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ON THE GENERALIZED LIFTING PROBLEM

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We construct curves for which the generalized lifting property does not hold, with high degree. We discuss the behaviour of the Hilbert function of the general plane section of these curves.

0. Introduction.

Let Y be an integral curve in $\mathbb{P}^3$, the projective three dimensional space over an algebraically closed field $\mathbf{k}$, that we will suppose of characteristic zero.

We denote, in the standard way, $d = \deg(Y)$, $g = p_a(Y)$, $e(Y) = \max\{t \mid h^2(\mathbb{P}^3, \mathscr{I}_Y(t)) \neq 0\}$, $s(Y) = \min\{t \mid h^0(\mathbb{P}^3, \mathscr{I}_Y(t)) \neq 0\}$, $M(Y)$ the Hartshorne-Rao module of Y and $\alpha(Y) = \min\{t \mid h^0(H, \mathscr{I}_\Gamma(t)) \neq 0\}$, where $\Gamma = Y \cap H$, H general plane.

We would like to know if it exists a function $f(\sigma, h)$ such that:

(*) If Y is a curve with $d > f(\sigma, h)$ and $h^0(H, \mathscr{I}_\Gamma(\sigma)) \neq 0$, then

$$h^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma + h)) \neq 0$$

(or, equivalently, $s(Y) \leq \sigma + h$) and there exists C of degree $f(\sigma, h)$ such that $h^0(H, \mathscr{I}_{C \cap H}(\sigma)) \neq 0$ and $h^0(\mathbb{P}^3, \mathscr{I}_C(\sigma + h)) = 0$ (note that now σ needs no longer to be *the minimal* value for which the plane section is contained in a curve of degree σ).

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This problem is closely related to the so-called "range-B problem", and now it is commonly called "the generalized lifting problem". Positive results are due to Laudal and Strano: it is known that $f(\sigma, 0) = \sigma^2 + 1$, and this bound is sharp (see [5], [7], [10]), and that the sharp $f(\sigma, 1)$ is between $\sigma^2 - 2\sigma + 6$ and $\sigma^2 - 2\sigma + 9$; i.e., if $d > \sigma^2 - 2\sigma + 9$, then property (*) holds with $h = 1$, and there are curves of degree $\sigma^2 - 2\sigma + 6$ (for every $\sigma \geq 4$) with $h^0(H, \mathscr{I}_\Gamma(\sigma)) \neq 0$ and $h^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma + 1)) = 0$. Moreover, for $h = 0$ boundary curves (i.e. with $d = \sigma^2 + 1$) are completely classified: they are all in the same liaison class and, for fixed σ, they have the same genus and form an irreducible family (see [4], [10], [11]).

In this paper we construct smooth curves with high degree not satisfying property (*), for every σ (provided it is enough large) and for every h, thus giving a lower bound for the function $f(\sigma, h)$. The basic idea is to use maximal rank curves and to bound σ looking to the range where the dimensions of the homogeneous components of $M(Y)$ are decreasing. From another point of view, we impose a gap between σ and s, and this forces $M(Y)$ to contain a substructure; we tried to get the "smallest" module having this property.

From the construction it turns out that it may happen that there are many possible genera for curves of this degree not having this generalized lifting property, but it is easy to pick out the curves with the maximal genus. This is maybe an evidence that this degree is not yet the best possible.

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1. Reflexive sheaves.

Up to now, one of the best available collection of good graded $\mathbf{k}[x_0, x_1, x_2, x_3]$-modules of finite lenght is given by the H^1-modules of rank-two reflexive sheaves with seminatural cohomology; for istance, they were used for constructing the conjectured boundary curves (i.e. with maximal $g(Y)$ with respect to the $d(Y)$ and $s(Y)$) in the so called range B. We collect here some fact about reflexive sheaves which will be used later on. This "choice" of good reflexive sheaves generalizes the construction done by Strano in [11] in the case $h = 1$.

