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ON THE POSSIBILITY OF APPLYING VARIATIONAL INEQUALITIES TO COMPUTER NETWORKS

LETIZIA PELLEGRINI

The possibility of applying a Variational Inequality model to the study of computer networks is analysed. With the intention of gaining for this subject the attention it deserves, some ideas for the management, the design and the reconfiguration of a computer network are examined. In particular it is shown how the concepts of duality and potential for a Variational Inequality naturally arise and are crucial in this analysis.

1. Introduction.

The number of national and worldwide telecommunications networks is continuously expanding and, over the last two decades, they have moved from being almost exclusively used for telephony to becoming a fully integrated services digital network.

Therefore, computer networks management represents a crucial and actual problem; this problem is of such wide and composite range that it is a nonsense to propose a unique model to deal with it, while a partial modelling of particular aspects can certainly improve the analysis of this problem.

One of the most important aspects in the network management concerns the study of the equilibrium flows which has been examined in past time by means of optimization models. These models catch reality only under very special circumstances, namely when all activities of the network are completely regulated by a central authority. Hence only few real networks are represented by an optimization model. Recently, Variational Inequality models have shown to be adequate to interpret the equilibrium flows in a network.

This note aims to propose a tentative formulation of a variational model for the equilibrium flow problem in a computer network; it obviously does not pretend to solve the problem, but merely attempts to call the attention of the scholars to such an interesting and challenging topic.

2. Variational Inequality models for computer networks.

Let us describe some characteristics of computer networks in order to introduce the variational model. In a computer network there are source nodes and destination nodes: a *source node* is a point (device) through which information enters the network, a *destination node* is a point (device) to which information is delivered.

Moreover, it is important to know the major classes of strategy of data transmission in computer networks; they are: circuit switching, message switching, packet switching, and broadcast [2], [5].

Circuit switching in the basic form consists of the setting up of a dedicated channel between the source and destination nodes for the duration of their interaction, hence this strategy is not mathematically interesting as far as the equilibrium flows are concerned. The same is for broadcast systems, because they use a single channel with all nodes on it able to see all traffic flowing.

Message-switched networks have no prearrangement or allocation of circuits and it is for this peculiarity that variational models are suitable for the management of this sort of computer networks. Messages are sent in a store and forward fashion from the source node to the destination node via some intermediary nodes. The messages traverse from one node to the other, and they are queued up at each node before transmission.

Packet-switched networks provide a similar form of service; that is, they take messages and send them in a store and forward fashion through the network. The major variant is that the messages are broken up into segments called *packets*; i.e., the message data is divided up into suitably dimensioned data packets, which enter the network through one of the nodes. They are then passed from node to node until they reach the node which serves the destination terminal. Packets forming part of the same message do not necessarily take the same route through the network or utilize the same circuits. Like message switching, packet-store buffers are required at each node to hold the packet when in transit.

In the basic frame structure of a packet, common elements are a fixed number of bits to indicate the source node, a variable field for the information and again a fixed number of bits for the destination.

The choice of path through the network for each packet is determined by the traffic on the network at the time the packet enters the network.

Since the traffic will be constantly changing, packets which form part of the same message may be routed through different nodes and circuits and may experience different delays in the store-and-forward procedure. Additionally, since the entire message is not sent as a whole, this scheme requires that we have mechanisms to break up the messages and reassemble them properly at the destination.

Let us suppose that a computer network be assigned in the form (N, A, T), where $N = \{N_1, N_2, \dots, N_p\}$, $A = \{A_1, A_2, \dots, A_n\}$ and $T = \{T_1, T_2, \dots, T_\ell\}$ are respectively the sets of nodes, links, and source-destination ordered pairs. The nodes of the pair T_j are connected by $r_j \geq 1$ paths, whose set is denoted by $\mathscr{P}_j = \{P_1, \dots, P_{r_j}\}$. Hence $P_1, \dots, P_m$ with $m := r_1 + \cdots + r_\ell$ are all the considered paths. The flow F_s, $s = 1, \dots, m$, on every path is the number of packets (and therefore of bits or bytes) passing through the path in a time unit; then $F := (F_1, \dots, F_m)$ is the vector of flows on paths and $C(F) = (C_1(F), \dots, C_m(F))$ the vector whose elements denote the transmission cost on every path as a function of the flows on all paths.

