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ERROR ESTIMATES FOR FINITE ELEMENT SOLUTION FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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In this paper we first study the stability of Ritz-Volterra projection and its maximum norm estimates, and then we use these results to derive some L^∞ error estimates for finite element methods for parabolic partial integro-differential equations.

1. Introduction.

In the study of finite element methods for parabolic partial integro-differential equations [1] [2], the following Ritz-Volterra projection has been introduced: for $u(t) \in \overset{\circ}{W}{}_2^1(\Omega)$, $t \in J = (0, T]$, its Ritz-Volterra projection

$$V_h(t) : C(\overline{J}; \overset{\circ}{W}{}_2^1(\Omega)) \to C(\overline{J}; S_h)$$

is defined by

$$A(t; V_h u - u, \chi) + \int_0^t B(t, \tau; V_h u(\tau) - u(\tau), \chi)\, d\tau = 0 \tag{1.1}$$

$\chi \in S_h$, $t \in \overline{J}$, where $A(t; \cdot, \cdot)$ and $B(t, \tau; \cdot, \cdot)$ are the bilinear forms associated with the positive symmetric definite elliptic operator $A(t)$ and an arbitrary second order operator $B(t, \tau)$, respectively, with smooth coefficients, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$

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is a bounded domain, and $S_h \subset \overset{\circ}{W}{}_2^1(\Omega)$, with a small parameter $h > 0$, are finite dimensional subspaces. $\|\cdot\|_p = \|\cdot\|_{0,p}$, $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_{r,p}$ denote the norm on the Sobolev spaces $W_p^r(\Omega)$ for $2 \leq p \leq \infty$.

Notice that when $t = 0$, we have $V_h(0) = R_h$, the standard Ritz projection associated with the operator $A(0)$.

It has been proved in [1], [2] that the Ritz-Volterra projection V_h defined by (1.1) exists and is unique, and it also enjoyes the following approximation properties: for $t \in \overline{J}$,

$$\|D_t^j(V_h u(t) - u(t))\| + h\,\|D_t^j(V_h u(t) - u(t))\|_{1,2} \leq \tag{1.2}$$

$$\leq Ch^r \sum_{l=0}^{j} |||D_t^l u(t)|||_{r,2}$$

for $u \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^r$, $j = 0, 1$, $1 \leq r \leq k$, provided that the approximation space S_h satisfies for some $k \geq 2$ the inequality

$$\inf_{\chi \in S_h} \left\{\|u - \chi\| + h\,\|u - \chi\|_{1,2}\right\} \leq Ch^s \|u\|_{s,2}\,, \quad 1 \leq s \leq k$$

where

$$|||u(t)|||_{r,p} = \|u(t)\|_{r,p} + \int_0^t \|u(\tau)\|_{\tau,p}\, d\tau.$$

Here and in what follows we denote by C the generic constants independent of u and h, if not stated otherwise.

Now we consider the finite element solution for the following parabolic integro-differential equation

$$\begin{aligned} u_t + A(t)\,u + \int_0^t B(t,\tau)\,u\,(\tau)\,d\tau &= f \quad \text{in} \quad \Omega \times J, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times J, \\ u &= v \quad \text{in} \quad \Omega \times \{0\}, \end{aligned} \tag{1.3}$$

and let $u_h(t)$ be its semi-discrete finite element analogue [1]. By using the Ritz-Volterra projection V_h defined by (1.1) the authors of [1] have shown for smooth data $u(0) = v$ that if $\|u_h(0) - v\| \leq Ch^r \|v\|_{r,2}$, then

$$\|u(t) - u_h(t)\| \leq Ch^r \left\{\|v\|_{r,2} + \int_0^t \|u_t(\tau)\|_{r,2}\, d\tau\right\}, \tag{1.4}$$

which is the same error as that for parabolic equations [11]. The estimates (1.4) was obtained also by Thomee and Zhang in [10] by employing the standard Ritz projection R_h [7]. A slightly weaker error estimate similar to (1.4) has been shown in [1], [2]. We know from [1], [2] that it is easier and more convenient to use the Ritz-Volterra projection V_h than the Ritz projection R_h in the study of finite element methods for problem (1.3), and moreover, this new projection V_h has a variety of other applications.

