LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. I, pp. 33-44

A DUALITY THEOREM BY MEANS OF RIEMANN STIELTJES INTEGRAL

OTTAVIO CALIGARIS - PIETRO OLIVA

Duality between the space of continuous functions and the space of bounded variations functions can be easily characterized by means of Riemann-Stieltjes integrals when we consider real valued functions defined, e.g., on [0,1]; here we give a self-contained exposition of Riemann-Stieltjes integration theory for functions which assumes values in infinite dimensional vector spaces and we show as the duality between the space of continuous functions and the space of bounded variation functions can be represented by means of such a theory.

Introduction.

Duality theory between the space of all V-valued absolutely continuous functions $\mathscr{A}(\mathbb{V})$ and the space of absolutely continuous functions with essentially bounded derivative is a powerful tool in calculus of variations. Many interesting results can be proved, by means of duality, about integral functionals of general type: let us quote relaxation theorems and necessary and sufficient conditions for a minimum above the other ones. However in some situations more complicated duality pairings are needed; typical examples are the duality between the dual and the bidual space of $\mathscr{A}(\mathbb{V})$ or the duality between $\mathscr{C}(\mathbb{V})$, the space of continuous functions, and the space $\mathscr{B}v(\mathbb{V}^*)$ of bounded variations functions.
The first one can be completely characterized by means of well known results

Entrato in Redazione il 30 settembre 1992.

due to Hewitt-Yosida [2] extended by Levin [6], [7], about the dual space of $L^\infty(\mathbb{V})$, the space of essentially bounded functions, even when functions are infinite dimensional valued, which is the case of our present interest. The second one is usually characterized by means of measure theoretic tools and involves non-trivial arguments.

Here we try to present a self-contained elementary Riemann-Stieltjes integration theory for functions which take values in an infinite dimensional reflexive separable Banach space $\mathbb{V}$; we prove some results, e.g. integration by parts, by means of this theory and we obtain an integral representation of the duality between $\mathscr{C}(\mathbb{V})$ and $\mathscr{B}v(\mathbb{V}^*)$.To this end we give an extension of the classical proof of the same result relative to finite dimensional case [4]. This setting appears to be very useful for obtaining necessary and sufficient conditions for the minimum of an integral functional of calculus of variations. As an application we expect to be able in future to derive necessary and sufficient conditions for the minimum of integral functionals in a very direct way.

We are grateful to the anonymous referee who pointed out some inaccuracies.

Notations and hypotheses.

In this section we state some notations and we enunciate the hypotheses which are assumed troughout all of this work. $\mathbb{V}$ is a reflexive separable Banach space whose dual space is indicated as $\mathbb{V}^*$. We use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to indicate the norm and the duality pairing between $\mathbb{V}$ and $\mathbb{V}^*$; we shall also use the same symbols to indicate norm and duality pairing regardless to the spaces involved.

Let us now introduce some functional spaces:

$\mathscr{M}(\mathbb{V})$ is the space of all strongly measurable functions

$$x : [0, 1] \to \mathbb{V}$$

where $[0, 1]$ is equipped with the Lebesgue σ-algebra;

$L^p(\mathbb{V})$ is the space of all $x \in \mathscr{M}(\mathbb{V})$ such that

$$\|x\|_{L^p(\mathbb{V})} = \left(\int_0^1 \|x(t)\|^p dt\right)^{\frac{1}{p}} < +\infty;$$

$L^\infty(\mathbb{V})$ is the space of all $x \in \mathscr{M}(\mathbb{V})$ which are essentially bounded in $[0, 1]$ with

$$\|x\|_{L^\infty(\mathbb{V})} = \inf\{\sup\{\|x(t)\| : t \in [0, 1] \setminus A\} : A \subset [0, 1],\ \text{meas}\, A = 0\};$$

$\mathscr{B}d(\mathbb{V})$ is the space of all bounded functions $x : [0, 1] \to \mathbb{V}$ normed with $L^\infty(\mathbb{V})$-norm;