The main criterion for the network efficiency is represented by the time that a message employs to go from the source node to the destination node; hence the cost on a path is expressed by a function of this time. Since the routing time on every link is negligible with respect to the time spent at every node, the cost results to be the sum of the waiting times at nodes.

In the case of a road network, the equilibrium flows have been initially characterized by the so-called Wardrop principle [4], which can be transferred to the present problem. It claims that a vector $H \in \mathbb{R}^m_+$ is an equilibrium pattern iff $\forall\, T_j$ and $\forall\, P_q, P_s \in \mathscr{P}_j$ it results

$$C_s(H) > C_q(H) \Rightarrow H_s = 0 \text{ (or, equivalently, } H_s > 0 \Rightarrow C_s(H) \leq C_q(H)).$$

For every pair T_j let ρ_j, $j = 1, \dots \ell$, be the transmission demand; such demand generally depends on the flows and hence it is denoted by $\rho_j(H)$. If we introduce a pairs-paths matrix $\varphi = (\varphi_{is})$, $i = 1, \dots, \ell$, $s = 1, \dots, m$ defined by

$$\varphi_{is} := \begin{cases} 1 & \text{if } P_s \in \mathscr{P}_i \\ 0 & \text{if } P_s \notin \mathscr{P}_i\,, \end{cases}$$

the so-called flow-conservation law can be written as $\varphi F = \rho(H)$, where $\rho(H) := (\rho_1(H), \dots, \rho_\ell(H))$.

Hence the set of feasible flows is $K(H) := \{F \in \mathbb{R}^m_+ : \varphi F = \rho(H)\}$ and in [4] it is proved that $H \in K(H)$ is an equilibrium pattern iff

$$\langle C(H), F - H\rangle \geq 0 \quad \forall F \in K(H). \tag{1}$$

This is a particular case of Quasi-Variational Inequality which collapses to a Variational Inequality when the demand ρ, and hence K, does not depend on H.

The variational model (1) does not take into account the capacities of links or nodes and consequently a solution of (1) may be unrealistic. Suppose that in the network there are upper bounds to the flows on links; let $f = (f_1, \dots, f_n)$ be the vector of flows on links and $d = (d_1, \dots, d_n)$ the vector of upper bounds on these flows. If we introduce the matrix $\Delta = (\delta_{is})$, $i = 1, \dots, n$, $s = 1, \dots, m$, where

$$\delta_{is} := \begin{cases} 1 & \text{if } A_i \in P_s \\ 0 & \text{if } A_i \notin P_s\,, \end{cases}$$

the flows on links are expressed in terms of flows on paths by $f = \Delta F$. Hence the set of feasible flows becomes

$$K_d(H) := \{F \in \mathbb{R}^m : \varphi F = \rho(H); \quad 0 \leq \Delta F \leq d\}.$$

Recently, generalized equilibrium principles are under investigation [6], [8] with the aim to establish that $H \in K_d(H)$ is an equilibrium pattern iff

$$\langle C(H), F - H\rangle \geq 0 \quad \forall F \in K_d(H). \tag{2}$$

Let us observe that a solution of (1), even if unrealistic because of the possibly positive difference between the flows and the capacities, is an important information; in fact, this positive quantity can be interpreted as demand in excess, while a solution of (2) spreads this surplus.

In computer networks, a high use of the system can produce traffic jam at the nodes (and not necessarily at the links); hence a realistic model should associate capacities also to the nodes. In this case, for any intermediate node N_j, let c_j be its capacity and E_j the set of indexes i such that A_i is a link which enters the node N_j. The capacity constraint at node N_j is $\sum_{i \in E_j} f_i \leq c_j$; hence it is expressed in terms of flows on paths by $\sum_{i \in E_j} (\Delta F)_i \leq c_j$, where $(\Delta F)_i$ is the i-th element of vector ΔF. In this case the set of feasible flows is modified too and new models are highly required to study the equilibrium problem.

