LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. I, pp. 53-75

INVESTIGATION ON SHOCK WAVES STABILITY IN RELATIVISTIC GAS DYNAMICS

ALEXANDER M. BLOKHIN - EVGENIY V. MISHCHENKO

This paper is devoted to investigation of the linearized mixed problem of shock waves stability in relativistic gas dynamics. The problem of symmetrization of relativistic gas dynamics equations is also discusssed.

The main result is derivation of a priory estimation without the loss of smoothness for the linearized mixed problem.

Introduction.

Investigation on the shock waves stability was of great interest in the last years. Here we should note the papers of A.M. Anile and G. Russo [1]-[3] on this subject. To study the shock waves stability the authors of the present paper use an "equation" approach. This means investigation of the well-posedness of the corresponding mixed problem on shock waves stability in relativistic gas dynamics. The most complete description of this approach is given in monograph [4] (relative to the study of shock waves stability in classical gas dynamics and in superfluid).

The main results of this paper are the following: the symmetrization of relativistic gas dynamics equations given in section 1 (being obtained by the authors in a previous paper, this symmetrization has not become widely known to occidental scientists) and so-called *a priori estimation without the loss of smoothness* for the linearized mixed problem of shock wave stability given in

Entrato in Redazione il 19 gennaio 1993.

section 3. It is known (see, e.g., monograph [4]) that existence of such a priori estimation for the linear mixed problem allows to prove (with the help of certain technique) a local theorem of existence and uniqueness of the classical solution to relativistic gas dynamics equations behind a curvilinear shock wave.

The results obtained are new, and, in our opinion, they should attract the attention of occidental mathematicians to such problems.

1. Symmetrization of relativistic gas dynamics equations.

First, we will discuss the question of the symmetrization of relativistic gas dynamics equations taking as the foundation an approach, given in monograph [4], chapter II. Following the notations and terminology, offered in the paper of A.M. Anile and G. Russo [2], we write out the system of relativistic gas dynamics equations (to obtain the equations of relativistic gas dynamics, see, e.g., [5], chapter XV):

$$\frac{\partial(\rho\Gamma)}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0,$$

(1.1) $$\frac{\partial(\rho h\Gamma u^j)}{\partial t} + \sum_{k=1}^{3}\frac{\partial\Pi_{jk}}{\partial x^k} = 0, \quad j = 1, 2, 3,$$

$$\frac{\partial(\rho h\Gamma^2 - p)}{\partial t} + \operatorname{div}(\rho h\Gamma\mathbf{u}) = 0.$$

Here ρ is the rest frame density,
$\Gamma = \dfrac{1}{\sqrt{1-|\mathbf{v}|^2}}$,
$\mathbf{v} = (v^1, v^2, v^3)^*$ is the velocity vector-column, $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v})$,
$\mathbf{u} = (u^1, u^2, u^3)^* = \Gamma\mathbf{v}$, $\Gamma^2 = 1 + |\mathbf{u}|^2$,
(Γ, u^1, u^2, u^3) is the four-velocity,
$\Pi_{jk} = \rho h u^j u^k + p\delta_{jk}$,
$h = 1 + e_0 + pV$, $V = \frac{1}{\rho}$, $e_0 = e_0(\rho, s)$;
p, e_0, s are the rest frame pressure, internal energy, entropy;
(x^1, x^2, x^3, t) are inertial coordinates (we restrict ourselves to the case of special relativity).

As in the case of classical gas dynamics, the following thermodynamical identity holds:

(1.2) $$T ds = de_0 + p dV,$$

where T is the temperature. With a state equation

$$e_0 = e_0(\rho, s)$$

and in view of (1.2) we can consider system (1.1) as a system to derive the components of the unknown variables vector:

$$\mathbf{U} = \begin{pmatrix} p \\ s \\ \mathbf{u} \end{pmatrix}.$$

We note also that in the model offered the light speed is taken to be unity, and x^1, x^2, x^3 are the Cartesian coordinates.