$\mathscr{C}(\mathbb{V})$ is the space of all continuous functions

$$x : [0, 1] \to \mathbb{V}$$

with the norm $||x||_{\mathscr{C}(\mathbb{V})} = \max\{||x(t)|| : t \in [0, 1]\}$;

$\mathscr{B}v(\mathbb{V})$ is the space of all bounded variations functions

$$x : [0, 1] \to \mathbb{V}$$

normed by total variation, i.e.

$$||x||_{\mathscr{B}v(\mathbb{V})} = ||x(0)|| + \mathrm{Var}\,(x, 0, 1) = ||x(0)|| +$$

$$+ \sup\left\{\sum_{i=1}^{n} ||x(t_i) - x(t_{i-1})|| : 0 = t_0 < t_1 < t_2 < \ldots < t_n = 1\right\};$$

$\mathscr{B}v_0(\mathbb{V})$ is the subspace of $\mathscr{B}v(\mathbb{V})$ defined by the condition $x(0) = 0$ with the same norm;

$\mathscr{A}(\mathbb{V})$ is the space of all absolutely continuous functions

$$x : [0, 1] \to \mathbb{V}.$$

$\mathscr{A}(\mathbb{V})$ can be normed by various standard equivalent norms:

$$||x||_{\mathscr{A}(\mathbb{V})} = ||x(0)|| + ||\dot{x}||_{L^1(\mathbb{V})}$$
$$||x||_{\mathscr{A}(\mathbb{V})} = ||x||_{\mathscr{C}(\mathbb{V})} + ||\dot{x}||_{L^1(\mathbb{V})}$$
$$||x||_{\mathscr{A}(\mathbb{V})} = ||x||_{L^1(\mathbb{V})} + ||\dot{x}||_{L^1(\mathbb{V})}$$

We also need some relations between the spaces we introduced, which can be easily found in literature [1], [3].

First of all let us recall that

$$(L^p(\mathbb{V}))^* = L^q(\mathbb{V}^*)$$

where p and q are conjugate exponents i.e. $1/p + 1/q = 1$ while $(L^\infty(\mathbb{V}))^*$ can be characterized by means of a result by Hewitt-Yosida [2] when $\mathbb{V}$ is finite dimensional and by Levin's [6], [7] results when $\mathbb{V}$ is an infinite dimensional space. It can be proved [6], [7] that every $z \in (L^\infty(\mathbb{V}))^*$ can be decomposed as $z = (z_a, z_s)$ where $z_a \in L^1(\mathbb{V}^*)$ and z_s is a singular functional in the sense that for every measurable subset $S \subset [0, 1]$ we can find a sequence of measurable sets A_n such that

$$S = \bigcup_{n=1}^{+\infty} A_n \quad , \quad A_n \subset A_{n+1} \quad , \quad z_s|_{A_n \cap A} = 0$$

for every measurable subset $A \subset [0, 1]$. Functions of bounded variations satisfy several useful properties which we need for the sequel [1].

- Every function of bounded variation is a bounded, strongly measurable, function.
- Every function of bounded variation has right and left limit at every interior point and right or left limit at extrema.
- Every function of bounded variation is continuous except at the points of a countable subset in $[0, 1]$.
- Every function of bounded variation is weakly differentiable for almost all $t \in [0, 1]$, moreover it results

$$\int_0^1 \|\dot{x}(t)\| \leq \mathrm{Var}(x, 0, 1).$$

The space $\mathscr{A}(\mathbb{V}) \subset \mathscr{B}v(\mathbb{V})$ can be characterized as

$$\mathscr{A}(\mathbb{V}) = \mathbb{V} \oplus L^1(\mathbb{V}).$$

Indeed every absolutely continuous function $x \in \mathscr{A}(\mathbb{V})$ has strong derivative $\dot{x}$ at almost all point in $[0, 1]$, [1], it results $\dot{x} \in L^1(\mathbb{V})$ and the fundamental theorem of calculus holds, so that

$$x(t) = x(0) + \int_0^t \dot{x}(s)ds.$$

Moreover we have

$$(\mathscr{A}(\mathbb{V}))^* = (\mathbb{V} \oplus L^1(\mathbb{V}))^* = \mathbb{V}^* \oplus L^\infty(\mathbb{V}^*)$$

and we shall often denote this space as $\mathscr{A}^\infty(\mathbb{V}^*)$.

