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ON SOME RECENT TOPOLOGICAL MINI-MAX THEOREMS

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We report some recent topological minimax theorems pointing out some applications and conjectures.

Let X, Y be two non-empty sets and let f be a real function defined on $X \times Y$. We are interested in the classical problem of finding suitable conditions under which the equality

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y) \tag{1}$$

does hold.

Perhaps, the most celebrated mini-max theorem, known also to non specialists, is the so called Fan-Sion theorem. Its statement is as follows:

Theorem 1. *Let X, Y be compact convex sets each in a Hausdorff topological vector space. If $f(\cdot, y)$ is lower semicontinuous and quasi-convex in X for each $y \in Y$, and $f(x, \cdot)$ is upper semicontinuous and quasi-concave in Y for each $x \in X$, then equality* (1) *does hold.*

The literature on mini-max theory is by now really impressive (see, for instance, [13]). It is rather easy, however, to recognize a few research currents within which practically each existing contribution can be located. In particular, one of these currents is that concerning the so called topological mini-max theorems. They are results where the assumptions are of purely topological nature. In proper sense, the first theorem of this kind was obtained by Wu ([15]). Precisely, he realized that, in Theorem 1, the convexity assumptions could be replaced by appropriate connectedness conditions. The original result by Hu, however, contained more stringent hypotheses on the continuity of f, so it could not cover completely Theorem 1. Wu's result was subsequently refined by other Authors (see, for instance, [14], [4]). Such a refinement process got recently the contribution by H. König who established the following result which covers both Theorem 1 and Wu's theorem:

Theorem 2 ([5], Theorem 1.2). *Let X, Y be topological spaces, with X compact. Assume that:*

(i_1) *for each $\lambda > \mu$, where $\mu = \sup_{y\in Y} \inf_{x\in X} f(x, y)$, and each non-empty finite set $H \subseteq Y$, the set*

$$\bigcap_{y\in H} \{x \in X : f(x, y) \leq \lambda\}$$

is connected;

(i_2) *for each $\lambda > \mu$ and each non-empty set $H \subseteq X$, the set*

$$\bigcap_{x\in H} \{y \in Y : f(x, y) > \lambda\}$$

is connected.

Moreover, suppose that either

(h_1) *for each $x \in X$, the function $f(x, \cdot)$ is upper semicontinuous in Y and, for each $y \in Y$, the function $f(\cdot, y)$ is lower semicontinuous in X;*

or

(h_2) *the function f is lower semicontinuous in $X \times Y$.*

Then, equality (1) *does hold.*

Remark 1. Always in [5], König asked whether, in Theorem 2, one can replace condition (i_2) by the following:

(i_2') for each $\lambda > \mu$ and each non-empty finite set $H \subseteq X$, the set

$$\bigcap_{x\in H} \{y \in Y : f(x, y) > \lambda\}$$

is connected.

Very recently [6], assuming condition (h_1), he showed that the answer to this question is, in general, negative, but it is positive if Y is compact. O. Naselli [7] has completed these investigations of König showing that, when condition (h_2) occurs, the answer to the above question can be negative even if Y is compact.

We wish now to discuss condition (i_1) (analogous considerations can be repeated with regard to (i_2)).

Apart from the case in which the space X has the very special property that the intersection of each pair of its connected subsets is connected too, it is apparent that the most natural setting in which condition (i_1) is automatically satisfied is when, X being a convex set in a topological vector space, the function $f(\cdot, y)$ is quasi-convex in X for each $y \in Y$. Out of that setting, it seems that the known literature does not contain any other intermediate condition (easy to be used) ensuring the validity of (i_1). In other words, the effective superiority of (i_1) with respect to the quasi-convexity of $f(\cdot, y)$, while is out of discussion from a theoretical point of view, becomes rather problematic in the practice. So, it is clear that, mainly from an operative point of view, a completely satisfactory alternative to the quasi-convexity of $f(\cdot, y)$ would be the natural improvement of condition (i_1), in the following terms:

(j_1) for each $y \in Y$ and each $\lambda > \mu$, the level set

$$\{x \in X : f(x, y) \leq \lambda\}$$

is connected.

We wish to stress at once that, in general, Theorem 2 is no longer true if (i_1) is replaced by (j_1). In this connection, consider the following

Example 1. Take:

$$X = \{(t, u) \in \mathbb{R}^2 : t^2 + u^2 = 1\}$$

$$Y = \{(v, z) \in \mathbb{R}^2 : v^2 + z^2 \leq 1\}$$

and, for each $(t, u) \in X$, $(v, z) \in Y$,

$$f(t, u, v, z) = tv + uz.$$

So, for fixed $(t, u) \in X$, we have

$$\sup_{(v,z)\in Y} f(t, u, v, z) = \sqrt{t^2 + u^2} = 1.$$

Moreover, for fixed $(v, z) \in Y$, we have

$$\inf_{(t,u)\in X} f(t, u, v, z) = -\sqrt{v^2 + z^2}.$$

Hence, it follows that

$$0 = \sup_{(v,z)\in Y} \inf_{(t,u)\in X} f(t, u, v, z) < \inf_{(t,u)\in X} \sup_{(v,z)\in Y} f(t, u, v, z) = 1.$$

Of course, except (i_1), each other assumption of Theorem 2, as well as (j_1), is satisfied.

