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CERTAIN TRANSFORMATIONS OF BASIC HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES - II

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This paper deals with the derivation of certain new transformations for the basic hypergeometric functions of two variables by making use of known summation formulas given earlier by Srivastava [7]. As $q \to 1$, known transformations for ordinary hypergeometric functions given earlier by Carlitz [2] and Jain [4] are obtained.

1. Introduction and preliminaries.

Let $|q| < 1$, and put

(1) $$(a;\, q)_n = \prod_{j=0}^{\infty} \frac{(1 - aq^j)}{(1 - aq^{n+j})}\,; \quad n > 0$$

for arbitrary a (real or complex).
Then the basic hypergeometric series is defined by [3]

(2) $${}_r\Phi_s\left[\begin{matrix} a_1, \dots, a_r; \\ b_1, \dots, b_s; \end{matrix}\; q;\; x\right] =$$

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$$= \sum_{n \geq 0} \frac{(a_1;q)_n \cdots (a_r;q)_n \, x^n \{(-1)^n q^{\frac{n}{2}(n-1)}\}^{1+s-r}}{(q;q)_n \, (b_1;q)_n \cdots (b_s;\, q)_n}$$

where, for convergence, $|x| < 1$.

Also, the generalized basic hypergeometric series of two variables is defined as, (cf. [8], p. 347, eq. 272)

(3) $$\Phi\begin{matrix} A: B'; B'' \\ C: D'; D'' \end{matrix}\left(\begin{matrix} (a): (b'); (b''); q; x, y \\ (c): (d'); (d''); i, j, k \end{matrix}\right) =$$

$$= \sum_{m,n \geq 0} \frac{\prod_{t=1}^{A} (a_t;q)_{m+n} \prod_{t=1}^{B'} (b'_t;q)_m \prod_{t=1}^{B''} (b''_t;q)_n \, x^m y^n}{\prod_{t=1}^{C} (c_t;q)_{m+n} \prod_{t=1}^{D'} (d'_t;q)_m \prod_{t=1}^{D''} (d''_t;q)_n}$$

$$\frac{q^{im(m-1)/2+jn(n-1)/2+kmn}}{(q;q)_m \, (q;q)_n}$$

where, for convergence, $|x| < 1$, $|y| < 1$ and i, j, k are arbitrary integers. In what follows the symbol (a) will represent the sequence of A parameters

$$a_1, \ldots, a_A \, .$$

Further, if we employ the notation

(4) $$\prod\left[\begin{matrix} (a) \\ (b) \end{matrix}; q\right] = \frac{\prod_{j=1}^{A} (a_j;\, q)_\infty}{\prod_{j=1}^{B} (b_j;\, q)_\infty}$$

where $(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$, then

(5) $$\lim_{q \to 1} \left\{ \frac{(q^\alpha;\, q)_n}{(q^\beta;\, q)_n} \right\} = \frac{(\alpha)_n}{(\beta)_n}$$

where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$; for arbitrary α and β; $\beta \neq 0, -1, -2, \ldots$.

The following summation formulae for double series obtained by Srivastava ([7], eqn. 3.8 and 3.9) will be needed in deriving the various transformations in the next section.

$$(6)\qquad \sum_{r,s\geq 0} \frac{(\gamma/\alpha;q)_r\,(\gamma/\beta;q)_r\,(q^{-m};q)_r\,(q^{-n};q)_s\,(\alpha;q)_s\,(\beta;q)_s\,q^{r+s}}{(\gamma;q)_{r+s}\,(\gamma q^{1-m}/\alpha\beta;q)_r\,(\alpha\beta q^{1-n}/\gamma;q)_s\,(q;q)_r\,(q;q)_s} =$$

$$= \frac{(\alpha;q)_m\,(\beta;q)_m\,(\gamma/\alpha;q)_n\,(\gamma/\beta;q)_n}{(\gamma;q)_{m+n}\,(\alpha\beta/\gamma;q)_m\,(\gamma/\alpha\beta;q)_n}$$

$$(7)\qquad \sum_{r,s\geq 0} \frac{(q^{-m};q)_r\,(q^{-n};q)_s\,(\alpha;q)_r\,(\beta;q)_s\,(\gamma;q)_{r+s}\,q^{r+s}}{(\alpha q^{1-m}/\beta;q)_r\,(\gamma;q)_r\,(\beta q^{1-n}/\alpha;q)_s\,(\gamma;q)_s\,(q;q)_r\,(q;q)_s} =$$

$$= \frac{(\gamma;q)_{m+n}\,(\beta;q)_m\,(\alpha;q)_n}{(\beta/\alpha;\,q)_m\,(\alpha/\beta;\,q)_n\,(\gamma;q)_m\,(\gamma;q)_n}\,.$$