By the use of the additional entropy conservation law

$$\frac{\partial(\rho\Gamma s)}{\partial t} + \operatorname{div}(\rho s\mathbf{u}) = 0$$

we can symmetrize (1.1). We remind briefly (see [4], chapter II) that to symmetrize system (1.1) we are to choose functions

$$L = L(\mathbf{Q}), \; M^{(k)} = M^{(k)}(\mathbf{Q}), \quad k = 1, 2, 3,$$

and new dependent variables

$$\mathbf{Q} = (q_1, q_2, q_3, q_4, q_5)^*,$$

such that system (1.1) can be written in the form

$$\frac{\partial(L_{qj})}{\partial t} + \sum_{k=1}^{3} \frac{\partial(M_{qj}^{(k)})}{\partial x^k} = 0, \quad j = \overline{1, 5},$$

or

(1.3)
$$A^0 \frac{\partial \mathbf{Q}}{\partial t} + \sum_{k=1}^{3} A^{(k)} \frac{\partial \mathbf{Q}}{\partial x^k} = 0,$$

where $A^{(0)} = (L_{q_j q_l})$, $A^{(k)} = (M_{q_j q_l}^{(k)})$, $j, l = \overline{1, 5}$ are symmetric matrices, and

(1.4)
$$A^{(0)} > 0 \quad (\text{or } A^{(0)} < 0).$$

It is clear that if condition (1.4) holds, then system (1.3) is symmetric t-hyperbolic (by Friedrichs) (in the case $A^{(0)} < 0$, we should multiply system (1.3) by -1).

We will not discuss in details the question on how to choose q_j, $j = \overline{1,5}$, L, $M^{(k)}$, $k = 1, 2, 3$. We write out immediately the concrete expressions for them:

$$q_1 = s - \frac{h}{T},$$

$$q_{1+k} = -\frac{u^k}{T}, \quad k = 1, 2, 3, \tag{1.5}$$

$$q_5 = \frac{\Gamma}{T},$$

$$L = -p\frac{\Gamma}{T}, \quad M^{(k)} = -p\frac{u^k}{T}, \quad k = 1, 2, 3.$$

Taking into account (1.5), we have:

$$d\mathbf{Q} = J d\mathbf{U}\,, \; d\mathbf{U} = J^{-1} d\mathbf{Q},$$

$$d\mathbf{L}_q = I^{(0)} d\mathbf{U},$$

$$d\mathbf{M}_q^{(k)} = I^{(k)} d\mathbf{U}, \quad k = 1, 2, 3,$$

where $\mathbf{L}_q = (L_{q1}, \ldots, L_{q5})^*$, $\mathbf{M}_q^{(k)} = M_{q1}^{(k)}, \ldots, M_{q5}^{(k)})^*$; J, $I^{(\alpha)}$, $\alpha = \overline{0,3}$ are some matrices. We don't give the concrete forms of these matrices. Then

$$A^{(0)} = (L_{q_j q_l}) = I^{(0)} J^{-1},$$

$$A^k = (M_{q_j q_l}^{(k)}) = I^{(k)} J^{-1}, \quad k = 1, 2, 3, \quad j, l = \overline{1,5}.$$

Following [6], we write out the matrices $A^{(\alpha)}$, $\alpha = \overline{0,3}$:

$$A^{(\alpha)} = \rho T (a_{ij}^{(\alpha)}), \quad i, j, = \overline{1,5}, \quad \alpha = \overline{0,3},$$

where

$$a_{11}^{(0)} = -\Gamma m_0\,, \; a_{11}^{(k)} = a_{11}^{(0)} v^k, \quad k = 1, 2, 3,$$

$$a_{15}^{(0)} = 1 + \Gamma^2 m_1\,, \; a_{15}^{(k)} = \Gamma m_1 u^k, \quad k = 1, 2, 3,$$

$$a_{55}^{(0)} = (3 + \Gamma^2 m_2)\Gamma h\,, \; a_{55}^{(k)} = (1 + \Gamma^2 m_2) h u^k, \quad k = 1, 2, 3,$$

$$a_{23}^{(2)} = h u^1 [m_2 (u^2)^2 - 1]\,, \; a_{24}^{(2)} = h u^1 u^2 u^3 m_2,$$