It is well known (see [7]) that if S_h are piecewise polynomial spaces imposed on quasi-uniform triangulations of Ω, the Ritz projection R_h satisfies the stability estimate

$$\|R_h u\|_{1,p} \leq C\,\|u\|_{1,p}\,, \quad 2 \leq p \leq \infty. \tag{1.5}$$

More importantly, this stability can be used to derive some optimal error estimates for finite element approximations for elliptic [7] and parabolic equations.

In this paper we study the stability of our Ritz-Volterra projection V_h. Due to the complexity of the problem, the integral term and the corresponding loss of ellipticity, we shall consider only a special case of (1.1). Namely, we assume that $\Omega \subset \mathbb{R}^2$,

$$A(t) = -\nabla \cdot a(\cdot, t)\nabla, \quad B(t, \tau) = -\nabla \cdot b(\cdot, t, \tau)\nabla \tag{1.6}$$

where $a(x, t) \geq a_0 > 0$ and $b = b(x, t, \tau)$ are smooth functions, and ∇ is the gradient operator in $\mathbb{R}^2$. Thus, the Ritz-Volterra projection V_h in (1.1) becomes

$$\left(a(\cdot, t)\nabla(V_h u(t) - u(t)) + \int_0^t b(\cdot, t, \tau) \cdot \nabla(V_h u(\tau) - u(\tau))\,d\tau, \nabla\chi\right) = 0,$$

$\chi \in S_h$, $t \in \overline{J}$, or for short,

$$a(t; V_h u - u, \chi) + \int_0^t b(t, \tau; V_h u - u, \chi)\,d\tau = 0\,, \tag{1.7}$$

$\chi \in S_h$, $t \in \overline{J}$, where $a(t; \cdot, \cdot)$ and $b(t, \tau; \cdot, \cdot)$ are bilinear forms associated with the above special operators in (1.6).

We shall show in Section 2 the following result for V_h defined in (1.7).

$$\|V_h u(t)\|_{1,p} \leq C|||u(t)|||_{1,p}\,, \qquad 2 \leq p \leq \infty. \tag{1.8}$$

Although (1.7) is a very simple case of (1.1) it still preserves the essential features for the general Ritz-Volterra projection V_h. That is, it is our conjecture that the stability result (1.8) will hold for the general form (1.1).

In Section 2 we state and prove our main theorems. In Section 3 we shall employ the results obtained in Section 2 to derive some optimal error estimates for finite element methods for parabolic integro-differential equations.

2. Stability of Ritz-Volterra projection.

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. For $k \geq 2$, $0 < h \leq 1$, let S_h^k be one parameter family of finite-dimensional subspaces of $\overset{\circ}{W}{}_2^1(\Omega)$, consisting of piecewise polynomial functions of degree at most $k-1$, defined on a quasi-uniform partition of Ω. It is required that S_h^k possesses the following approximation property: for all $w \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_p^k(\Omega)$,

$$\inf_{\chi \in S_h^k} (\|w - \chi\|_p + h\,\|w-\chi\|_{1,p}) \leq Ch^s \|w\|_{s,p}, \quad p \geq 2, \; 1 \leq s \leq k. \tag{2.1}$$

Lemma 2.1. *Let $P_h : L^2(\Omega) \to S_h^k$ be the L^2-projection, then*

$$\|P_h w\|_{s,p} \leq C\,\|w\|_{s,p}, \qquad s = 0, 1, \quad 2 \leq p \leq \infty\,. \tag{2.2}$$

Proof. See [6]. □

Let $z \in \Omega$ and let $\delta_h^z \in S_h^k$ be the discrete δ -function at z such that

$$(\delta_h^z, \chi) = \chi(z)\,, \qquad \chi \in S_h^k\,. \tag{2.3}$$

Let G^z be the smooth Green's function at z that

$$\begin{aligned} -\nabla \cdot a \nabla G^z &= \delta_h^z \quad &&\text{in} \quad \Omega \\ G^z &= 0 \quad &&\text{on} \quad \delta\Omega\,. \end{aligned} \tag{2.4}$$