It is also worth to notice that

$$\mathscr{A}^{**}(\mathbb{V}) = (\mathbb{V}^* \oplus L^\infty(\mathbb{V}^*))^* = \mathbb{V} \oplus (L^\infty(\mathbb{V}^*))^* = \mathbb{V} \oplus (L^\infty)^*(\mathbb{V})$$

where $(L^\infty(\mathbb{V}^*))^*$ is the precedingly quoted space.

Riemann-Stieltjes integration in Banach spaces.

Now we want to define a Riemann-Stieltjes integral for vector valued integrands with respect to vector valued integrators. Similar types of integrals can be found in [3] and [1]. We shall follow classical patterns in theory of integration which can be found e.g. in [4].

Let us state now some more notations.

Definition 1.1. *We call partition of* $[0, 1]$ *every finite set* P *of points in* $[0, 1]$ *i.e.*

$$P = \{t_i : i = 0, 1, \ldots, n\}.$$

We always suppose to name points in P *in such a way that*

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$$

$\mathscr{P}$ *shall indicate the set of all partitions of* $[0, 1]$*; in* $\mathscr{P}$ *we can define a partial order by subset inclusion: we shall say that a partition* P *is more refined than a partition* Q *when* $P \supset Q$*. To every* $P \in \mathscr{P}$ *we also associate a mesh of points*

$$M = \{\tau_i : i = 1, \ldots, n\}$$

in such a way that

$$t_{i-1} \leq \tau_i \leq t_i.$$

Definition 1.2. *Let* $x : [0, 1] \to \mathbb{V}$*,* $y : [0, 1] \to \mathbb{V}^*$*, let* $P \in \mathscr{P}$ *and let* M *be a mesh of points relative to* P*. Let us define*

$$RS(x, y, P, M) = \sum_{i=1}^{n} \langle x(\tau_i), y(t_i) - y(t_{i-1}) \rangle.$$

We say that $RS(x, y, P, M)$ *is a Riemann-Stieltjes sum for the function* x *with respect to the function* y*, the partition* P *and the mesh* M*. We say that* x *is Riemann-Stieltjes integrable with respect to* y*, or simply* RS*-integrable, when we can find* $I \in \mathbb{R}$ *such that* $\forall \varepsilon \in \mathbb{R}_+ \ \exists P_\varepsilon \in \mathscr{P}$ *such that*

$$|RS(x, y, P, M) - I| < \varepsilon$$

for all partition $P \in \mathscr{P}$*,* P *more refined than* P_ε *and for every mesh of points in* P*. In this case we define*

$$I = \int_0^1 \langle x, dy \rangle.$$

Using definitions 1.1 and 1.2 we can prove that

- *when x_1, x_2 are both integrable with respect to y, then $\alpha x_1 + \beta x_2$ is also integrable with respect to y for any $\alpha, \beta \in \mathbb{R}$ and it results*
$$\int_0^1 \langle \alpha x_1 + \beta x_2, dy \rangle = \alpha \int_0^1 \langle x_1, dy \rangle + \beta \int_0^1 \langle x_2, dy \rangle$$
- *when x is integrable with respect to y_1, y_2, then x is also integrable with respect to $\alpha y_1 + \beta y_2$ and it results*
$$\int_0^1 \langle x, d(\alpha y_1 + \beta y_2) \rangle = \alpha \int_0^1 \langle x, dy_1 \rangle + \beta \int_0^1 \langle x, dy_2 \rangle.$$