$$a_{34}^{(2)} = hu^3[m_2(u^2)^2 - 1];$$

$$a_{1,l+1}^{(0)} = \Gamma m_1 u^l \,,\; a_{l+1,5}^{(0)} = hu^l(\Gamma^2 m_2 + 1),$$

$$a_{l+1,l+1}^{(0)} = \Gamma h[m_2(u^l)^2 - 1] \,,\; l = 1,2,3;$$

$$a_{1,l+1}^{(k)} = (u^k u^l m_1 - \delta_{kl}) \,,\; a_{l+1,5}^{(k)} = h\Gamma(m_2 u^k u^l - \delta_{kl}),$$

$$a_{l+1,l+1}^{(k)} = hu^k[m_2(u^l)^2 - 1 - 2\delta_{kl}] \,,\; k,l = 1,2,3;$$

$$a_{r+1,\sigma+1}^{(0)} = \Gamma h u^r u^\sigma m_2 \,,\; a_{r+1,\sigma+1}^{(1)} = hu^\sigma(m_2 u^r u^1 - \delta_{1r}),$$

$$a_{r+1,\sigma+1}^{(3)} = hu^r(m_2 u^\sigma u^3 - \delta_{3\sigma}) \,, \quad r = 1,2, \quad \sigma = \overline{r+1,3};$$

$$m_0 = \frac{(e_0)_{ss}}{\Delta} \,,\; m_1 = -1 + \frac{\rho T(e_0)_{\rho s} - h(e_0)_{ss}}{\Delta} \,,$$

$$m_2 = -3 - \frac{h}{c^2} - \frac{(c^2 T - h\rho(e_0)_{\rho s})^2}{hc^2\Delta} \,,$$

$$\Delta = c^2(e_0)_{ss} - \rho^2(e_0)_{\rho s}^2 \,,\; c^2 = (\rho^2(e_0)_\rho)_\rho;$$

$$T = (e_0)_s \,,\; p = -(e_0)_V \quad \text{(see (1.2))}.$$

Let

$$u^2 = u^3 = 0$$

and let the condition

$$-A^{(0)} > 0$$

be fulfilled. That implies the fulfilment of the following inequalities:

$$\begin{gathered} m_0 > 0, \\ hm_0 - (u^1)^2(m_0 m_2 h + m_1^2) > 0, \\ (u^1)^2[m_2 - hm_0 - 2m_1 + \Gamma^2(m_0 m_2 h + m_1^2)] - \\ - \Gamma^2 h m_0(3 + \Gamma^2 m_2) - (1 + m_1\Gamma^2)^2 > 0. \end{gathered} \tag{1.6}$$

It is easy to show that, as in classical dynamics, inequalities (1.6) are fulfilled if the state equation $e_0 = e_0(\rho, s)$ satisfies the inequalities (see [7], chapter IV)

$$(e_0)_V < 0 \,,\; (e_0)_s > 0,$$

$$(e_0)_{VV} > 0 \,,\; (e_0)_{VV}(e_0)_{ss} - (e_0)_{Vs}^2 > 0.$$

We linearize system (1.3) with respect to the constant solution of this system:

$$\mathbf{U} = \widehat{\mathbf{U}} = \begin{pmatrix} \hat{p} \\ \hat{s} \\ \hat{\mathbf{u}} \end{pmatrix}, \quad \hat{\mathbf{u}} = (\hat{u}^1, 0, 0)^*,$$

where $\hat{p}$ = const, $\hat{s}$ = const, $\hat{u}^1$ = const > 0, and

$$\hat{p} = -(e_0)_V(\hat{\rho}, \hat{s}) \ , \ \hat{c}^2 = \widehat{V}^2(e_0)_{VV}(\hat{\rho}, \hat{s}) \ , \ \widehat{V} = \frac{1}{\hat{\rho}}.$$

