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ON THE CONNECTIONS BETWEEN OPTIMALITY CONDITIONS, VARIATIONAL INEQUALITIES AND EQUILIBRIUM PROBLEMS

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In this note we underline, once more, the relationships between necessary optimality conditions for constrained extremum problems and variational inequalities in finite dimension spaces. All is done passing through the concept of variational principle for equilibrium problems.

1. Introduction.

The aim of this note is to put in evidence, once more, the strict connections between the study of variational inequalities in a finite-dimensional space, that of necessary optimality conditions for constrained extremum problems in $\mathbb{R}^n$ and "equilibrium problems" [3]. In particular we want to underline how necessary optimality conditions and variational inequalities are particular cases of equilibrium problems and we want to investigate under what conditions to obtain variational inequalities from necessary optimality conditions and viceversa.
Following [2], by an equilibrium problem we understand the problem of finding

(EP) $\qquad \bar{x} \in K \quad \text{such that} \quad g(\bar{x}, y) \geq 0 \quad \forall y \in K$

where K is a given set and $g : K \times K \to \mathbb{R}$ is a given function with $g(x, x) = 0$, $\forall x \in K$. It has been proved [2] that these types of problems contain, as special cases, several other classical problems: Nash equilibria, optimization problems, fixed point problems, complementarity problems and variational inequalities. This first observation confirms the strict relationship between variational inequalities and optimization problems. In the sequel we shall go further into the details.

2. General scheme for necessary optimality conditions.

Given the problem

$$\text{(P)} \qquad \min_{x \in Q} f(x)$$

with $f : \mathbb{R}^n \to \mathbb{R}$, $Q \subseteq \mathbb{R}^n$; necessary and sufficient optimality conditions for $\bar{x} \in Q$ is the impossibility of the system

$$\text{(S)} \qquad \begin{cases} f(\bar{x}) > f(x) \\ x \in Q. \end{cases}$$

To verify the impossibility of (S) is, in general, a too hard task; therefore the purpose of optimizers is to build a new system ($\tilde{\text{S}}$) more tractable of (S) from the computational viewpoint whose impossibility is implied by the impossibility of (S) (necessary optimality condition) or implies the impossibility of (S) (sufficient optimality condition). The general form of ($\tilde{\text{S}}$) will be

$$(\tilde{\text{S}}) \qquad \begin{cases} g(\bar{x}; z) < 0 \\ z \in G(Q; \bar{x}) \end{cases}$$

where $g(\bar{x}; z)$ is an approximation of $f(x) - f(\bar{x})$ and $G(Q; \bar{x})$ is one of Q. G is a multifunction defined on $2^{\mathbb{R}^n} \times \mathbb{R}^n$ and taking values in $2^{\mathbb{R}^n}$, while g is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and takes values in $\mathbb{R}$. The couple (G, g) will be called "admissible" for problem (P) if the impossibility of ($\tilde{\text{S}}$) implies the impossibility of (S). Therefore an admissible pair produces a necessary optimality condition. The structure of system ($\tilde{\text{S}}$) clarifies why we require that the function g be positive homogeneous of degree 1 with respect to the variable z i.e.

$$g(\bar{x}; \lambda z) = \lambda g(\bar{x}; z), \quad \forall \lambda > 0.$$

In fact, in this case, we have $g(\bar{x}; z) \geq 0$ if and only if $g(\bar{x}, \lambda z) \geq 0$, $\forall \lambda > 0$, and therefore the inequalities in ($\tilde{\text{S}}$) must be tested only for vectors in the unitary ball.

Moreover from this remark we observe that the analogous requirement for $G(Q; \bar{x})$ is the fact that it is a cone. This means, in fact

$$z \in G(Q; \bar{x}) \iff \lambda z \in G(Q; \bar{x}) \quad \forall \lambda > 0.$$

These properties of positive homogeneity of g and G make system ($\tilde{\text{S}}$) more tractable but yet difficult to solve. A good property, not too strong, but sufficient for our purpose and, in particular, for establishing connections with variational inequalities is the following one:

Definition.

a) *The function g has the "dualization" property at the point $\bar{x}$ if the set*

$$R(\bar{x}) = \{w \in \mathbb{R}^n : g(\bar{x}; z) \geq \langle w, z \rangle, \quad \forall z\}$$

is not empty.

b) *The function g has the "partial dualization" property at the point $\bar{x}$ with respect the multifunction G and to the set Q if the set*

$$R_G(\bar{x}) = \{w \in \mathbb{R}^n : g(\bar{x} : z) \geq \langle w, z \rangle, \quad \forall z \in G(Q; \bar{x})\}$$

is not empty.