It is obvious that $G^z \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$ exists and is unique, and it follows by (2.3) that

$$a(t; G^z, w) = P_h w(z)\,, \qquad w \in \overset{\circ}{W}{}_2^1(\Omega). \tag{2.5}$$

Let $G_h^z \in S_h^k$ be the Ritz projection of G^z, i.e.,

$$a(t; G^z - G_h^z, \chi) = 0, \qquad \chi \in S_h^k. \tag{2.6}$$

It is well known [9] that

$$\|G^z - G_h^z\|_{1,1} \leq Ch(\log(1/h))^{k^*}, \qquad k^* = \begin{cases} 1 & \text{if} \quad k = 2 \\ 0 & \text{if} \quad k \geq 3 \end{cases}. \tag{2.7}$$

Define

$$\partial_z G^z = \lim_{\Delta z \to 0,\; \Delta z \,\|\, L} \frac{G^{z+\Delta z} - G^z}{|\Delta z|}\,,$$

where L is an arbitrary fixed direction. We know from (2.4)–(2.6) that $\partial_z G^z \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega)$ exists and is such that

$$a(t; \partial_z G^z, w) = \partial_z w(z)\,, \qquad w \in \overset{\circ}{W}{}_2^1(\Omega), \tag{2.8}$$

$$a(t; \partial_z G^z - \partial_z G_h^z, \chi) = 0, \qquad \chi \in S_h^k. \tag{2.9}$$

Let $\phi(x) = (|x - z|^2 + \rho^2)^{-1}$, with $\rho = \gamma h$ and γ large enough, be the weight. We define the weighted norms for $\alpha \in \mathbb{R}$,

$$\|f\|_{\phi^\alpha} = \left(\int_\Omega \phi^\alpha |f|^2 \, dx \right)^{1/2},$$

$$|f|_{1,\phi^\alpha} = \left(\int_\Omega \phi^\alpha (|f|^2 + |\nabla f|^2) \, dx \right)^{1/2}.$$

It follows from a direct computation that

$$\int_\Omega \phi^\alpha(x)\, dx \le C(\alpha - 1)^{-1} \rho^{-2(\alpha-1)}\,, \qquad \alpha > 1.$$

We now recall the following results concerning the estimates for Green's function G^z and its Ritz projection G_h^z [7].

Lemma 2.2. *Under our assumptions on S_h^k, we have*

$$\|\partial_z G^z - \partial_z G_h^z\|_{1,\phi^{-1-\varepsilon}} \le C h^\varepsilon\,, \quad \varepsilon \in (0, 1), \tag{2.10}$$

$$\|\partial_z G^z - \partial_z G_h^z\|_{1,1} + \|G^z\|_{1,1} + \|G_h^z\|_{1,1} + \|G_h^z\| \le C\,, \tag{2.11}$$

$$\|\partial_z G^z\|_q \le C, \qquad 1 \le q \le 3/2. \tag{2.12}$$

Proof. (2.10)–(2.11) can be found in [7]. For (2.12), let w satisfy

$$-\nabla \cdot a \nabla w = g\,, \qquad x \in \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega$$

and

$$\|w\|_{2,p} \le C_p \|g\|_p, \qquad 1 < p < \infty.$$

Let $p = 3$, we see from (2.7), stability of P_h and Sobolev imbedding theorem that

$$(\partial_z G^z, g) = a(t; \partial_z G^z, w) = \partial_z P_h w(z) \le$$
$$\le C\,\|w\|_{1,\infty} \le C\,\|w\|_{2,3} \le C_3 \|g\|_3 \le C\,\|g\|_p\,, \quad 3 \le p \le \infty.$$

Thus, (2.12) follows. □

We now state and show our main result in this section.

Theorem 2.1. *Assume that* $u \in L^1(J; \overset{\circ}{W}{}_2^1(\Omega))$. *Then the following stability estimate for our Ritz-Volterra projection* V_h *holds,*

$$\|V_h u(t)\|_{1,p} \le C\,|||u(t)|||_{1,p}, \quad t \in \overline{J}, \quad 2 \le p \le \infty. \tag{2.13}$$

Remark. When $t = 0$, (1.2) is just the stability estimate (1.5) obtained by Ranacher and Scott [7] for Ritz projection R_h.