Finally we obtain the following system with constant coefficients:

$$\widehat{A}^{(0)} \frac{\partial(\delta \mathbf{Q})}{\partial t} + \sum_{k=1}^{3} \widehat{A}^{(k)} \frac{\partial(\delta \mathbf{Q})}{\partial x^k} = 0, \tag{1.7}$$

where $\widehat{A}^\alpha = A^\alpha(\widehat{\mathbf{Q}})$, $\widehat{\mathbf{Q}} = \mathbf{Q}(\widehat{\mathbf{U}})$ (see (1.5)), $\alpha = \overline{0, 3}$;
$\delta \mathbf{Q} = (\delta q_1, \ldots, \delta q_5)^*$,
δq_i, $j = \overline{1, 5}$ are small perturbations of components of the vector $\mathbf{Q}$.
Since

$$\delta \mathbf{Q} = \widehat{J} \delta \mathbf{U} \ , \ \widehat{J} = J(\widehat{\mathbf{U}}),$$

then system (1.7) can be rewritten as follows:

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} \widehat{A}_k \frac{\partial \mathbf{U}}{\partial x^k} = 0 \tag{1.7'}$$

(we denote the vector $\delta \mathbf{U}$ by $\mathbf{U}$ again). Here

$$\widehat{A}_1 = \begin{pmatrix} a_1 & 0 & a_2 & 0 & 0 \\ 0 & \hat{v}^1 & 0 & 0 & 0 \\ a_3 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & \hat{v}^1 & 0 \\ 0 & 0 & 0 & 0 & \hat{v}^1 \end{pmatrix},$$

$$\widehat{A}_2 = \begin{pmatrix} 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 \\ \frac{1}{\hat{\rho}\hat{h}\widehat{\Gamma}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\hat{\rho}\hat{h}\widehat{\Gamma}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a_1 = \frac{\hat{\upsilon}^1(1-\hat{c}_s^2)}{\widehat{\Delta}}\ ,\ a_2 = \frac{\hat{\rho}\hat{h}\hat{c}_s^2}{\widehat{\Gamma}^3\widehat{\Delta}}\ ,\ a_3 = \frac{1}{\hat{\rho}\hat{h}\widehat{\Gamma}\widehat{\Delta}},$$

$$\widehat{\Delta} = 1-(\hat{\upsilon}^1\hat{c}_s)^2\ ,\ \hat{c}_s^2 = \frac{\hat{c}^2}{\hat{h}}\ ,\ b_1 = \frac{\hat{\rho}\hat{c}^2}{\widehat{\Gamma}\widehat{\Delta}},$$

$$b_2 = -\frac{\hat{\upsilon}^1\hat{c}_s^2}{\widehat{\Delta}}\ ,\ \widehat{\Gamma}^2 = 1+(\hat{u}^1)^2\ ,\ \hat{\upsilon}^1 = \frac{\hat{u}^1}{\widehat{\Gamma}}.$$

For the sake of convenience we rewrite system (1.7′) in the form:

$$L_p - \frac{\hat{\upsilon}^1\hat{c}_s^2}{\widehat{\Gamma}^2\widehat{\Delta}}\xi_1 p + \frac{\hat{c}_s^2}{\widehat{\Gamma}^2\widehat{\Delta}}\left\{\frac{1}{\widehat{\Gamma}^2}\xi_1 u^1 + \xi_2 u^2 + \xi_3 u^3\right\} = 0,$$

$$Lu^1 + \hat{\upsilon}^1\widehat{\Gamma}^2 Lp + \xi_1 p = 0,$$

$$Lu^2 + \xi_2 p = 0, \tag{1.8}$$

$$Lu^3 + \xi_3 p = 0,$$

$$Ls = 0,$$

where $L = \tau + \hat{\upsilon}^1\xi_1$, $\tau = \frac{\partial}{\partial t}$, $\xi_k = \frac{\partial}{\partial x^k}$, $k = 1,2,3$. Besides, in system (1.8) the pressure p is related to the value $\hat{\rho}\hat{h}\widehat{\Gamma}$. It is easy to see that system (1.8) is an analog of the system of acoustic equations in classical gas dynamics.
And similarly to this case (see [4], chapter III), the function p satisfies the wave equation:

$$L^2 p - 2\frac{\hat{\upsilon}^1\hat{c}_s^2}{\widehat{\Gamma}^2\widehat{\Delta}}L\xi_1 p - \frac{\hat{c}_s^2}{\widehat{\Gamma}^2\widehat{\Delta}}\left\{\frac{1}{\widehat{\Gamma}^2}\xi_1^2 p + \xi_2^2 p + \xi_3^2 p\right\} = 0. \tag{1.9}$$