We say that a rank two reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$ is normalized if $c_1(\mathscr{F}) = 0$ or -1. If $\mathscr{F}$ is a normalized rank two reflexive sheaf on $\mathbb{P}^3$, then Riemann-Roch gives

$$\chi(\mathscr{F}(t)) = \frac{1}{3}(t+1)(t+2)(t+3) - (t+2)c_2 + \frac{c_3}{2} \quad \text{if} \quad c_1 = 0$$

and

$$\chi(\mathscr{F}(t)) = \frac{1}{6}(t+1)(t+2)(2t+3) - (2t+3)c_2 + \frac{c_3}{2} \quad \text{if} \quad c_1 = -1.$$

Note that this implies that $\chi(\mathscr{F}(t))$ attains its minimum value (in the range $[-2, +\infty[$) at p, where p is the unique integer such that $(p+1)(p+2) \leq c_2 < (p+2)(p+3)$ if $c_1 = 0$, and $(p+1)^2 \leq c_2 < (p+2)^2$ if $c_1 = -1$.

We say that a normalized rank two reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$ has *seminatural cohomology* if for every $n \geq -2 - c_1$ at most one of the groups $H^i(\mathbb{P}^3, \mathscr{F}(n))$ is different from zero, $n = 0, 1, 2, 3$.

We say that a rank two reflexive sheaf $\mathscr{F}$ on $\mathbb{P}^3$ is *curvilinear* if it has the following property: $\mathscr{F}(t)$ globally generated $\Rightarrow$ the zero-set of a general section of $\mathscr{F}(t)$ is a smooth curve.

Several existence results for such sheaves are known; in particular, we need the following theorem by Hirschowitz:

Theorem 1.1. [6] *Let*

$$\begin{array}{lll} c_1 = 0, & c_2 \text{ even}, & c_3 \leq 4c_2, \text{ or} \\ c_1 = 0, & c_2 \text{ odd}, & c_3 \leq 4c^2 - 6, \text{ or} \\ c_1 = -1, & & c_3 \leq 4c_2 - 5 \text{ and } (c_2, c_3) \neq (2,0) \text{ and } (4,0). \end{array}$$

Then there exists a curvilinear reflexive sheaf with Chern classes (c_1, c_2, c_3) *and seminatural cohomology.*

The zero-sets of the sections of a reflexive sheaf (that we suppose from now on to be of rank two), if two-codimensional, are curves whose invariants are easily computable from the Chern classes and the cohomology of the sheaf; in particular, if $\mathscr{F}$ has Chern classes $(0, c_2, c_3)$ and $f \in H^0(\mathbb{P}^3, \mathscr{F}(p))$ is a section whose zero set Y is two-codimensional, then there is an exact sequence

$$(*) \qquad 0 \to \mathscr{O}_{\mathbb{P}}(-p) \to \mathscr{F} \to \mathscr{I}_Y(p) \to 0$$

and $\deg(Y) = c^2 + p^2$, $p_a(Y) = (1/2)[c_3 + 2 - (4 - 2p)(c_2 + p^2)]$.

If $\mathscr{F}$ has Chern classes $(-1, c_2, c_3)$, and $f \in H^0(\mathbb{P}^3, \mathscr{F}(p))$ is a section whose zero set Y is two-codimensional, then there is an exact sequence

$$(**) \qquad 0 \to \mathscr{O}_{\mathbb{P}}(-p) \to \mathscr{F} \to \mathscr{I}_Y(p-1) \to 0$$

and deg $(Y) = c^2 + p^2 - p$, $p_a(Y) = (1/2)[c_3 + 2 - (5-2p)(c_2 + p^2 - p)]$.

Note that if $H^0(\mathbb{P}^3, \mathcal{F}(p-1)) = 0$ and f is general, then automatically its zero-set is two-codimensional.

From the sequences above (and their twistings) it follows that $\oplus H^1(\mathbb{P}^n, \mathcal{F}(t))$ is isomorphic to $\oplus H^1(\mathbb{P}^n, \mathcal{I}_Y(t + p + c_1))$, hence the H^1-cohomology of $\mathcal{F}$ determines the Hartshorne-Rao module of the zero sets of the sections of $\mathcal{F}$.