Proof. It has been shown by a duality argument in [5] that

$$\|V_h u - u\|_{1,p} \le C_p |||u|||_{1,p}, \qquad 2 \le p < \infty.$$

Thus, the case of $2 \le p \le 3$ follows.

For $3 \le p < \infty$, let $\eta = u(t) - V_h u(t)$, then we see from the definition of V_h and Green's functions that

$$\partial_z P_h \eta(z,t) = a(t;\eta,\partial_z G^z) + \int_0^t b(t,\tau;\eta(\tau),\partial_z G^z)\,d\tau -$$

$$-\int_0^t b(t,\tau;\eta(\tau),\partial_z G^z)\,d\tau = a(t;u,\partial_z G^z - \partial_z G_h^z) +$$

$$+\int_0^t b(t,\tau;\eta(\tau),\partial_z G^z - \partial_z G_h^z)\,d\tau - \int_0^t b(t,\tau;\eta(\tau),\partial_z G^z)\,d\tau =$$

$$= a(t;u,\partial_z G^z - \partial_z G_h^z) + \int_0^t b(t,\tau;u(\tau) - P_h u(\tau),\partial_z G^z - \partial_z G_h^z)\,d\tau +$$

$$+\int_0^t b(t,\tau;P_h\eta(\tau),\partial_z G^z - \partial_z G_h^z)\,d\tau - \int_0^t b(t,\tau;\eta(\tau),\partial_z G^z)\,d\tau =$$

$$= I_1 + \int_0^t (I_2 + I_3 + I_4)\,d\tau.$$

We see from Lemma 2.1 and Hölder inequality [7] that for I_1,

$$|I_1| \le C\left(\int_\Omega \phi^{1+\varepsilon}\,dx\right)^{(p-2)/2p}\left(\int_\Omega \phi^{1+\varepsilon}(|u|^p + |\nabla u|^p)\,dx\right)^{1/p} \cdot \tag{2.14}$$

$$\cdot \|\partial_z G^z - \partial_z G_h^z\|_{1,\phi^{-1-\varepsilon}} \le C h^{2\varepsilon/p}\left(\int_\Omega \phi^{1+\varepsilon}(|u|^p + |\nabla u|^p)\,dx\right)^{1/p}.$$

Similarly, we have

$$(2.15)\qquad |I_2| \le Ch^{2\varepsilon/p}\Big(\int_\Omega \phi^{1+\varepsilon}\big(|u(\tau)-P_hu(\tau)|^p + |\nabla(u(\tau)-P_hu(\tau))|^p\big)\,dx\Big)^{1/p},$$

$$(2.16)\qquad |I_3| \le Ch^{2\varepsilon/p}\Big(\int_\Omega \phi^{1+\varepsilon}\big(|P_h\eta(\tau)|^p + |\nabla P_h\eta(\tau)|^p\big)\,dx\Big)^{1/p}.$$

We can write I_4 as

$$I_4 = -b(t,\tau;u(\tau)-P_hu(\tau),\partial_zG^z) - b(t,\tau;P_h\eta(\tau),\partial_zG^z) = -M_1 - M_2.$$

Thus, it follows from the structure of the two operators in (1.6) and by integration by parts that

$$\begin{aligned}|M_2| &= \Big|\Big(a(\cdot,t)\nabla\Big[\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big)P_h\eta(\tau)\Big],\nabla\partial_zG^z\Big) - \\ &\quad - \Big(a(\cdot,t)P_h\eta(\tau)\nabla\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big),\nabla\partial_zG^z\Big)\Big| = \\ &= \Big|\partial_zP_h\Big[\Big(\frac{b(z,t,\tau)}{a(z,t)}\Big)P_h\eta(z,\tau)\Big] + \Big(\nabla\cdot a(\cdot,t)P_h\eta(\tau)\nabla\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big),\partial_zG^z\Big)\Big| \le \\ &\le \Big|\partial_zP_h\Big[\Big(\frac{b(z,t,\tau)}{a(z,t)}\Big)P_h\eta(z,\tau)\Big]\Big| + C\|P_h\eta(\tau)\|_{1,p}\,\|\partial_zG^z\|_q \le \\ &\le \Big|\partial_zP_h\Big[\Big(\frac{b(z,t,\tau)}{a(z,t)}\Big)P_h\eta(z,\tau)\Big]\Big| + C\|P_h\eta(\tau)\|_{1,p},\end{aligned}$$