Remark 1.1. Let $\hat{\upsilon}^1 < \hat{c}_s$. Introducing new operators τ', ξ_1' instead of τ, ξ_1 by the rule:

$$\tau = p_0\tau'\ ,\ \xi_1 = p_1\xi_1' + p_2\tau',$$

where $p_0 = \widehat{\Gamma}\widehat{\beta}$, $p_1 = \frac{\hat{c}_s}{p_0}$, $p_2 = \hat{\upsilon}^1\widehat{\Gamma}\frac{1-\hat{c}_s^2}{\hat{\beta}}$, $\widehat{\beta}^2 = \hat{c}_s^2 - (\hat{\upsilon}^1)^2$, we present equation (1.9) in the form

$$\{(\tau')^2 - (\xi_1')^2 - \xi_2^2 - \xi_3^2\}p = 0. \tag{1.9′}$$

Remark 1.2. The matrix $\widehat{A}_1$ has the following eigen-values:

$$\lambda_{1,2,3} = \hat{\nu}^1 \,, \ \lambda_{4,5} = \frac{\hat{\nu}^1(1-\hat{c}_s^2) \pm \frac{\hat{c}_s}{\hat{\Gamma}^2}}{\widehat{\Delta}},$$

and if $\hat{\nu}_1 > \hat{c}_s$, then $\lambda_j > 0,\ j = \overline{1,5}$; if $\hat{\nu}^1 < \hat{c}_s$, then $\lambda_{1,2,3,4} > 0,\ \lambda_5 < 0$.

Remark 1.3. Provided that the inequalities obtained above

$$(e_0)_V(=-p) < 0\,, \ (e_0)_s(=T) > 0,$$

$$(e_0)_{VV}(=\rho^2c^2) > 0,$$

$$(e_0)_{VV}(e_0)_{ss} - (e_0)^2_{Vs} > 0,$$

are fulfilled, system (1.1) can be rewritten as a symmetric t-hyperbolic (by Friedrichs) system. The last inequality means *convexity* of the state equation

$$e_0 = e_0(\rho, s).$$

It was shown in [7], chapter IV that the well-posedness of Cauchy problem for the equations system which describes the propagation of sound in usual heat-conducting gas is a consequence of the convexity of its state equation. We have an analogous situation in relativistic gas dynamics.

2. Formulation of the problem on shock waves stability in relativistic gas dynamics.

First, we write out the conditions which should hold on a shock wave in relativistic gas dynamics. We consider a discontinuity with an equation:

$$\tilde{f}(t, x^1, x^2, x^3) = f(t, x^2, x^3) - x^1 = 0.$$

From common reasons (see [8], § 4) we conclude that in the case of relativistic hydrodynamics the following values have to be continuous on the discontinuity front:

$$[j] = 0\,, \ j = \rho\Gamma(\nu_N - D_N),$$

$$[hu_N j + p] = 0\,, \ [hu_{\tau_{1,2}} j] = 0, \tag{2.1}$$

$$[h\Gamma j + pD_N] = 0.$$

Here

$$v_N = (\mathbf{v}, \mathbf{N}) \ , \ u_{\tau_{1,2}} = (\mathbf{u}, \tau_{1,2}),$$