Remark. If the function g is also convex with respect to z then g has the dualization property. This can be proved taking into account a Hahn-Banach type theorem.

When g has the partial dualization property the very general necessary optimality condition takes the form of the following theorem:

Theorem (General necessary optimality condition). *Let $\bar{x}$ be an optimal solution of* (P), *let g be a function with the partial dualization property, let G be a multifunction such that the couple (G, g) be admissible for problem* (P), *then*

$$(\bar{S}) \qquad O \in R_G(\bar{x}).$$

System $(\bar{S})$ remains difficult to test if we don't suppose anything about the multifunction G. Till now, in fact, dualization property regards mainly g. The right property for G is the following:

Definition. *An admissible pair (G, g) has the "decomposition" property if g has the partial dualization property and if there exists a multifunction $P : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ (which does not depend on Q) such that, for every Q and $\bar{x}$, one has:*

$$R_G(\bar{x}) = P(\bar{x}) + G^\circ(Q; \bar{x})$$

(where $\circ$ denotes the negative polar).

Remark. If the pair (G, g)has the decomposition property then $G^\circ(Q; \bar{x})$ is not empty.

In this case the general theorem has the following form:

Theorem (Dualizable necessary optimality condition). *Let $\bar{x}$ be an optimal solution; let (G, g) be an admissible pair for* (P) *and having the "decomposition property", then the system*

$$(\hat{\mathrm{S}}) \qquad w \in P(\bar{x}) \cap (-G^{\circ}(Q, \bar{x}))$$

has solution.

System $(\hat{\mathrm{S}})$ represents the theoretical paradigm of necessary optimality conditions. In this field the problem becomes how to obtain couples (G, g) satisfying the assumptions of these two theorems. Starting from the pioneering works of Rockafellar [9], several results [4], [6], have been obtained in this direction. Let us summarize one of them [4]. Let (G, g) be an admissible pair (of approximations) satisfying the following properties

i) G and epi g are local cone approximation in the sense of [6];
ii) $G(Q \times]-\infty, a[, (q, a)) \supseteq G(Q, q) \times]-\infty, 0[$, $\forall Q \subseteq \mathbb{R}^n$, $\forall q \in \mathrm{cl}\, Q$, $\forall a \in \mathbb{R}$ (1).

In this case, in every local optimal solution $\bar{x}$, we have:

$$(\mathrm{SG}) \qquad g(\bar{x}; z) \geq 0, \quad \forall z \in G(Q, \bar{x}).$$

For the details of this theory we refer to [4], [6]. In particular, we recall that for passing from a general necessary optimality condition to a dualizable one it is sufficient to have that G is contained in a halfspace (for example the convexity of G is sufficient).

In this way, in this section, we have showed how necessary optimality conditions are particular cases of equilibrium problems.

3. General scheme for variational inequalities.

If the system $(\hat{\mathrm{S}})$ represents a very general necessary optimality condition we can understand why it is possible to see it as a very general scheme for variational inequalities. This can be done starting from the smooth convex case. It has been proved that by choosing $g(\bar{x}; z) \triangleq \nabla f(\bar{x}) z$ and $G(Q; \bar{x}) \triangleq T(Q; \bar{x})$ (the Bouligand tangent cone) the couple (G, g) satisfy the assumptions of the

(1) To be formally correct I should use a different notation since $Q \subseteq \mathbb{R}^n$ and $(Q \times]-\infty, a[) \subseteq \mathbb{R}^{n+1}$.

general theorem of Section 2. In particular, if Q is convex, one obtains that $P(\bar{x}) = \{\nabla f(\bar{x})\}$ and therefore ($\hat{\text{S}}$) becomes

$$(\hat{\text{S}}) \qquad -\nabla f(\bar{x}) \in N(Q; \bar{x});$$

where $N(Q; \bar{x})$ is the normal cone of the convex analysis. If we write ($\hat{\text{S}}$) in the other equivalent manner