A priori, the condition of "seminatural cohomology" gives the dimensions $h^1(\mathbb{P}^n, \mathcal{F}(t))$ only for $t \geq -2$ ($h^1(\mathbb{P}^n, \mathcal{F}(t)) = -\chi(\mathcal{F}(t))$, whenever the value of $\chi(\mathcal{F}(t))$ is negative, and zero otherwise); but for $t \leq -1$ the dimension $h^1(\mathbb{P}^n, \mathcal{F}(t))$ is easy to compute from the spectrum of $\mathcal{F}$, and this spectrum is completely determined for a sheaf with seminatural cohomology (Lemma at page 408, [1]).

If x is a real number, we denote with $\lceil x \rceil$ the minimum integer $\geq x$. Let us fix $h \geq 0$, and let $t = t(h) = \left\lceil \left(\frac{2h - 9 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$. Note that $t \geq -1$. Then consider a rank two reflexive sheaf $\mathcal{F}$ with Chern classes $(0, c_2, c_3)$, where

$$c_2 = t^2 + 5t + 5$$

and

$$c_3 = 2(t+h+3)(t^2+5t+5) - \frac{2(t+h+2)(t+h+3)(t+h+4)}{3}.$$

Claims.

1) $c_3 \geq 0$
2) $c_3 \leq 4c^2 - 6$
3) $\chi(\mathcal{F}(t+h+1)) = 0$.

Proof. 1) $c_3 \geq 0$ is equivalent to $(t^2+5t+5) - \frac{(t+h+2)(t+h+4)}{3} \geq 0$, and this happens precisely if $t \geq \frac{2h - 9 + \sqrt{12h^2 + 12h + 25}}{4}$. Note moreover that by definition t is the minimum integer k for which

$$(k^2 + 5k + 5)(k+h+3) - \frac{(k+h+2)(k+h+3)(k+h+4)}{3} \geq 0;$$

hence

$$(t^2 + 3t + h)(t+h+2) - \frac{(t+h+1)(t+h+2)(t+h+3)}{3} \leq -1.$$

2) $c_3 \leq 4c_2 - 6$ means

$$2(t+h+3)(t^2+5t+5) - \frac{2(t+h+2)(t+h+3)(t+h+4)}{3} \leq$$

$$\leq 4(t^2+5t+5) - 6,$$

and this is equivalent to

$$\frac{(t+h+2)(t+h+3)(t+h+4)}{3} - (t+h+1)(t^2+5t+5) - 3 \geq 0.$$

But this is true, since (from 1))

$$0 \leq \left[(t^2+3t+h)(t+h+2) - \frac{(t+h+1)(t+h+2)(t+h+3)}{3} - 1\right] +$$

$$+ [2t+1+h+h^2] = \frac{(t+h+2)(t+h+3)(t+h+4)}{3} -$$

$$-(t+h+1)(t^2+5t+5) - 3$$

(after some manipulation).

3) It is a simple computation, using Riemann-Roch.

As a consequence of these claims and of theorem 1.1, the sheaf $\mathcal{F}$ can be chosen curvilinear with seminatural cohomology. The cohomology table of $\mathcal{F}$ hence looks as follows:

$\mathcal{F}$	$\cdots$	t	$t+1$	$\cdots$	$t+h$	$t+h+1$	$t+h+2$	$\cdots$
h^0	$\cdots$	0	0	$\cdots$	0	0	$*$	$\cdots$
h^1	$\cdots$	$*$		$\cdots$	$*$	0	0	$\cdots$
h^2	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$
h^3	$\cdots$	0	0	$\cdots$	0	0	0	$\cdots$

2. Curves.

Let now again h be fixed and $\mathcal{F}$ the sheaf constructed in the previous paragraph, depending on h. By Castelnuovo-Mumford, $H^0(\mathbb{P}^3, \mathcal{F}(t+h+2))$ is globally generated, and therefore for every $r \geq 0$ we have that a general section of $H^0(\mathbb{P}^3, \mathcal{F}(t+h+2+r))$ gives an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-(t+h+r+2)) \to \mathcal{F} \to \mathcal{I}_Y(t+h+2+r)) \to 0$$

where Y is a smooth curve with maximal rank.