Theorem 1.3. *When $x \in \mathscr{C}(\mathbb{V})$ and $y \in \mathscr{B}v(\mathbb{V}^*)$ then x is RS-integrable with respect to y. Moreover it results*
$$\left| \int_0^1 \langle x, dy \rangle \right| \leq \|x\|_{\mathscr{C}(\mathbb{V})} \operatorname{Var}(y, 0, 1).$$

Proof. Let $P_n \in \mathscr{P}$ defined as follows $P_n = \{i\, 2^{-n} : i = 0, 1, \ldots, 2^n\}$. P_n is a sequence of partitions which is completely ordered by inclusion i.e. P_{n+1} is more refined than P_n; let us moreover define a mesh of points in P_n as
$$M_n = \{i\, 2^{-n} : i = 0, 1, \ldots, 2^n - 1\}.$$
Let us call $\rho_n = RS(x, y, P_n, M_n)$; by uniform continuity of x on $[0, 1]$ for every $\varepsilon \in \mathbb{R}$ we can find δ_ε such that when $|t' - t''| < \delta_\varepsilon$ it results $\|x(t') - x(t'')\| < \varepsilon$ so, if we choose $m > n$ sufficiently large so that $2^{-n} < \delta_\varepsilon$ it results
$$|\rho_m - \rho_n| \leq \varepsilon \operatorname{Var}(y, 0, 1).$$
We can define $I = \lim \rho_n$ and we have
$$|I| \leq \|x\|_{\mathscr{C}(\mathbb{V})} \operatorname{Var}(y, 0, 1).$$
Now, if we choose a partition P_ε refined enough and we take n sufficiently large we can assert that, for every partition P finer than P_ε and for every mesh of points M in P, we have
$$|RS(x, y, P, M) - I| \leq |RS(x, y, P, M) - \rho_n + \rho_n - I| \leq$$
$$\leq |\rho_n - I| + |RS(x, y, P, M) - \rho_n| < \varepsilon. \qquad \square$$

Theorem 1.4. *Let $x : [0, 1] \to \mathbb{V}$ and let $y : [0, 1] \to \mathbb{V}^*$, then x is RS-integrable with respect to y if and only if y is RS-integrable with respect to x and it results*
$$\int_0^1 \langle x, dy \rangle + \int_0^1 \langle y, dx \rangle = \langle x(1), y(1) \rangle - \langle x(0), y(0) \rangle.$$

Proof. Clearly it is sufficient to prove only one implication. Therefore let us assume that x is integrable with respect to y. We have: $\forall \varepsilon > 0 \, \exists P_\varepsilon \in \mathscr{P}$ such that when $P \in \mathscr{P}$, P more refined than P_ε

$$\left| RS(x, y, P, M) - \int_0^1 \langle x, dy \rangle \right| < \varepsilon, \quad \forall M.$$

Now let us choose $Q \in \mathscr{P}$, Q more refined than P_ε

$$Q = \{t_0, \dots, t_n\}$$

and let $M = \{\tau_1, \dots, \tau_n\}$ a mesh of points in Q; obviously $t_{i-1} \leq \tau_i \leq t_i$ and we have

$$RS(y, x, Q, M) = \sum_{i=1}^{n} \langle y(\tau_i), [x(t_i) - x(t_{i-1})] \rangle.$$

Let us consider $P_1 \in \mathscr{P}$ defined as

$$P_1 = Q \cup M = \{t_0, \tau_1, t_1, \tau_2, \dots, \tau_n, t_n\}$$

and a mesh of points M_1 obtained choosing t_{i-1} in every interval $[t_{i-1}, \tau_i]$ and t_i in every interval $[\tau_i, t_i]$.

