptions hold:*

(i) *There exist* $\alpha > 0$, $\beta > 0$ *such that*

$$\langle u - v, T(u) - T(v)\rangle \geq \alpha \|u - v\|^2 \quad \forall u, v \in H,$$
$$\|T(u) - T(v)\| \leq \beta \|u - v\| \quad \forall u, v \in H;$$

(ii) *there exists* $k \geq 0$ *such that*

$$\|P_{K(u)}(z) - P_{K(v)}(z)\| \leq k\|u - v\| \; \forall u, v, z \in H;$$

(iii) $k + \sqrt{1 - \alpha^2/\beta^2} < 1$.

Then Problem (18) *has a unique solution.*

Proof. It follows from (i) that $\alpha/\beta \leq 1$, and moreover

$$\|u - \tau T(u) - (v - \tau T(v))\|^2 \leq \|u - v\|^2(1 - 2\alpha\tau + \beta^2\tau^2).$$

With the aid of (ii) we obtain then, since the projection mapping is nonexpansive, that

$$\begin{aligned}\|\Phi_\tau(u) - \Phi_\tau(v)\| &= \|P_{K(u)}(u - \tau T(u)) - P_{K(v)}(u - \tau T(u)) + \\ &+ P_{K(v)}(u - \tau T(u)) - P_{K(v)}(v - \tau T(v))\| \leq \\ &\leq k \cdot \|u - v\| + \|u - \tau T(u) - (v - \tau T(v))\| \leq \\ &\leq \|u - v\| \cdot \left(k + \sqrt{1 - 2\alpha\tau + \beta^2\tau^2}\right).\end{aligned}$$

Choosing in particular $\overline{\tau} := \alpha/\beta^2$ we obtain

$$\|\Phi_{\overline{\tau}}(u) - \Phi_{\overline{\tau}}(v)\| \leq \|u - v\| \cdot \left(k + \sqrt{1 - \alpha^2/\beta^2}\right).$$

Hence by (iii) $\Phi_{\overline{\tau}}$ is a contraction mapping and has a unique fixed point $\overline{u}$, which at the same time is the unique solution of (18). $\square$

If $K(u) \equiv K$ is independent of u, then we may choose $k = 0$ in assumption (ii), and consequently assumption (iii) is satisfied. It follows that in this case problem (18) has a unique solution under assumption (i) alone, without further restrictions on α, β. If T is linear-affine, this is a classical result due to Stampacchia [7], p. 82.

It should be seen, however, that Corollary 2 gives the same existence result under considerably weaker assumptions. Thus the assumptions of Theorem 9 should be mainly seen as conditions which ensure that (18) can be solved constructively by means of successive approximations $u^{n+1} := \Phi_{\overline{\tau}}(u^n)$.

Let us now specialize Theorem 9 to the case

$$K(u) := u - S(u) + K, \tag{20}$$

where $K \subset H$ is a closed convex cone, and $S : H \to H$. Let

$$K^* := \{y \in H \,|\, \langle x, y\rangle \geq 0 \quad \forall x \in K\}$$

denote the polar cone of K, which implies that for any $x \in K$,

$$\langle \xi - x, y\rangle \geq 0 \quad \forall \xi \in K \iff \big(y \in K^* \text{ and } \langle x, y\rangle = 0\big). \tag{21}$$

Under (20) Problem (18) can be rewritten as

$$S(\overline{u}) \in K, \;\langle u - S(\overline{u}), T(\overline{u})\rangle \geq 0 \quad \forall u \in K,$$

hence by (21) is the same as the complementarity problem

$$S(\overline{u}) \in K, \; T(\overline{u}) \in K^*, \; \langle S(\overline{u}), T(\overline{u})\rangle = 0. \tag{22}$$

Theorem 9 gives therefore conditions for the existence of a unique solution to (22). Assumption (ii) is satisfied in this case when $u - S(u)$ is Lipschitz continuous with Lipschitz constant k. This follows from the well-known fact that $\|P_{a+K}(z) - P_K(z)\| \leq \|a\|$.

Let us set for abbreviation

$$g(u) := u - S(u), \, h_\tau(u) := u - \tau \cdot T(u).$$

The function $\Phi_\tau(u)$, given by (19), can then be rewritten as

$$\Phi_\tau(u) = g(u) + P_K(h_\tau(u) - g(u)). \tag{23}$$

Since for all $z \in H$, $P_K(z) = z + P_{K^*}(-z)$ [37], p. 256, we obtain from (23) the "dual" representation

$$\Phi_\tau(u) = h_\tau(u) + P_{K^*}(g(u) - h_\tau(u)). \tag{24}$$

We note that from (23) together with (17) and (21) it follows that $v = \Phi_\tau(u)$ if, and only if,

$$v - g(u) \in K, \ v - h_\tau(u) \in K^*, \ \langle v - g(u), v - h_\tau(u) \rangle = 0.$$

Thus for u given, $\Phi_\tau(u)$ itself is the solution of a quasi-complementarity problem.

REFERENCES

[1] G. Allen, *Variational inequalities, complementarity problems, and duality theorems,* J. Math. Anal. Appl., 58 (1977), pp 1-10.

[2] J.P. Aubin, *Mathematical Methods of Game and Economic Theory,* North Holland, Amsterdam, 1979.

[3] M.S. Bazaraa - J.J. Goode - M.Z. Nashed, *A nonlinear complementarity problem in mathematical programming in Banach space,* Proc. Amer. Math. Soc., 35 (1972), pp. 165-170.