$\mathbf{N} = \frac{1}{|\nabla \tilde{f}|}\nabla \tilde{f}$ is the normal to the discontinuity surface,

$$\nabla \tilde{f} = \left(-1, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3} \right)^* \ , \ D_N = -\frac{f_t}{|\nabla \tilde{f}|} \ ,$$

$$\tau_1 = \left(\frac{\partial f}{\partial x^2}, 1, 0 \right)^* \ , \ \tau_2 = \left(\frac{\partial f}{\partial x^3}, 0, 1 \right)^* ,$$

$$(\tau_{1,2}, \mathbf{N}) = 0.$$

By $[F]$ we, as usual, denote $[F] = F^+ - F^-$, where F^+, F^- are the values of a variable F on the right side ($\tilde{f} \to -0$) and on the left side ($\tilde{f} \to +0$) of a discontinuity surface. Below we will write F instead of F^+, and F_∞ instead of F^-. Besides, by $\langle F \rangle$ we denote the following expression:

$$\frac{F^+ + F^-}{2} \quad \left(\text{or} \quad \frac{F + F_\infty}{2}\right).$$

Taking into account the continuity of the variable j, we obtain from (2.1)

$$(2.1') \qquad \begin{gathered} [j] = 0 \ , \ j^2[hV] + (1 - D_N^2)[p] = 0, \\ j[hu_{\tau_{1,2}}] = 0 \ , \ j[h\Gamma] + D_N[p] = 0. \end{gathered}$$

In the case of shock wave ($j \neq 0$) we have the following representation of conditions (2.1):

$$(2.1'') \qquad \begin{gathered} [j] = 0 \ , \ [hu_N] + \frac{1}{j}[p] = 0, \\ [h\Gamma] + \frac{D_N}{j}[p] = 0 \ , \ [h\mathbf{u}_\tau] = 0. \end{gathered}$$

It follows from the last equality that the vectors $\mathbf{u}_\tau$ and $(\mathbf{u}_\infty)_\tau$ are collinear, i.e., the equation

$$(2.2) \qquad [h^2|\mathbf{u}_\tau|^2] = 0$$

is true. Here $\mathbf{u}_\tau$ is the tangential to the surface of the strong discontinuity component of the vector $\mathbf{u}$.

Now we will obtain a relativistic equation of the shock adiabat – analog of the Hugoniot adiabat from classical gas dynamics (see [8], § 5). Multiplying the expression

$$[h\Gamma] + \frac{D_N}{j}[p] = 0$$

by $2\langle h\Gamma\rangle$, we have:

$$[h^2\Gamma^2] + 2[p]\left\langle hV\frac{D_N}{\nu_N - D_N}\right\rangle = 0.$$

Since

$$\Gamma^2 = |\mathbf{u}|^2 + 1 = u_N^2 + |\mathbf{u}_\tau|^2 + 1,$$

then, by (2.2),

$$[h^2\Gamma^2] = [h^2u_N^2] + [h^2],$$

and, consequently,

(2.3)
$$[h^2] + [h^2u_N^2] + 2[p]\left\langle \frac{hVD_N}{\nu_N - D_N}\right\rangle = 0.$$

Multiplying the condition

$$[hu_N] + \frac{1}{j}[p] = 0$$

by $2\langle hu_N\rangle$, we have

(2.4)
$$[h^2u_N^2] + 2[p]\left\langle \frac{hV\nu_N}{\nu_N - D_N}\right\rangle = 0.$$

From (2.3) and (2.4) we finally derive the desired equation of the shock adiabat (the Taub adiabat):

(2.5)
$$[h^2] = 2[p]\langle hV\rangle.$$

Note that any equation from (2.1″) could be replaced by the Taub adiabat (2.5). Accounting the equality (see (1.2))

$$p = \rho^2(e_0)_\rho(\rho, s) = -(e_0)_V(\rho, s),$$

we obtain another representation of (2.5):

$$p = H(V; p_\infty, V_\infty). \tag{2.5'}$$

We linearize system (1.1) and conditions (2.1) with respect to the following piece-wise basic solution. When $x^1 < 0$:

$$p = \hat{p}_\infty = \text{const}\,,\ s = \hat{s}_\infty = \text{const},$$

$$\nu^1 = \hat{\nu}^1_\infty = \text{const} > 0\,,\ \nu^2 = \nu^3 = 0,$$

$$\rho = \hat{\rho}_\infty = \text{const}\,,\ \hat{p}_\infty = -(e_0)_V(\hat{\rho}_\infty, \hat{s}_\infty),$$