3. Further applications of variational models to computer networks.

In the previous section variational models dealing with the management of a computer network have been described. Other quite interesting problems are represented by the design of a network or by its reconfiguration when the network is no longer sufficient to fulfil the requests of the users. In this case it is reasonable to suppose that source and destination nodes are assigned, because they represent the users, while intermediate nodes and links have to be chosen.

A class of these problems, i.e. the design of electric, hydraulic or telecommunications networks, has been dealt with by Steiner-Weber models.

These models tackle the problem by constructing a minimal cost tree connecting the assigned nodes with the unknown extra nodes, added to the tree to reduce the cost; the cost is the weighted sum of the distances between nodes which are connected among them. The trees are chosen in a class defined by a connection scheme that specifies which of the assigned nodes are connected with the same additional node; these latter are expressed by their unknown coordinates that are supposed to be continuous variables running in the Euclidean space.

Some of the features of this type of models are not completely satisfactory. First of all, since the connection scheme is assigned, only particular types of solution are obtained. Secondly, Steiner-Weber models take into account only the distance between nodes to evaluate the network performance, while in computer networks this aspect is not the most meaningful, as well as it is not realistic to suppose that the coordinates of the unknown nodes are continuous variables.

A variational model could well overcome these difficulties. One of its applications consists in fixing both intermediate nodes at possible positions and arcs linking these nodes. For any possible configuration of arcs and nodes, the variational model gives a solution that is an equilibrium flow. Since we are in a design or reconfiguration stage, the comparison between equilibrium flows relative to different configurations of the network is now fundamental. A suitable criterion for the comparison could be a measure of the flow stability.

Let us consider model (1); as we already pointed out, one of its solutions furnishes the demand in excess, because (1) does not take into account capacities. Let $H \in \mathbb{R}^m$ be a flow solution of (1) and d the vector of upper bounds to the flows on links present in model (2); therefore

$$D = \sum_{s=1}^{m} \max \left\{0, (\Delta H)_s - d_s\right\}$$

is the sum of the demands in excess corresponding to the solution H and hence to the given network. This real number can be used for the comparison between different configurations of the network, in the sense that "the best" configuration is the one that corresponds to the minimum value of D. An analogous criterion can be established when there are capacities on the nodes and a solution of (1) is available.

In the case of networks with capacities, a criterion of comparison is not immediately obtainable. What follows is a proposal arising from some recent and not yet definitive results concerning the concepts of potential and duality for a Variational Inequality [1], [3].

When the flows in a network are analysed by means of a mathematical model, the concept of duality arises and consequently that of potential. Roughly speaking, the dual variable associated to a node or to a link represents the pressure (i.e. the potential or the difference of potential) that the flow produces on the node or on the link in presence of capacities. Hence, if a variable with this meaning would be available in the variational model, it could be used to define a criterion of comparison between flows that are solution of different networks configurations; for example, by summing all the positive differences of potential.

In [1] an extension of the concept of potential is proposed for a particular Variational Inequality model, in which the capacities are on the links and are expressed as upper bounds to the flows on links. First of all, it would be interesting to extend the analysis to the variational formulation (2) in which the capacities are on the flows on paths instead of links.

Unfortunately, since (2) does not contain a particular constraint, i.e. the conservation of flow at every node, the mere extension of the above analysis to model (2) cannot provide the potential at nodes. This lack is particularly serious as regards the application to computer networks, owing to the above remarked importance of nodes in message and packet switching strategy, which we described above.

As a conclusion we can underline that the study of duality theory for Variational Inequality models needs to be carried on, due to the crucial importance that it has in equilibrium problems.

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Istituto di Matematica,
Università di Verona,
Via dell'Artigliere 19,
37129 Verona (ITALY)
e-mail: letpel@chiostro.univr.it