From the cohomology table of $\mathcal{F}$ it follows that

$$s(Y) \geq t+h+2+t+h+r+3 = 2t+2h+4+r.$$

On the other hand the module $M(Y)$ is isomorphic – up to shifting by $t+h+r+2$ the degrees – to the H^1 module of the sheaf $\mathcal{F}$, hence

$$\chi(\mathcal{F}(t)) > \chi(\mathcal{F}(t+1))$$

implies

$$h^1(\mathbb{P}^3, \mathcal{I}_Y(2t+h+2+r)) > h^1(\mathbb{P}^3, \mathcal{I}_Y(2t+h+3+r)).$$

Then for every plane H we have that $H^0(H, \mathcal{I}_\Gamma(2t+h+r+3)) \neq 0$, where $\Gamma = Y \cap H$.
Therefore $\sigma(Y) \leq 2t+h+r+3$.
The conclusion is that Y is a smooth curve such that, if we let $\sigma = 2t+h+r+3$, then $H^0(H, \mathcal{I}_\Gamma(\sigma)) \neq 0$; $H^0(\mathbb{P}^3, \mathcal{I}_Y(\sigma+h)) = 0$, and

$$\begin{aligned}\deg(Y) &= t^2+5t+5+(t+2+h+r)^2 = \\ &= t^2+5t+5+(\sigma-t-1)^2 = \sigma^2-2(t+1)\sigma+2t^2+7t+6,\end{aligned}$$

$$\begin{aligned}p_a(Y) &= (1/2)[c_3+2-(4-2p)(c_2+p^2)] = \\ &= (1/2)\Big[2(t+h+3)(t^2+5t+5) - \frac{2(t+h+2)(t+h+3)(t+h+4)}{3} + \\ &\qquad +2+(2t+2h+2r)(\sigma^2-2(t+1)\sigma+2t^2+7t+6\Big] = \\ &= (1/2)\Big[2(t+h+3)(t^2+5t+5) - \frac{2(t+h+2)(t+h+3)(t+h+4)}{3} + \\ &\qquad +2+(2t+2h+2r)d\Big],\end{aligned}$$

where $t = \left\lceil \left(\frac{2h - 9 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$. (The case $h = 0$, $r = 0$ is not connected: Y is the union of two skew lines).

The cohomology table of $\mathscr{I}_Y$ looks as follows (for $r = 0$, and $h \geq 1$):

$\mathscr{I}_Y$	$\cdots$	$2t+h+2$	$2t+h+3$	$\cdots$	$2t+2h$	$2t+2h+1$	$2t+2h+2$	$2t+2h+3$	$2t+2h+4$	$\cdots$
h^0	$\cdots$	0	0	$\cdots$	0	0	0	0	$*$	$\cdots$
h^1	$\cdots$	$*$		$\cdots$			$*$	0	0	$\cdots$
h^2	$\cdots$			$\cdots$	1	0	0	0	0	$\cdots$

Note that all examples (for fixed h) are in the same liaison class, and that for fixed h and r they form an irreducible family ([2]).

Note moreover that if $c_3 > 0$, then it may be possible to chose a lower integer between 0 and c_3 as a third Chern class for $\mathscr{F}$ (with the same c_1 and c_2). This gives a curve Z with the same degree, again satisfying $H^0(H, \mathscr{I}_{Z \cap H}(\sigma)) \neq 0$, $H^0(\mathbb{P}^3, \mathscr{I}_Z(\sigma + h)) = 0$. The only difference is that now $p_a(Z)$ is lower than $p_a(Y)$.

It is possible to do exactly the same construction with sheaves with $c_1 = -1$. In this situation, if $t = \left\lceil \left(\frac{2h - 7 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$, then there exists a curvilinear rank-two reflexive sheaf with seminatural cohomology and Chern classes $(-1, c_2, c_3)$, where

$$\begin{aligned} c_2 &= t^2 + 4t + 3 \\ c_3 &= (2t + 2h + 5)(t^2 + 4t + 3) - \frac{(t + h + 2)(t + h + 3)(2t + 2h + 5)}{3}. \end{aligned}$$

Again by Castelnuovo-Mumford, $H^0(\mathbb{P}^3, \mathscr{F}(t + h + 2))$ is globally generated, and therefore for every $r \geq 0$ we have that a general section of $H^0(\mathbb{P}^3, \mathscr{F}(t + h + 2 + r))$ gives an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-(t + h + r + 2)) \to \mathscr{F} \to \mathscr{I}_Y(t + h + 1 + r)) \to 0$$

where Y is a smooth curve with maximal rank.