where we have used (2.12) for $1 \le q \le 3/2$ since $p \ge 3$ and $p^{-1}+q^{-1}=1$. Also, for the same reason we have

$$|M_1| \le \Big|\partial_zP_h\Big[\Big(\frac{b(z,t,\tau)}{a(z,t)}\Big)\big(u(z,\tau)-P_hu(z,\tau)\big)\Big]\Big| + C\,\|u(\tau)\|_{1,p}.$$

Thus, we obtain from (2.14)–(2.16)

$$\|I_1\|_p \le Ch^{2\varepsilon/p}\Big(\max_{x\in\Omega}\int_\Omega \phi^{1+\varepsilon}\,dz\Big)^{1/p}\|u\|_{1,p} \le 1C\,\|u\|_{1,p},$$

$$\|I_2\|_p \le C\,\|u - P_h u\|_{1,p} \le C\,\|u\|_{1,p},$$

$$\|I_3\|_p \le C\,\|P_h \eta\|_{1,p},$$

and by estimates for M_i's, we have for I_4,

$$\|I_4\|_p \le C\|P_h\eta\|_{1,p} + C\,\|u\|_{1,p}.$$

Notice that if

$$H(x) = N(x) + \int_0^t K(x,\tau)\,d\tau,$$

then

$$\|H\|_p \le \|N\|_p + \int_0^t \|K(\tau)\|_p\,d\tau\,, \quad 2 \le p \le \infty.$$

Thus, we see from the estimates for I_i's that

$$\|P_h\eta\|_{1,p} \le C\,|||u(t)|||_{1,p} + C\int_0^t \|P_h\eta\|_{1,p}\,d\tau\,, \quad 3 \le p < \infty.$$

Notice that the above inequality also holds for $p = \infty$ by using (2.7) [7]. Thus, Gronwall's lemma implies

$$\|P_h\eta\|_{1,p} \le C\,|||u(t)|||_{1,p}$$

and

$$\|V_h u\|_{1,p} \le \|P_h\eta\|_{1,p} + \|P_h u\|_{1,p} \le C\,|||u(t)|||_{1,p}. \tag{2.17}$$

Hence, Theorem 2.1 follows. □

As a direct application of Theorem 2.1 we show the following result.

Corollary. *For any function* $u \in L^1(J; \overset{\circ}{W}{}^1_p(\Omega) \cap W^k_p)$ *we have*

$$\|u(t) - V_h u(t)\|_{1,p} \le Ch^{k-1}|||u(t)|||_{k,p}\,, \quad 2 \le p \le \infty, \tag{2.18}$$

$$\|u(t) - V_h u(t)\|_p \le C_p h^k |||u(t)|||_{k,p}\,, \quad 2 \le p < \infty \tag{2.19}$$

Remark. (2.19) has been shown in [4] by a different method and (2.18) is an improvement of the estimates obtained in [4].

Proof. Let I_h be the interpolant operator on S_h^k. We apply Theorem 2.1 for $u - I_h u$ and observe that $V_h \equiv \text{id}$ on S_h^k to obtain

$$\|V_h u(t) - I_h u(t)\|_{1,p} \le C\,|||u(t) - I_h(u(t)|||_{1,p}\,, \quad 2 \le p \le \infty. \tag{2.20}$$

Then, (2.18) follows from the approximation properties of the interpolant operator I_h.