It results

$$\begin{aligned} RS(x, y, P_1, M_1) &= \sum_{i=1}^{n} \langle x(t_{i-1}), [y(\tau_i) - y(t_{i-1})] \rangle + \\ &+ \sum_{i=1}^{n} \langle x(t_i), [y(t_i) - y(\tau_i)] \rangle = \\ &= \sum_{i=1}^{n} \langle y(\tau_i), [x(t_{i-1}) - x(t_i)] \rangle + \\ &+ \sum_{i=1}^{n} \langle [x(t_i), y(t_i) \rangle - \langle x(t_{i-1}), y(t_{i-1})] \rangle = \\ &= \langle x(1), y(1) \rangle - \langle x(0), y(0) \rangle - RS(y, x, Q, M). \end{aligned}$$

Therefore we have

$$\begin{aligned} &\left| RS(y, x, Q, M) - [\langle x(1), y(1) \rangle - \langle x(0), y(0) \rangle - \int_0^1 \langle x, dy \rangle] \right| = \\ &= \left| RS(x, y, P_1, M_1) - \int_0^1 \langle x, dy \rangle \right| < \varepsilon \end{aligned}$$

as soon as we recall that P_1 is more refined than Q which, in turn is more refined than P_ε. □

Now we can prove a representation theorem for continuous linear functional on $\mathscr{C}(\mathbb{V})$.

Theorem 1.5. *Every continuous linear functional $F \in (\mathscr{C}(\mathbb{V}))^*$ can be represented as*

$$F(x) = \int_0^1 \langle x, dy \rangle$$

where $y \in \mathscr{B}v_0(\mathbb{V}^*)$ and we take the integral in the sense of Riemann-Stieltjes. Moreover

$$||F|| = \text{Var}(y, 0, 1).$$

So we can assert that

$$(\mathscr{C}(\mathbb{V}))^* = \mathscr{B}v_0(\mathbb{V}^*).$$

Proof. First of all let us remark that, for every $y \in \mathscr{B}v_0(\mathbb{V}^*)$, the correspondence

$$x \mapsto \int_0^1 \langle x, dy \rangle$$

defines a continuous linear functional on $\mathscr{C}(\mathbb{V})$.

On the contrary let us take $F \in (\mathscr{C}(\mathbb{V}))^*$ and let us show that we can find $y \in \mathscr{B}v_0(\mathbb{V}^*)$ such that

$$F(x) = \int_0^1 \langle x, dy \rangle.$$

By Hahn-Banach theorem (see [5] e.g.) we can extend F to the space $\mathscr{B}d(\mathbb{V})$ of all bounded functions, preserving its norm and we can define, for any $t \in [0, 1]$ and for any $v \in V$

$$\Phi(t, v) = F(\chi_{[0,t]} v)$$

as soon as we recall that $\chi_{[0,t]} v \in \mathscr{B}d(\mathbb{V})$ ($\chi_{[0,t]}$ being the characteristic function of the interval $[0, t]$).

Now, by linearity of F we can easily verify that $\Phi(t, \cdot)$ is linear and moreover it results, when $t > 0$,

$$\begin{aligned} |\Phi(t, v)| &= |F(\chi_{[0,t]} v)| \leq \\ &\leq ||F|| ||\chi_{[0,t]} v||_{\mathscr{B}d(\mathbb{V})} = ||F|| \, ||v||. \end{aligned}$$

So, for $t \in [0, 1]$ we can find $y(t) \in \mathbb{V}^*$ such that

$$\Phi(t, v) = \langle y(t), v \rangle.$$

Let us show that $y \in \mathscr{B}v_0(\mathbb{V}^*)$. First of all we have

$$\langle y(0), v\rangle = \Phi(0, v) = 0 \qquad \forall v \in \mathbb{V}$$

and $y(0) = 0$; moreover we can prove that $\mathrm{Var}(y, 0, 1) \leq \|F\|$.