The same argument as above shows that if we let $\sigma = 2t + h + r + 3$, then $H^0(H, \mathscr{I}_\Gamma(\sigma)) \neq 0$, $H^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma + h)) = 0$, and

$$\deg(Y) = \sigma^2 - (2t + 1)\sigma + 2t^2 + 5t + 3,$$

$$p_a(Y) = (1/2)[c_3 + 2 - (5-2p)(c_2 + p^2 - p)] =$$
$$= (1/2)[(2t+2h+5)(t^2+4t+3) - \frac{(t+h+2)(t+h+3)(2t+2h+5)}{3} +$$
$$+ 2 + (2t+2h+2r-1)(\sigma^2 - (2t+1)\sigma + 2t^2 + 5t + 3)]$$

where $t = \left\lceil \left(\frac{2h - 7 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$.

We have now two families of curves, but it is easy to chose, for every h, the highest degree. Let us denote

$$t_0(h) = \left\lceil \left(\frac{2h - 9 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$$
$$t_{-1}(h) = \left\lceil \left(\frac{2h - 7 + \sqrt{12h^2 + 12h + 25}}{4} \right) \right\rceil$$
$$d(0) = \sigma^2 - 2(t+1)\sigma + 2t^2 + 7t + 6$$
$$d(-1) = \sigma^2 - (2t+1)\sigma + 2t^2 + 5t + 3.$$

If $t_0(h) = t_{-1}(h)$, then $d(-1) > d(0)$; if $t_0(h) < t_{-1}(h)$, then $d(-1) < d(0)$. Hence we can unify the constructions in the following

Proposition 2.1. *Let* $h \geq 0$ *and* $k = \left\lceil \left(\frac{2h - 7 + \sqrt{12h^2 + 12h + 25}}{2} \right) \right\rceil$. *Then for every* $\sigma \geq k + h + 2$ *there exists a smooth connected curve* Y *such that*

$$H^0(H, \mathcal{I}_\Gamma(\sigma)) \neq 0\,, \quad H^0(\mathbb{P}^3, \mathcal{I}_Y(\sigma + h)) = 0\,,$$

and

$$\deg(Y) = \sigma^2 - (k+1)\sigma + \frac{k^2 + 5k + 6}{2}.$$

Note that $k \sim (1 + \sqrt{3})h$.

3. Further examples, comments about the Hilbert function of Γ and a conjecture.

In paragraph 2, for a fixed h, we constructed a sequence of curves of large degree, one for each value of σ, provided it is enough large, all belonging to the same liaison class.

Let us consider more closely the value $h = 1$ (see [11]). In this case, our construction starts (§ 1) with a rank-two reflexive sheaf $\mathcal{F}$ (actually, a vector bundle) with Chern classes (0,5,0), whose cohomolgy is

$\mathcal{F}$	-2	-1	0	1	2	3	...
h^0	0	0	0	0	0	15	...
h^1	0	5	8	7	0	0	...
h^2	0	0	0	0	0	0	...
h^3	0	0	0	0	0	0	...

In § 2, we saw that a general section of $\mathcal{F}(3)$ gives a smooth curve Y with $d(Y) = 14$, $g(Y) = 15$ whose cohomology is

$\mathcal{I}_Y$	0	1	2	3	4	5	6	...
h^0	0	0	0	0	0	0	$*$	...
h^1	0	0	5	8	7	0	0	...
h^2	15	4	1	0	0	0	0	...

which is minimal in its liaison class (7]). The argument in § 2 shows that for every plane H

$$h^0(H, \mathcal{I}_{Y \cap H}(4)) \neq 0, \quad \text{hence} \quad \sigma(Y) \leq 4.$$

A general section of $\mathcal{F}(3 + r)$, $r > 0$, gives a curve Y_r with

$$h^0(H, \mathcal{I}_{Y_r \cap H}(4 + r)) \neq 0 \quad \text{and} \quad h^0(\mathbb{P}^3, \mathcal{I}_Y(4 + r + 1)) = 0,$$

all belonging to the same liaison class.