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in Q$$

we obtain the very classical paradigm of variational inequality. With this classical example in mind we can observe that (SG) is exactly a variational inequality when the function $g(\bar{x}; z)$ admits a Riesz type representation theorem ($g(\bar{x}; z) = \langle M(\bar{x}), z \rangle$). In this case (SG) becomes

$$\langle M(\bar{x}), z \rangle \geq 0 \quad \forall z \in G(Q; \bar{x}).$$

In this way we can define exactly a variational inequality; when a representation theorem does not hold the necessary optimality condition remains in the form (SG) which is an equilibrium problem. In this framework the other important inverse problem is how to recognize that a systems (SG) derives from an optimization problem. In this sense we can speak, in a free sense, how to "integrate" a system (SG). The results in this field are less strong and more difficult to obtain, but some efforts have been done in this direction. We consider an equilibrium problem of type (SG):

$$(\text{ES}) \qquad \text{find } \bar{x} \in Q \text{ such that } g(\bar{x}; z) \geq 0 \quad \forall z \in G(Q; \bar{x}).$$

Adopting the terminology of [2] we say that a variational principle holds for the equilibrium problem (ES) iff there exists a function $f : Q \to \mathbb{R}$ such that the solution set of (ES) coincides with the solution set of the optimization problem (P); we say that a weak variational principle holds iff the solution set of (ES) contains the solution set of (P). Let us remember that if there exists f such that $g(\bar{x}; z) = \nabla f(\bar{x})z$ then a weak variational principle holds for (ES); in particular, it is known that if f is convex a variational principle holds.

Some more general variational principles have been established in the last years. Let us describe one of them [3]. Let $g(x, y) = \ell(x, y) + h(x, y)$ be and assume that

i) *$K \subset X$ is closed, convex, non empty;*
ii) *$\ell(x, x) = 0$, $\ell(x, y) + \ell(y, x) \leq 0$ $\forall x, y \in K$, $t \in [0, 1] \to \ell(ty + (1 - t)x; y)$ is upper semicontinuous at $t = 0$, ℓ is convex and lower semicontinuous in the second argument;*

iii) $h(x, x) = 0\ \forall x \in K$, *h is upper semicontinuous in the first argument and convex in the second argument;*

We define $G : K \to \mathbb{R}$ by means of

$$G(x) = \inf_{y \in K} \big(-\ell(y, x) + h(x, y) + \pi(x, y) \big)$$

where $\pi : K \times K \to \mathbb{R}$ satisfies $\forall x, y \in K$:

$$\pi(x, y) \geq 0,\ \pi(x, x) = 0,\ \pi(x, ty + (1-t)x) = O(t) \quad 0 \leq t \leq 1.$$

Then

Theorem. $\bar{x}$ *is a solution of* (SG) *if and only if* $\bar{x}$ *is a solution of*

$$\max_{x \in K} G(x).$$

In this framework it is important to refer other efforts in this direction by using the concept of gap-function [6], [7].

REFERENCES

[1] G. Allen, *Variational inequalities, complementarity problems and duality theorems,* J. Math. Anal. Appl., 58 (1977), pp. 1-10.

[2] G. Auchmuthy, *Variational principles for variational inequalities,* Numer. Funct. Anal. Opt., 10 (1989), pp. 863-874.

[3] E. Blum - W. Oettli, *From optimization and variational inequalities to equilibrium problems,* 57th Annual Conference of the Indian Math. Soc., (1991).

[4] M. Castellani - M. Pappalardo, *First order cone approximation and necessary optimality conditions,* Accepted for publication in Optimization.

[5] K.H. Elster - J. Thierfelder, *Abstract cone approximations and generalized differentiability in nonsmooth optimization,* Optimization, 19 (1988), pp. 315-341.

[6] F. Giannessi, *Separation of sets and gap functions for quasi variational inequalities,* in "Variational inequalities and network equilibrium problems", F. Giannessi - A. Maugeri editors, Plenum Publishing, New York, 1995, pp. 101-121.

[7] P.T. Harker - J.S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications,* Math. Progr. Ser. B, 48 (1990), pp. 161-220.

[8] D. Kinderlehrer - G. Stampacchia, *An introduction to variational inequalities and their applications,* Academic Press, 1980.

[9] T.T. Rockafellar, *Convex Analysis,* Princeton Press, 1970.

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