2. Transformations.

The following two transformations for the basic double series are to be established here.

$$(8)\qquad \Phi^{A:B'+2;B''+2}_{C+1:D'+1;D''+1}\begin{pmatrix}(a):(b'),\alpha,\beta;(b''),\gamma/\alpha,\ \gamma/\beta;q;x,y\\ (c),\gamma:(d'),\alpha\beta/\gamma;(d''),\ \gamma/\alpha\beta;-,-,-\end{pmatrix} =$$

$$= \sum_{r,s\geq 0} \frac{\prod_{t=1}^{A}(a_t;q)_{r+s}\ \prod_{t=1}^{B'}(b'_t;q)_r\,(\gamma/\alpha;q)_r\,(\gamma/\beta;q)_r\ \prod_{t=1}^{B''}(b''_t;q)_s}{\prod_{t=1}^{C}(c_t;q)_{r+s}\,(\gamma;q)_{r+s}\ \prod_{t=1}^{D'}(d'_t;q)_r\,(\alpha\beta/\gamma;\,q)_r\ \prod_{t=1}^{D''}(d''_t;q)_s}$$

$$\frac{(\alpha;q)_s\,(\beta;q)_s\,(x\alpha\beta/\gamma)^r\,(y\gamma/\alpha\beta)^s}{(\gamma/\alpha\beta;\,q)_s\,(q;q)_r\,(q;q)_s}\ \Phi^{A:B'+1;B''+1}_{C:D'+1;D''+1}$$

$$\begin{pmatrix}(a)\,q^{r+s}:\ (b')q^r,\ \alpha\beta/\gamma;\,(b'')q^s,\ \gamma/\alpha\beta;q;x,y\\ (c)\,q^{r+s}:\ \ (d')q^r,\ \alpha\beta q^r/\gamma;(d'')q^s,\ \gamma q^s/\alpha\beta;-,-,-\end{pmatrix}$$

where $|x|<1,\ |y|<1$.

$$(9)\qquad \Phi^{A+1:B'+1;B''+1}_{C:D'+2;D''+2}\begin{pmatrix}(a),\gamma\ :(b'),\beta;(b''),\alpha; & q;x,y\\ (c)\ :(d'),\beta/\alpha,\gamma;(d''),\alpha/\beta,\gamma;-,-,-\end{pmatrix} =$$

$$= \sum_{r,s\geq 0} \frac{\prod_{t=1}^{A}(a_t;q)_{r+s}\,(\gamma;q)_{r+s}\,\prod_{t=1}^{B'}(b'_t;q)_r\,(\alpha;q)_r}{\prod_{t=1}^{C}(c_t;q)_{r+s}\,\prod_{t=1}^{D'}(d'_t;q)_r\,(\gamma;q)_r\,(\beta/\alpha;q)_r}$$

$$\frac{\prod_{t=1}^{B''}(b''_t;q)_s\,(\beta;q)_s\,(x\beta/\alpha)^r\,(y\alpha/\beta)^s}{\prod_{t=1}^{D''}(d''_t;q)_s\,(\gamma;q)_s\,(\alpha/\beta;q)_s\,(q;q)_r(q;q)_s}\,.$$

$$\cdot\,\Phi^{A:B'+1;B''+1}_{C:D'+1;D''+1}\left(\begin{matrix}(a)q^{r+s} : (b')q^r,\,\beta/\alpha;(b'')q^s,\,\alpha/\beta;q; \quad x,\,y\\ (c)q^{r+s} : (d')q^r,\,\beta q^r/\alpha;(d'')q^s,\,\alpha q^s/\beta;-,\,-,\,-\end{matrix}\right)$$

where $|x| < 1$, $|y| < 1$.