What can be said for small values of σ? We can find curves with $d = \sigma^2 - 2\sigma + 6$ and the desired property in the so-called FW-range. In fact, Ch. Walter communicated to us the following existence result:

Proposition 3.1. [12] *Let* $F(d, n) = dn + 1 - \dfrac{(n + 3)(n + 2)(n + 1)}{6}$. *Then there exists a smooth curve in* $\mathbb{P}^3$ *of degree* d *and genus* g *with seminatural cohomology for*

$$0 \leq g \leq \max\{F(d, n) \mid n \geq 0 \ \textit{is an integer}\} = F(d, n(d)),$$

where $n(d)$ *is the integer such that* $\dfrac{(n + 2)(n + 1)}{2} \leq d < \dfrac{(n + 3)(n + 2)}{2}$ *(i.e. when the pair* (d, g) *is in the so-called FW-range).*

Example 1. $\sigma = 3$.
Take $d = 9$ and $g = 2$. This pair is the FW-range, hence there exists a smooth curve with degree 9 and genus 2 with natural cohomology, that is to say

$\mathcal{I}_C$	0	1	2	3	4	5	$\cdots$
h^0	0	0	0	0	0	$*$	$\cdots$
h^1	0	4	7	6	0	0	$\cdots$
h^2	2	0	0	0	0	0	$\cdots$

(Note that a maximal rank curve of degree of degree 9 and genus 2 has automatically seminatural cohomology, and the same is true in the next example). Just by counting dimensions, for the general plane H we have

$$h^0(H, \mathcal{I}_{C\cap H}(3)) \neq 0, \quad \text{whilst} \quad h^0(\mathbb{P}^3, \mathcal{I}_C(4)) = 0.$$

Note moreover that one has $h^1(H, \mathcal{I}_{C\cap H}(4)) = 0$, hence $\Delta H(5) = 0$ where ΔH is the first difference of the Hilbert function of $C \cap H$). It is customary to draw ΔH in a diagram whose behaviour is subjected to the following rules (if $Y \cap H$ is the general plane section of a smooth irreducible curve), see [10]:

$$\Delta H(t) = t + 1 \quad \text{if} \quad t = 0, \dots, a_1 - 1,$$

where $a_1 = \sigma(Y)$ is the least degree of a curve D containing $C \cap H$

$$\Delta H(t) = \sigma(Y) \quad \text{if} \quad t = \sigma(Y) \dots a_2 - 1,$$

where a_2 is the least degree of a curve containing $C \cap H$ and not containing D ΔH is strictly decreasing for $t > a_2$, and the sum of the "dots" in the diagram is equal to $d(C)$.

Hence the only possibilities in our situation are (on the x-axis we have t and on the y-axis the value of $\Delta H(t)$):

(1)

3						
2		•	•	•	•	
1	•	•	•	•	•	
	0	1	2	3	4	5

(which is clearly impossible, since the general plane section of our curve should be contained in a conic)

(2)

3			•	•		
2		•	•	•		
1	•	•	•	•		
	0	1	2	3	4	5

and

(3)

3			•			
2		•	•	•		
1	•	•	•	•	•	
	0	1	2	3	4	5

Diagram (3) corresponds to the ΔH diagram of a complete intersection of two plane cubics; hence a theorem of Strano ([9]) implies that C itself is a complete intersection of two cubics surfaces, which is clearly not our situation.
So the only possibility is diagram (2), and this implies that

$$h^0(H, \mathscr{I}_{C\cap H}(3)) = 1$$
$$h^1(H, \mathscr{I}_{C\cap H}(3)) = 0$$
$$h^1(H, \mathscr{I}_{C\cap H}(2)) = 3\,;$$

hence the multiplication induced in $M(C)$ by a general plane has maximal rank in every degree.