To prove (2.19), let $p \in [2,\infty)$, $q = p/(p-1) \in (1,2]$, and $w \in \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2$ be such that

$$Aw = g = \operatorname{sgn}(u - V_h u)\,|u - V_h u|^{p-1} \quad \text{in } \Omega, \tag{2.21}$$

and

$$\|w\|_{2,q} \le C_p \|g\|_q \le C_p \|u - V_h u\|_p^{p-1}. \tag{2.22}$$

Thus, by (2.21), (2.22) and Hölder inequality we have

$$\begin{aligned} \|u - V_h u\|_p^p &= a(t; u - V_h u, w - I_h w) + a(t; u - V_h u, I_h w) \le \\ &\le C\,\|u - V_h u\|_{1,p}\,\|w - I_h w\|_{1,q} + a(t; u - V_h u, I_h w) \end{aligned} \tag{2.23}$$

and by (1.8)

$$\begin{aligned} a(t; u - V_h u, I_h w) &= -\int_0^t b(t,\tau; u(\tau) - V_h u(\tau), I_h w - w)\,d\tau - \\ &\quad - \int_0^t b(t,\tau; u(\tau) - V_h u(\tau), w) d\tau = \\ &= -\int_0^t b(t,\tau; u(\tau) - V_h u(\tau), I_h w - w)\,d\tau + \\ &\quad + \int_0^t \big(u(\tau) - V_h u(\tau), B(t,\tau) w\big)\,d\tau \le \\ &\le C \int_0^t \|u - V_h u\|_{1,p}\,d\tau\, \big(\|w - I_h w\|_{1,q} + \|w\|_{2,q}\big), \end{aligned}$$

so that we see from (2.22)–(2.23) that

$$\|u - V_h u\|_p \le C_p h^k |||u(t)|||_{k,p} + C_p \int_0^t \|u - V_h u\|_p\,d\tau. \tag{2.24}$$

Hence, the proof is complete by Gronwall's lemma. □

We now consider the case of $p = \infty$, the maximum norm estimates, and show:

Theorem 2.2. *Under the assumption of Theorem 2.1, we have*

$$\|u(t) - V_h u(t)\|_{s,\infty} \le Ch^{k-s}\big(\log(1/h)\big)^{(1-s)k^*} |||u(t)|||_{k,\infty}\,. \tag{2.25}$$

$$s = 0, 1, \qquad k^* = \begin{cases} 1 & if \;\; k = 2 \\ 0 & if \;\; k \ge 3 \end{cases},$$

Proof. For $s = 1$, this is a special case of (2.18) with $p = \infty$. For $s = 0$, we have as shown in Theorem 2.1,

$$P_h\eta(z,t) = a(t;\eta, G^z - G_h^z) + \int_0^t b(t,\tau;\eta, G^z - G_h^z)\,d\tau -$$
$$- \int_0^t b(t,\tau; u(\tau) - P_h u(\tau), G^z)\,d\tau - \int_0^t b(t,\tau; P_h\eta\,(\tau), G^z)\,d\tau =$$
$$= J_1 + J_2 + J_3 + J_4.$$

From (2.7) and Theorem 2.1 we obtain

$$|J_1 + J_2| \le C\,|||\eta|||_{1,\infty}\,\|G^z - G_h^z\|_{1,1} \le Ch^k\big(\log(1/h)\big)^{k^*} |||u(t)|||_{k,\infty}\,,$$

and for J_3 we see from the stability of P_h that

$$J_3 = \int_0^t \Big(a(\cdot,t)\nabla\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big)(u(\tau) - P_h u(\tau)), \nabla G^z\Big)\,d\tau -$$
$$- \int_0^t \Big(a(\cdot,t)\big(u(\tau) - P_h u(\tau)\big)\nabla\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big), \nabla G^z\Big)\,d\tau =$$
$$= \int_0^t P_h\Big[\Big(\frac{b(z,t,\tau)}{a(z,t)}\Big)\big(u(z,\tau) - P_h u(z,\tau)\big)\Big]\,d\tau +$$
$$+ \int_0^t \Big(a(\cdot,t)\big(u(\tau) - P_h u(\tau)\big)\nabla\Big(\frac{b(\cdot,t,\tau)}{a(\cdot,t)}\Big), \nabla G^z\Big)\,d\tau \le$$
$$\le C\int_0^t \|u - P_h u\|_{0,\infty}\,d\tau + C\int_0^t \|u - P_h u\|_{0,\infty}\,d\tau\,\|G^z\|_{1,1} \le$$
$$\le Ch^k \int_0^t \|u\|_{k,\infty}\,d\tau\,.$$