Likewise, if we consider the case in which $X =]0, 1]$, $Y = [0, +\infty[$ and $f(x, y) = xy$ for all $(x, y) \in X \times Y$, we realize that, in Theorem 2, compactness of X cannot be removed.

Observe that, in the above examples, the space Y does not admit any continuous bijection onto $[0, 1]$. When, on the contrary, this instance does occur, we can obtain, in a unified way, a large number of mini-max theorems in which the other assumptions are practically optimal from the point of view of the topological theory. This was precisely shown in [9]. For instance, we have the following two results:

Theorem 3 ([9], Theorem 1.1). *Let X, Y be topological spaces, with Y connected and admitting a continuous bijection onto* $[0, 1]$. *Assume that, for each $\lambda > \mu$, $x_0 \in X$, $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) \leq \lambda\}$$

and

$$\{y \in Y : f(x_0, y) > \lambda\}$$

are connected. In addition, assume that at least one of the following three sets of conditions is satisfied:

(h_1) *$f(x, \cdot)$ is upper semicontinuous in Y for each $x \in X$, and $f(\cdot, y)$ is lower semicontinuous in X for each $y \in Y$;*
(h_2) *Y is compact, and f is upper semicontinuous in $X \times Y$;*
(h_3) *X is compact, and f is lower semicontinuous in $X \times Y$.*

Under such hypotheses, equality (1) *does hold.*

Theorem 4 ([9], Theorem 1.2). *Let X, Y be topological spaces, with X connected and Y admitting a continuous bijection onto* $[0, 1]$. *Assume that, for each $x_0 \in X$, $y_0 \in Y$, one has*

$$\inf_{y \in Y} f(x_0, y) < \mu < \sup_{x \in X} f(x, y_0).$$

In addition, assume that at least one of the following two sets of conditions is satisfied:

(k_1) *for each $x \in X$, the sets*

$$\{y \in Y : f(x, y) < \mu\}$$

and

$$\{y \in Y : f(x, y) > \mu\}$$

are connected, and, for each $y \in Y$, the sets

$$\{x \in X : f(x, y) < \mu\}$$

and

$$\{x \in X : f(x, y) > \mu\}$$

are open;

(k_2) *Y is compact, for each $x \in X$, the sets*

$$\{y \in Y : f(x, y) < \mu\}$$

and

$$\{y \in Y : f(x, y) \geq \mu\}$$

are connected, and the set

$$\{(x, y) \in X \times Y : f(x, y) \geq \mu\}$$

is closed.

Under such hypotheses, equality (1) *does hold.*

Theorems 3 and 4 were obtained making essential use of the following principle:

Theorem 5 ([9], Theorem 2.3). *Let X, Y be two topological spaces, with Y admitting a continuous bijection onto* $[0, 1]$, *and let S, T be two subsets of $X \times Y$, with S connected and, for each $x \in X$, $\{y \in Y : (x, y) \in T\}$ connected. Moreover, assume that either $\{x \in X : (x, y) \in T\}$ is open for each $y \in Y$, or Y is compact and T is closed.*

Then, at least one of the following assertions does hold:

(α) *There exists $x_0 \in X$ such that*

$$(\{x_0\} \times Y) \cap T = \emptyset.$$

(β) *There exist $y_1, y_2 \in Y$ such that*

$$(X \times \{y_1\}) \cap S = \emptyset$$

and

$$(p_X(S) \times \{y_2\}) \cap T = \emptyset,$$

where $p_X(S)$ is the projection of S on X.

(γ) $S \cap T \neq \emptyset$.

Theorem 5 has numerous further applications (see, for instance, [1], [2], [3], [11]).

It is important to note that Theorem 3 has been the key tool to obtain a certain variational property of integral functionals [10]. We conclude just recalling the results of [10] (see also [12]), and pointing out some related conjectures.

We first fix some notation. Throughout the sequel, $(T, \mathcal{F}, \mu)$ is a σ-finite non-atomic measure space, $(E, \|\cdot\|)$ is a real Banach space, whose Borel family is denoted by $\mathcal{B}(E)$, and p is a real number in $[1, +\infty[$.

For simplicity, we denote by X the usual space $L^p(T, E)$ of (equivalence classes of) strongly measurable functions $u : T \to E$ such that $\int_T \|u(t)\|^p d\mu < +\infty$, equipped with the norm $\|u\|_X = (\int_T \|u(t)\|^p d\mu)^{1/p}$. X^* will denote the topological dual of X.