So this gives an example of a smooth curve of degree $\sigma^2 - 2\sigma + 6$ with the desired properties also for $\sigma = 3$. Note that this example does not propagate by liaison (at least, not automatically).

Example 2.
Take $d = 9$ and $g = 1$. As before, this pair of integers is in the FW-range, hence there exists a smooth curve X with $d(X) = 9$, $g(X) = 1$ and cohomology given by

X	0	1	2	3	4	5	6	...
h^0	0	0	0	0	0	$*$	$*$	...
h^1	0	5	8	7	1	0	0	...
h^2	1	0	0	0	0	0	0	...

As before, $h^0(H, \mathcal{I}_{X\cap H}(3)) \neq 0$ for the general plane H. Moreover, there exists a plane K for which the induced multiplication

$$M(X)_3 \to M(X)_4$$

is not zero map, since if not $M(X)$ should have a generator in degree 4. But $M(X)$ is generated in degrees 1 and 2, thanks to [3], and this is a contradiction. Hence this map should be surjective and $h^1(K, \mathcal{I}_{X\cap K}(4)) = 0$ and the same (by semicontinuity) is true for the general plane section. Now one can argue as in the previous example, getting the same diagram for the ΔH of the general plane section of X:

3			•	•		
2		•	•	•		
1	•	•	•	•		
	0	1	2	3	4	5

Note again that the multiplication induced in $M(X)$ by a general plane has maximal rank in every degree and that this example too does not propagate by liaison.

Example 3. ($\sigma = 4$).
Take $d = 14$ and $g = 15$. This pair is in the FW-range, and as before Walter's result gives a curve W whose cohomology is

$\mathcal{I}_W$	0	1	2	3	4	5	6	...
h^0	0	0	0	0	0	0	*	...
h^1	0	0	4	8	7	0	0	...
h^2	15	4	0	0	0	0	0	...

(note the relation of this table with the table, "less general", of the curve that we constructed with our procedure).