Let $\{t_0, t_1, t_2, \dots, t_n\} \in \mathscr{P}$ we can find $v_i \in \mathbb{V}^*$ such that $\|v_i\| = 1$ and

$$\|y(t_{i+1}) - y(t_i)\| = \langle y(t_{i+1}) - y(t_i), v_i\rangle$$

so that

$$\sum_{i=1}^{n} \|y(t_i) - y(t_{i-1})\| = \sum_{i=1}^{n} \langle y(t_i) - y(t_{i-1}), v_i\rangle =$$

$$= \sum_{i=1}^{n} (\Phi(t_i, v_i) - \Phi(t_{i-1}, v_i)) = \sum_{i=1}^{n} (F(\chi_{[0,t_i]} v_i) - F(\chi_{[0,t_{i-1}]} v_i)) =$$

$$= \sum_{i=1}^{n} F(\chi_{[t_{i-1},t_i]} v_i) = F\left(\sum_{i=1}^{n} \chi_{[t_{i-1},t_i]} v_i\right) \leq \|F\| \left\|\sum_{i=1}^{n} \chi_{[t_{i-1},t_i]} v_i\right\| = \|F\|$$

as soon as we observe that $\|\chi_{[t_{i-1},t_i]} v_i\| = \|v_i\| = 1$. So we can deduce that

$$\mathrm{Var}(y, 0, 1) \leq \|F\|.$$

Now, let $x \in \mathscr{C}(\mathbb{V})$ let $\varepsilon \in \mathbb{R}_+$ and let us choose a partition $P \in \mathscr{P}$ such that

$$\|x(t'') - x(t')\| \leq \varepsilon$$

for every $t'', t' \in [t_i, t_{i+1}]$ and for every $i = 0, 1, \dots, n-1$.

Let

$$\sigma(t) = \sum_{i=1}^{n} \chi_{[t_{i-1},t_i]} x(t_{i-1})$$

it results $\sigma \in \mathscr{B}d(\mathbb{V})$ and moreover $\|x - \sigma\|_{\mathscr{C}(\mathbb{V})} < \varepsilon$, whence

$$|F(x) - F(\sigma)| \leq \varepsilon \|F\|.$$

We also have

$$F(\sigma) = F\left(\sum_{i=1}^{n} \chi_{[t_{i-1},t_i]} x(t_{i-1})\right) = \sum_{i=1}^{n} F(\chi_{[t_{i-1},t_i]} x(t_{i-1})) =$$

$$= \sum_{i=1}^{n} \langle y(t_i) - y(t_{i-1}), x(t_{i-1})\rangle.$$

So $F(\sigma)$ is a RS-sum for x with respect to y and, since x is RS-integrable with respect to y we can deduce that

$$\left|\int_0^1 \langle x, dy\rangle - F(\sigma)\right| < \varepsilon$$

as soon as partition P is choosen refined enough. We finally obtain

$$\left|F(x) - \int_0^1 \langle x, dy\rangle\right| = \left|F(x) - F(\sigma) + F(\sigma) - \int_0^1 \langle x, dy\rangle\right| \leq$$
$$\leq \varepsilon(1 + \|F\|)$$

for every $\varepsilon \in \mathbb{R}_+$, whence

$$F(x) = \int_0^1 \langle x, dy\rangle.$$

By definition of RS-integrability it results

$$\left|\int_0^1 \langle x, dy\rangle\right| \leq \mathrm{Var}(y, 0, 1)\|x\|_{\mathscr{C}(\mathbb{V})}$$

so that $\|F\| \leq \mathrm{Var}(y, 0, 1)$ and since $\mathrm{Var}(y, 0, 1) \leq \|F\|$ equality $\|F\| = \mathrm{Var}(y, 0, 1)$ holds. □

Theorem 1.6. *Let $x \in \mathscr{C}(\mathbb{V})$, $y \in \mathscr{A}(\mathbb{V}^*)$ then x is integrable with respect to y, y is integrable with respect to x and it results*

$$\int_0^1 \langle x, dy\rangle = \int_0^1 \langle x, \dot{y}\rangle dt.$$

Proof. Let $P \in \mathscr{P}$ defined by

$$P = \{t_i : i = 0, 1, \ldots, n\}$$

and let us choose a mesh of points $M = \{t_i : i = 0, 1 \ldots, n-1\}$ in P. We have