Similarly, we have

$$|J_4| \le C\int_0^t \|P_h\eta\|_{0,\infty}\,d\tau\,.$$

Collecting the above estimates for J_i's we obtain

$$\|P_h\eta\|_{0,\infty} \leq Ch^k\big(\log(1/h)\big)^{k^*}|||u(t)|||_{k,\infty} + C\int_0^t \|P_h\eta\|_{0,\infty}\,d\tau\,.$$

Thus, Gronwall's lemma implies

$$\|P_h\eta\|_{0,\infty} \leq Ch^k\big(\log(1/h)\big)^{k^*}|||u(t)|||_{k,\infty}\,. \tag{2.26}$$

Hence, Theorem 2.2 follows from the inequality

$$\|V_hu - u\|_{0,\infty} \leq \|P_h(V_hu - u)\|_{0,\infty} + \|P_hu - u\|_{0,\infty}$$

and relation (2.26). □

3. An application to parabolic integro-differential equations.

In this section we consider someL^∞ error estimates for finite element methods for the parabolic integro-differential equation (1.3). As before we assume that the operators $A(t)$ and $B(t,\tau)$ have the special forms (1.6).

Let $u_h(t) : \bar{J} \to S_h^k$ be the finite element solution of problem (1.3) defined by

$$(u_{h,t}, \chi) + a(t; u_h, \chi) + \int_0^t b(t,\tau; u_h(\tau), \chi)\,d\tau = (f, \chi)\,, \qquad \chi \in S_h^k\,,$$
$$u_h(0) = v_h \in S_h^k\,.$$

It has been shown in [4] that the finite element approximations of parabolic integro-differential equations have *weak* L^∞ error estimates. That is, for any $\varepsilon > 0$ there exists a $C(\varepsilon, u) > 0$ such that

$$\|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq C(\varepsilon, u)h^{k-\varepsilon} \tag{3.1}$$

which is not optimal. Here we shall show the following result assuming sufficient regularity of the solution u at $t = 0$.

Theorem 3.1. *For $k = 2$, we assume that $u \in L^1(J; \overset{\circ}{W}{}^1_\infty(\Omega) \cap W^2_\infty)$ $u_t \in L^2(J, W^2_2)$ and $v_h = V_h(0)v = R_h(0)v$. Then we have*

$$\|u(t) - u_h(t)\|_{0,\infty} \leq Ch^2\Bigg\{\log(1/h)\big(\|v\|_{2,\infty} + |||u(t)|||_{2,\infty}\big) + \\ + \left(\log(1/h)\int_0^t \|u_t\|^2_{2,2}\,d\tau\right)^{1/2}\Bigg\}. \tag{3.2}$$

For $k \geq 3$, we assume that $u \in L^1(J; \overset{\circ}{W}{}^1_\infty(\Omega) \cap W^k_\infty)$, $u_t \in L^2(J; W^k_2)$ and $v_h = V_h(0)v = R_h(0)v$, $u_{tt} \in L^2(J; W^k_2)$, we have

$$\|u(t) - u_h(t)\|_{0,\infty} \leq Ch^k \Big\{ \|v\|_{k,\infty} + |||u(t)|||_{k,\infty} + \|u_t(0)\|_{k,2} + \int_0^t \|u_{tt}\|_{k,2}\, d\tau \Big\}. \tag{3.3}$$

Proof. As usual we write the error

$$e(t) = u(t) - u_h(t) = (u - V_h u) + (V_h u - u_h) = \eta + \theta.$$

Thus, we see from Theorem 2.2 that we need to estimate θ only.