Moreover, we denote by $\mathcal{A}(T \times E)$ the set of all functions $f : T \times E \to \mathbb{R}$ such that, for each $u \in X$, the function $t \to f(t, u(t))$ belongs to $L^1(T)$. If $f \in \mathcal{A}(T \times E)$, we put

$$\Phi_f(u) = \int_T f(t, u(t))\, d\mu$$

for all $u \in X$.

We denote by $\mathcal{E}(T \times E)$ the set of all $\mathcal{F} \otimes \mathcal{B}(E)$-measurable functions $f : T \times E \to \mathbb{R}$ which are upper semicontinuous in $x (x \in E)$.

We denote by $\mathcal{G}(T \times E)$ the set of all functions $f : T \times E \to \mathbb{R}$ for which there are $\alpha \in L^1(T)$, $\gamma_i \in]0, 1[$ and $\beta_i \in L^{\frac{p}{p-\gamma_i}}(T)$ $(i = 1, \ldots, k)$ such that

$$-\alpha(t) \leq f(t, x) \leq \alpha(t) + \sum_{i=1}^{k} \beta_i(t) \|x\|^{\gamma_i}$$

for almost every $t \in T$ and for every $x \in E$.

We denote by $\mathcal{V}(X)$ the family of all sets $V \subseteq X$ such that

$$V = \{u \in X : \Psi(u) = \Phi_g(u)\}$$

where $\Psi \in X^*$, $g \in \mathcal{A}(T \times E)$ and Φ_g is Lipschitzian on X, with Lipschitz constant strictly smaller than $\|\Psi\|_{X^*}$.

Finally, we denote by $\mathcal{I}(T \times E)$ the set of all functions $f \in \mathcal{A}(T \times E)$ such that

$$-\infty < \inf_{u \in X} \Phi_f(u) = \inf_{u \in V} \Phi_f(u)$$

for every $V \in \mathcal{V}(X)$.

In [10], we got the following results:

Theorem 6 ([10], Theorem 1). *Let E be separable and μ be complete. Then, one has*

$$\mathcal{E}(T \times E) \cap \mathcal{G}(T \times E) \subseteq \mathcal{I}(T \times E).$$

Theorem 7 ([10], Theorem 3). *Let $\gamma \in]0, 1[$, $v \in L^{\gamma}(T, E)$ and $\beta \in L^{\frac{p\gamma}{p-\gamma}}(T)$. Then, for every $V \in \mathcal{V}(X)$, one has*

$$\inf_{u \in V} \int_T \|v(t) - \beta(t)u(t)\|^{\gamma} \, d\mu = \int_{\beta^{-1}(0)} \|v(t)\|^{\gamma} \, d\mu.$$

Theorem 8 ([10], Theorem 4). *Let E be separable and let $f : T \times E \to \mathbb{R}$ be a function which is measurable with respect to t and continuous with respect to $x (t \in T, x \in E)$. Moreover, assume that there exists some $\alpha \in L^1(T)$ such that*

$$|f(t, x)| \leq \alpha(t)$$

for almost every $t \in T$ and for every $x \in E$.

Then, for every $V \in \mathcal{V}(X)$, one has

$$]\inf_X \Phi_f, \sup_X \Phi_f[\subseteq \Phi_f(V).$$

So, in particular, for each $r \in]\inf_X \Phi_f, \sup_X \Phi_f[$, the convex hull of the set $\Phi_f^{-1}(r)$ is dense in X.

Theorem 9 ([10], Theorem 5). *Let E be reflexive and separable, let $p > 1$, and let $f \in \mathcal{E}(T \times E) \cap \mathcal{G}(T \times E)$. Moreover, assume that, for some $\psi \in L^{\frac{p}{p-1}}(T)$ and for almost every $t \in T$, $f(t, \cdot) \in C^1(E)$, $f'_x(t, \cdot)$ is uniformly continuous in E and*

$$\sup_{x \in E} \|f'_x(t, x)\|_{E^*} \leq \psi(t).$$

Then, for every $V \in \mathcal{V}(X)$, there exists a sequence $\{u_n\}$ in V such that

$$\lim_{n \to \infty} \Phi_f(u_n) = \inf_{u \in X} \Phi_f(u)$$

and

$$\lim_{n \to \infty} \int_T \|f'_x(t, u_n(t))\|_{E^*}^{\frac{p}{p-1}} \, d\mu = 0.$$

Assuming that E and μ are as in Theorem 6, our conjectures can be stated as follows:

Conjecture 1. Let $U \subseteq X$ be a closed set such that

$$\inf_X \Phi_f = \inf_U \Phi_f$$

for every $f \in \mathcal{E}(T \times E) \cap \mathcal{G}(T \times E)$. Then, there exists some $V \in \mathcal{V}(X)$ such that

$$V \subseteq U.$$

Conjecture 2. There exists some function $f : T \times E \to \mathbb{R}$ such that

$$f \in \mathcal{G}(T \times E) \setminus \mathcal{I}(T \times E)$$

and

$$-f \in \mathcal{E}(T \times E).$$

Finally, we refer to [8] for another application of Theorem 3.

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