To prove (8), we consider its L.H.S., and replace it by its equivalent series (3), then on using (6), it becomes

$$\sum_{m,n\geq 0} \frac{\prod_{t=1}^{A}(a_t;q)_{m+n}\,\prod_{t=1}^{B'}(b'_t;q)_m\,\prod_{t=1}^{B''}(b''_t;q)_n\,x^m y^n}{\prod_{t=1}^{C}(c_t;q)_{m+n}\,\prod_{t=1}^{D'}(d'_t;q)_m\,\prod_{t=1}^{D''}(d''_t;q)_n\,(q;q)_m\,(q;q)_n}\,.$$

$$\cdot\,\sum_{r,s\geq 0}\frac{(\gamma/\alpha;q)_r\,(\gamma/\beta;q)_r\,(q^{-m};q)_r\,(\alpha;q)_s\,(\beta;q)_s\,(q^{-n};q)_s\,q^{r+s}}{(\gamma;q)_{r+s}\,(\gamma q^{1-m}/\alpha\beta;q)_r\,(\alpha\beta q^{1-n}/\gamma;q)_s\,(q;q)_r\,(q;q)_s}\,.$$

Now, replacing m by $(m + r)$ and n by $(n + s)$ and interchanging the order of summation, which is permissible under the aforesaid conditions, the R.H.S. follows at once.

Similarly, we can prove (9) by making use of (7).

3. Special cases.

If we set $A = C = D' = D'' = 0$; $B' = B'' = 1$, $b'_1 = \alpha\beta/\gamma$, $b''_1 = \gamma/\alpha\beta$ in (8), and sum the inner ${}_1\Phi_0(\cdot)$ with the help of Heine's theorem, ([6], p. 248, IV. 11), namely,

$${}_1\Phi_0\left[\begin{matrix}a;\\ -;\end{matrix}\;q;\;z\right] = \prod\left[\begin{matrix}az;\\ z;\end{matrix}\;q\right]$$

We obtain an interesting transformation associated with basic Appell series $\Phi^{(3)}(\cdot)$

$$\Phi^{(3)}[\alpha,\ \gamma/\alpha;\ \beta,\ \gamma/\beta;\gamma;\ q;\ x,y] = \prod\left[\begin{matrix}\alpha\beta x/\gamma, & \gamma y/\alpha\beta; & \\ x, & y; & q\end{matrix}\right]\cdot \tag{11}$$

$$\cdot\,\Phi^{(3)}[\gamma/\alpha,\ \alpha;\ \gamma/\beta,\ \beta;\gamma;\ q;\ x\alpha\beta/\gamma,\ y\gamma/\alpha\beta]$$

where $\Phi^{(3)}(\cdot)$ is defined as, ([1], p. 618),

$$\Phi^{(3)}[a,\ a';\ b,\ b';\ c;\ q;\ x,\ y] = \tag{12}$$

$$= \sum_{m,n\geq 0} \frac{(a;q)_m\,(a';q)_n\,(b;q)_m\,(b';q)_n\,x^m y^n}{(q;q)_m\,(q;q)_n\,(c;q)_{m+n}}.$$

Again, if we let $y \to 0$ in (11) we arrive at another known transformation, ([8], p. 348, Eqn. 281)

$${}_2\Phi_1\left[\begin{matrix}\alpha,\beta; & \\ \gamma; & q;x\end{matrix}\right] = \prod\left[\begin{matrix}\alpha\beta x/\gamma; & \\ x; & q;\end{matrix}\right]{}_2\Phi_1\left[\begin{matrix}\gamma/\alpha,\ \gamma/\beta; & \\ \gamma; & q;\ \dfrac{\alpha\beta x}{\gamma}\end{matrix}\right] \tag{13}$$

which is q-analogue of well-known Euler transformation

$${}_2F_1\left[\begin{matrix}\alpha,\beta; & \\ \gamma; & x\end{matrix}\right] = (1-x)^{\gamma-\alpha-\beta}\,{}_2F_1\left[\begin{matrix}\gamma-\alpha,\ \gamma-\beta; & \\ \gamma; & x\end{matrix}\right]. \tag{14}$$