$$RS(x, y, P, M) = \sum_{i=1}^{n} \langle x(t_{i-1}), y(t_i) - y(t_{i-1})\rangle =$$
$$= \sum_{i=1}^{n} \langle x(t_{i-1}), \int_{t_{i-1}}^{t_i} \dot{y}(s)ds\rangle = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \langle x(t_{i-1}), \dot{y}(s)ds\rangle =$$
$$= \sum_{i=1}^{n} \int_0^1 \chi_{[t_{i-1}, t_i]} \langle x(t_{i-1}), \dot{y}(s)\rangle \, ds = \int_0^1 \sum_{i=1}^{n} \langle \chi_{[t_{i-1}, t_i]} x(t_{i-1}), \dot{y}(s)\rangle \, ds =$$
$$= \int_0^1 \langle \sigma(s), \dot{y}(s)\rangle \, ds$$

as soon as we define

$$\sigma(t) = \sum_{i=1}^{n} \chi_{[t_{i-1}, t_i]} x(t_{i-1}).$$

Now if we choose $P \in \mathscr{P}$ fine enough, we can assert that

$$\left| RS(x, y, P, M) - \int_0^1 \langle x, dy \rangle \right| < \varepsilon$$

moreover,

$$\left| \int_0^1 \langle \sigma, \dot{y} \rangle ds - \int_0^1 \langle x, \dot{y} \rangle ds \right| \leq \left| \int_0^1 \langle \sigma - x, \dot{y} \rangle ds \right| \leq$$
$$\leq \|\sigma - x\|_{L^\infty(\mathbb{V})} \, \|\dot{y}\|_{L^1(\mathbb{V})} < \varepsilon.$$

Therefore

$$\left| \int_0^1 \langle x, dy \rangle - \int_0^1 \langle x, \dot{y} \rangle \, dt \right| \leq$$
$$\leq \left| \int_0^1 \langle x, dy \rangle - RS(x, y, P, M) + RS(x, y, P, M) - \int_0^1 \langle x, \dot{y} \rangle \, dt \right| \leq$$
$$\leq \left| \int_0^1 \langle x, dy \rangle - RS(x, y, P, M) \right| + \left| RS(x, y, P, M) - \int_0^1 \langle x, \dot{y} \rangle \, dt \right| \leq$$
$$\leq \left| \int_0^1 \langle x, dy \rangle - RS(x, y, P, M) \right| + \left| \int_0^1 \langle \sigma, \dot{y} \rangle \, dt - \int_0^1 \langle x, \dot{y} \rangle dt \right| \leq 2\varepsilon$$

and

$$\int_0^1 \langle x, dy \rangle = \int_0^1 \langle x, \dot{y} \rangle \, dt.$$

REFERENCES

[1] V. Barbu - T. Precupanu, *Convexity and optimization in Banach spaces,* Sijthoff & Nordhoff, Alphen aan Rijn The Netherlands, 1974.

[2] E. Hewitt - K. Yosida, *Finitely additive measures,* Trans. Amer. Math. Soc., 72 (1952), pp. 46-66.

[3] E. Hille - R.S. Philips, *Functional analysis and semigroups,* Amer. Math. Soc. Colloquium Pubbl. 31, Providence, 1957.

[4] A. Kolmogoroff - S. Fòmine, *Elèments de la théorie des fonctions et de l'analyse fonctionelle,* Èditions MIR, Moscou, 1974.

[5] R. Larsen, *Functional analysis an introduction,* M. Dekker, New York, 1973.

[6] V.L. Levin, *The Lebesgue decomposition for functionals of vector function space* L_X^∞, Functional Anal. and Appl., 8 (1974), pp. 314-317.

[7] V.L. Levin, *Convex integral funtionals and the theory of lifting,* Russian Math. Surveys, (1975), pp. 119-183.

Università di Torino,
Sede di Alessandria,
Via Cavour 84,
15100 Alessandria (Italy)
and
Università di Genova,
Istituto Matematico di Ingegneria,
P.le Kennedy pad. D,
16129 Genova (Italy)