We first show the case of $k = 2$. Since $v_h = V_h(0)v = R_h(0)v$, then $\theta(0) = 0$. It has been shown in [4] that

$$\|\theta\|_{1,2} \leq Ch^2 \Big(\|v\|_{2,2} + \Big(\int_0^t \|u_t\|^2_{2,2}\, d\tau \Big)^{1/2} \Big). \tag{3.4}$$

Thus, (3.2) follows from the *weak* Sobolev inequality on S^k_h [8],

$$\|\theta\|_{0,\infty} \leq C(\log(1/h))^{1/2} \|\theta\|_{1,2}$$

and the triangle inequality.

Now for the case of $k \geq 3$, we see that θ satisfies

$$a(t; \theta, \chi) + \int_0^t b(t, \tau; \theta(\tau), \chi) = -(e_t, \chi), \quad \chi \in S^k_h.$$

Letting $\chi = G^z_h$, it follows

$$\theta(z, t) = a(t; \theta, G^z_h) = -(e_t, G^z_h) - \int_0^t b(t, \tau; \theta(\tau), G^z_h)\, d\tau = K_1 + K_2,$$

and as before by the Lemma 2.1 we write K_2 as

$$K_2 = -\int_0^t a\Big(t; \frac{b(\cdot, t, \tau)}{a(\cdot, t)} \theta(\tau), G^z_h\Big)\, d\tau + \\ + \int_0^t \Big(a(\cdot, t)\, \theta(\tau) \nabla \Big(\frac{b(\cdot, t, \tau)}{a(\cdot, t)} \Big), \nabla G^z_h \Big)\, d\tau \leq$$

$$\le -\int_0^t a\Big(t; \frac{b(\cdot,t,\tau)}{a(\cdot,t)}\theta(\tau), G_h^h - G^z\Big)\,d\tau - \int_0^t a\Big(t; \frac{b(\cdot,t,\tau)}{a(\cdot,t)}\theta(\tau), G^z\Big)\,d\tau +$$

$$+ C\int_0^t ||\theta||_{0,\infty}\,d\tau\,||G^z||_{1,1} \le -\int_0^t a\Big(t; \frac{b(\cdot,t,\tau)}{a(\cdot,t)}\theta(\tau), G_h^h - G^z\Big)\,d\tau -$$

$$- \int_0^t P_h\Big[\frac{b(z,t,\tau)}{a(z,t)}\theta(z,\tau)\Big]\,d\tau + C\int_0^t ||\theta||_{0,\infty}\,d\tau\,||G^z||_{1,1} \le$$

$$\le -\int_0^t P_h\Big[\frac{b(z,t,\tau)}{a(z,t)}\theta(z,\tau)\Big]\,d\tau +$$

$$+ C\int_0^t ||\theta||_{1,\infty}\,d\tau\,||G_h^z - G^z||_{1,1} + C\int_0^t ||\theta||_{0,\infty}\,d\tau\,.$$

By the inverse assumption (quasi-uniformity), stability of P_h, (2.7) and Lemma 2.1, we obtain

$$K_2 \le C\int_0^t ||\theta||_{0,\infty}\,d\tau$$

and

$$K_1 \le ||e_t||\,||G_h^z|| \le C||e_t||\,.$$

Thus, we have

$$||\theta||_{0,\infty} \le C||e_t|| + C\int_0^t ||\theta||_{0,\infty}\,d\tau$$

and Gronwall's lemma implies

$$||\theta||_{0,\infty} \le C\Big(||e_t|| + \int_0^t ||e_t||\,d\tau\Big)\,.$$

However, we have from [4] that

$$||e_t|| \le ||\eta_t|| + ||\theta_t|| \le Ch^k\Big\{f\,|||u|||_{k,2} + |||u_t|||_{k,2} + \\ + ||v||_{k,2} + ||u_t(0)||_{k,2} + \int_0^t ||u_{tt}||_{k,2}\,d\tau\Big\}\,.$$

Hence, we have

$$||\theta||_{0,\infty} \le Ch^k\Big\{|||u|||_{k,2} + |||u_t|||_{k,2} + \\ + ||v||_{k,2} + ||u_t(0)||_{k,2} + \int_0^t ||u_{tt}||_{k,2}\,d\tau\Big\} \tag{3.5}$$

so that (3.3) follows from (3.5), Theorem 2.2 and the triangle inequality. □

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