Further, if we apply a known transformation due to Andrews ([1], p. 621, eqn. 3.4) to (12), namely,

$$\Phi^{(3)}[a,\ a';\ b,\ b';\ aa';\ q;\ x,y] = \tag{15}$$

$$= \prod\left[\begin{matrix}a,\ bx; & \\ aa',\ x; & q\end{matrix}\right]\Phi^{(2)}[a';\ x,\ b';\ bx,\ 0;\ q;\ a,y]$$

then we arrive at a new transformation for $\Phi^{(2)}(\cdot)$ series:

$$\Phi^{(2)}[\gamma/\alpha;\ x,\ \gamma/\beta;\ \beta x,\ 0;\ q;\ \alpha,y] = \prod\left[\begin{matrix}\gamma/\alpha,\ \alpha x,\ y\gamma/\alpha\beta; & \\ \alpha,\ \beta x,\ y; & q\end{matrix}\right]\cdot \tag{16}$$

$$\cdot\,\Phi^{(2)}[\alpha;\ x\alpha\beta/\gamma,\ \beta;\ \alpha x,\ 0;\ q;\ \gamma/\alpha, y\gamma/\alpha\beta]$$

where $\Phi^{(2)}(\cdot)$ is defined as [1].

(17) $$\Phi^{(2)}\left[a;\ b,\ b';\ c,\ c';\ q;\ x,\ y\right] = \sum_{m,n\geq 0} \frac{(a;q)_{m+n}\,(b;q)_m\,(b';q)_n\,x^m y^n}{(q;q)_m\,(q;q)_n\,(c;q)_m\,(c';q)_n}\,.$$

It is interesting to observe that (12) is a q-analogue of a known transformation due to Jain ([4], p. 300, eqn. 5) namely,

(18) $$F_3\left[\alpha,\ \gamma-\alpha;\ \beta,\ \gamma-\beta;\ \gamma;\ x, y\right] =$$
$$= \left(\frac{1-x}{1-y}\right)^{\gamma-\alpha-\beta} F_3\left[\gamma-\alpha,\ \alpha;\ \gamma-\beta,\ \beta;\ \gamma;\ x, y\right].$$

If we set $A = C = D' = D'' = 0$; $B' = B'' = 1$, $b_1' = \beta_{/\alpha}$, $b_1'' = \alpha_{/\beta}$ in (9) and then using (10), we find that

(19) $$\Phi^{(2)}\left[\gamma;\ \beta;\ \alpha;\ \gamma,\ \gamma;\ q;\ x,\ y\right] = \prod\left[\begin{matrix}\beta x/\alpha,\ \alpha y/\beta; \\ x,\ y;\end{matrix}\ q\right]\cdot$$
$$\cdot\,\Phi^{(2)}\left[\gamma;\ \alpha,\ \beta;\ \gamma,\ \gamma;\ q; x\beta/\alpha,\ y\alpha/\beta\right]$$

which, for $q \to 1$, reduces to a known result due to Carlitz ([2], p. 140, eqn. 10) namely,

(20) $$F_2\left[\gamma;\ \beta,\ \alpha; \gamma,\ \gamma;\ x, y\right] = \left(\frac{1-x}{1-y}\right)^{\alpha-\beta} F_2\left[\gamma;\ \beta,\ \alpha;\ \gamma,\ \gamma;\ y,\ x\right].$$

Finally, if we employ a known transformation due to Andrews ([1], p. 620, eqn. 3.2) namely,

(21) $$\Phi^{(2)}\left[a;\ b,\ b';\ a,\ a;\ q;\ x,\ y\right] =$$
$$= \prod\left[\begin{matrix}b,\ ax; \\ a, x;\end{matrix}\ q\right]\Phi^{(3)}\left[a/b,\ 0;\ x,\ b';\ ax;\ q;\ b,\ y\right]$$

in (19), the following new transformation for $\Phi^{(3)}(\cdot)$ series is obtained

(22) $$\Phi^{(3)}\left[\gamma/\beta,\ 0;\ x,\ \alpha;\ \gamma x;\ q;\ \beta,\ y\right] =$$
$$= \prod\left[\begin{matrix}\alpha,\ \alpha y/\beta,\ \beta\gamma x/\alpha; \\ \beta,\ y,\ \gamma x\end{matrix}\ ;\ q\right]\Phi^{(3)}\left[\gamma/\alpha,\ 0;\ x\beta/\alpha,\ \beta;\ \beta\gamma x/\alpha;\ q;\ \alpha,\ \alpha y/\beta\right].$$

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