[4] C. Berge, *Espaces topologiques - fonctions multivoques (deuxième édition),* Dunod, Paris, 1966.

[5] E. Blum - W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Student, 63 (1994), pp. 123-145.

[6] D.W. Boyd - J.S.W. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc., 20 (1969), pp. 458-464.

[7] H. Brézis, *Analyse fonctionnelle,* Masson, Paris, 1987.

[8] D. Chan - J.-S. Pang, *The generalized quasi-variational inequality problems,* Math. Operations Res., 7 (1982), pp. 211-222.

[9] L.V. Chóng, *On the existence of solutions for a general form of variational and quasi-variational inequalities,* Z. Analysis Anw., 3 (1984), pp. 541-548.

[10] R.W. Cottle, *Complementarity and variational problems,* Symposia Mathematica, 19 (1976), pp. 177-208.

[11] P. Cubiotti, *Finite-dimensional quasi-variational inequalities associated with discontinuous functions,* J. Optimization Theory Appl., 72 (1992), pp. 577-582.

[12] B.C. Eaves, *On the basic theorem of complementarity,* Math. Programming, 1 (1971), pp. 68-75.

[13] K. Fan, *A minimax inequality and applications,* in: O. Shisha (ed.), *Inequalities III*, pp. 103-113, Academic Press, New York, 1972.

[14] S.C. Fang, *An iterative method for generalized complementarity problems,* IEEE Trans. Automatic Control, AC-25 (1980), pp. 1225-1227.

[15] S.C. Fang - E.L. Peterson, *A fixed-point representation of the generalized complementarity problem,* J. Optimization Theory Appl., 45 (1985), pp. 375-381.

[16] M.S. Gowda - J.-S. Pang, *Some existence results for multivalued complementarity problems,* Math. Operations Res., 17 (1992), pp. 657-669.

[17] M.S. Gowda, *Complementarity problems over locally compact cones,* SIAM J. Control Optimization, 27 (1989), pp. 836-841.

[18] G.J. Habetler - A.L. Price, *Existence theory for generalized nonlinear complementarity problems,* J. Optimization Theory Appl., 7 (1971), pp.223-239.

[19] P.T. Harker - J.-S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems,* Math. Programming, 48 (1990), pp. 161-220.

[20] G. Isac, *Fixed point theory and complementarity problems in Hilbert space,* Bull. Austral. Math. Soc., 36 (1987), pp. 295-310.

[21] S. Karamardian, *Generalized complementarity problem,* J. Optimization Theory Appl., 8 (1971), pp. 161-168.

[22] M. Kojima, *A unification of the existence theorems of the nonlinear complementarity problem,* Math. Programming, 9 (1975), pp. 257-277.

[23] G. Luna, *A remark on the nonlinear complementarity problem,* Proc. Amer. Math. Soc., 48 (1975), pp. 132-134.

[24] O.L. Mangasarian, *Solutions of symmetric linear complementarity problems by iterative methods,* J. Optimization Theory Appl., 22 (1977), pp. 465-485.

[25] L. McLinden, *The complementarity problem for maximal monotone multifunctions,* in "Variational Inequalities and Complementarity Problems", R.W. Cottle - F. Giannessi, J.-L. Lions editorss, Wiley, Chichester, 1980, pp. 251-270.

[26] J.J. Moré, *Coercivity conditions in nonlinear complementarity problems,* SIAM Rev., 16 (1974), pp. 1-16.

[27] U. Mosco, *Implicit variational problems and quasi-variational inequalities,* in Lecture Notes in Mathematics, Vol. 543, Springer-Verlag, Berlin, 1976, pp. 83-156.

[28] X.T. Nguyen, *Quasi-variational inequalities in topological linear locally-convex Hausdorff spaces,* Math. Nachr., 122 (1985), pp. 231-245.

[29] M.A. Noor, *General nonlinear variational inequalities,* J. Math. Anal. Appl., 126 (1987), pp. 78-84.

[30] M.A. Noor, *Quasivariational inequalities,* Appl. Math. Letters, 1 (1988), pp. 367-370.

[31] W. Oettli, *Monoton-konvexe Funktionen – Eine Bemerkung zum Satz von Browder-Minty,* in Mathematical Research, Vol. 45, Akademie-Verlag, Berlin, 1988, pp. 130-136.

[32] R. Saigal, *Extension of the generalized complementarity problem,* Math. Operations Res., 1 (1976), pp. 260-266.

[33] M.-H. Shih - K.-K. Tan, *Generalized quasi-variational inequalities in locally convex topological vector spaces,* J. Math. Anal. Appl., 108 (1985), pp. 333-343.

[34] A.H. Siddiqi - Q.H. Ansari, *Strongly nonlinear quasivariational inequalities,* J. Math. Anal. Appl., 149 (1990), pp. 444-450.

[35] G. Tian - J. Zhou, *Quasi-variational inequalities with non-compact sets,* J. Math. Anal. Appl., 160 (1991), pp. 583-595.

[36] J.-C. Yao, *The generalized quasivariational inequality problem with applications,* J. Math. Anal. Appl., 158 (1991), pp. 139-160.

[37] E.H. Zarantonello, *Projections on convex sets in Hilbert space and spectral theory,* in "Contributions to Nonlinear Functional Analysis", E.H. Zarantonello editor, Academic Press, New York, 1971, pp. 237-424.

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