The numerical behavior is hence

$$d(W) = 14, \ g(W) = 15, \ h^0(H, \mathcal{I}_{W\cap H}(4)) \neq 0$$

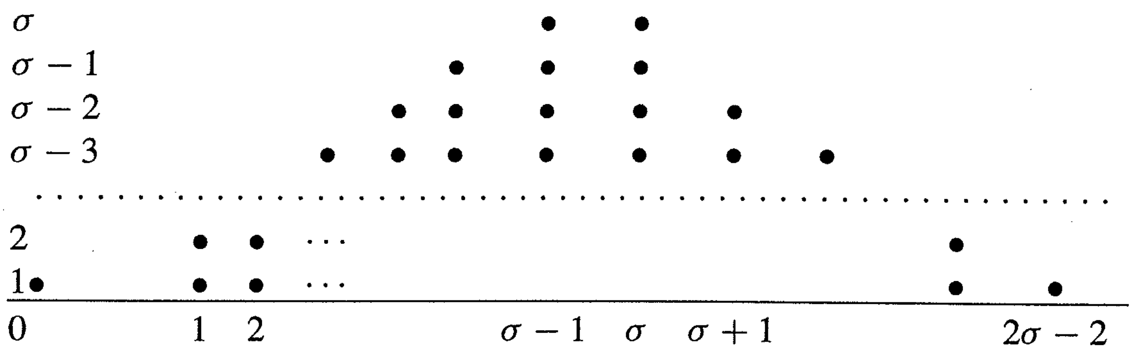
and

$$h^1(H, \mathcal{I}_{W\cap H}(5)) = 0$$

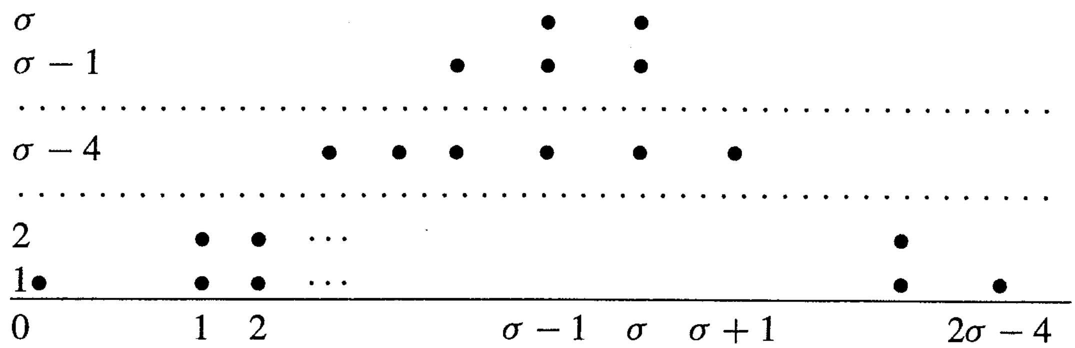
(which implies $\Delta H(6) = 0$) for the general plane H, and $h^0(\mathbb{P}^3, \mathcal{I}_W(5)) = 0$.

One would like to describe the Hilbert functions of these general hyperplane sections of the curves that we constructed, and to draw the diagram of ΔH. Of course, this is possible only if one knows something about the multiplicative structure of the Hartshorne-Rao module $M(Y)$.

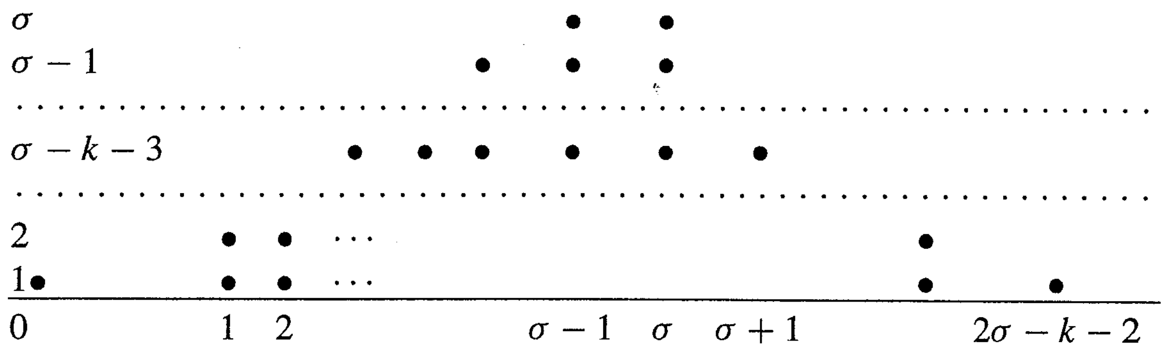
For $h = 0$ there is no problem, since these curves are zero-sets of sections of the null-correlation bundle; hence $M(Y) = \mathbf{k}$ and the diagram of ΔH is as follows (a):



So, let us suppose, just for stating a conjecture, that it is possible to perform the construction of the sheaves above in such a way to have that for the general hyperplane the multiplication $H^1(\mathbb{P}^3, \mathscr{F}(t)) \to H^1(\mathbb{P}^3, \mathscr{F}(t+1))$ has maximal rank for every t. With this assumption, the ΔH is as follows for $h = 1$ (b):



and for arbitrary h we have (c):



where $k = k(h)$ is as above.

So, a good point for verifying which degree is the best possible, is the study of the difference $\Delta H(\sigma) - \Delta H(\sigma + 1)$ (the *second* difference of the Hilbert function).

Note in fact that the statement

$$\begin{pmatrix} H^0(h, \mathscr{I}_\Gamma(\sigma)) \neq 0 \\ H^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 0 \end{pmatrix} \Rightarrow \Delta H(\sigma) - \Delta H(\sigma + 1) \geq 2$$

is equivalent to Laudal's lemma: diagram (a) maximizes areas subjected to the conditions $\Delta H(\sigma) - \Delta H(\sigma + 1) \geq 2$, $H^0(H, \mathscr{I}_\Gamma(\sigma)) \neq 0$; hence a curve satisfying $H^0(H, \mathscr{I}_\Gamma(\sigma)) \neq 0$, $H^0(\mathbb{P}^3, \mathscr{I}_Y(\sigma)) = 0$ must have degree less or equal to that "area", hence to $\sigma^2 + 1$.

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