$$\hat{c}^2_\infty = \widehat{V}^2_\infty (e_0)_{VV}(\hat{\rho}_\infty, \hat{s}_\infty)\,,\ \widehat{V}_\infty = 1/\hat{\rho}_\infty,$$

$$\hat{c}^2_{s\infty} = \frac{\hat{c}^2_\infty}{\hat{h}_\infty}\,,\ \hat{h}_\infty = 1 + e_0(\hat{\rho}_\infty, \hat{s}_\infty) + \hat{p}_\infty \widehat{V}_\infty,$$

$$\widehat{\Gamma}^2_\infty = \frac{1}{\sqrt{1 - (\hat{\nu}^1_\infty)^2}};$$

when $x^1 > 0$:

$$p = \hat{p} = \text{const}\,,\ s = \hat{s} = \text{const},$$

$$\nu^1 = \hat{\nu}^1 \text{const} > 0\,,\ \nu^2 = \nu^3 = 0,$$

$$\rho = \hat{\rho} = \text{const}\,,\ \hat{p} = -(e_0)_V(\hat{\rho}, \hat{s}),$$

$$\hat{c}^2_s = \frac{\hat{c}^2}{\hat{h}}\,,\ \hat{h} = 1 + e_0(\hat{\rho}, \hat{s}) + \hat{p}\widehat{V},$$

$$\widehat{\Gamma}^2 = \frac{1}{\sqrt{1 - (\hat{\nu}^1)^2}}\,;$$

and when $x^1 = 0$ conditions (2.1) hold on the discontinuity surface (we assume that the discontinuity front does not move and it is described by the equation $x^1 = 0$):

$$\hat{\rho}\hat{u}^1 = \hat{\rho}_\infty \hat{u}^1_\infty = \hat{j} \neq 0,$$

$$\hat{\rho}\hat{h}(\hat{u}^1)^2 + \hat{p} = \hat{\rho}_\infty \hat{h}_\infty (\hat{u}^1_\infty)^2 + \hat{p}_\infty,$$

$$\hat{h}\widehat{\Gamma} = \hat{h}_\infty \widehat{\Gamma}_\infty,$$

and any relation here could be replaced by the Taub adiabat (2.5′)

$$\hat{p} = H(\widehat{V}, \hat{p}_\infty, \widehat{V}_\infty).$$

We require the fulfilment of the necessary conditions:

$$\hat{p} > \hat{p}_\infty \, , \; \hat{\rho} > \hat{\rho}_\infty \, , \; \hat{s} > \hat{s}_\infty ,$$

$$(e_0)_{VV}(\hat{\rho}_\infty, \hat{s}_\infty) > 0, \; (e_0)_{VV}(\hat{\rho}, \hat{s}) > 0,$$

$$(e_0)_{VV}(\hat{\rho}_\infty, \hat{s}_\infty)(e_0)_{ss}(\hat{\rho}_\infty, \hat{s}_\infty) - (e_0)^2_{Vs}(\hat{\rho}_\infty, \hat{s}_\infty) > 0,$$

$$(e_0)_{VV}(\hat{\rho}, \hat{s})(e_0)_{ss}(\hat{\rho}, \hat{s}) - (e_0)^2_{Vs}(\hat{\rho}, \hat{s}) > 0,$$

$$\hat{v}^1_\infty > \hat{c}_{s\infty} \, , \; \hat{v}^1 < \hat{c}_s \quad (\hat{c}_{s\infty}, \hat{c}_s < 1!).$$

Accounting that (see Remark 1.2)

$$\hat{v}^1_\infty > \hat{c}_{s\infty},$$

we obtain the following formulation of the problem on stability of relativistic shock waves: in the domain $x^1 > 0$, $(x^2, x^3) \in R^2$, $t > 0$ we seek the solution to linear system (1.8), satisfying the initial data on $t = 0$ and the following boundary conditions on $x^1 = 0$:

$$\begin{aligned} &F_t = \mu p \, , \; u^1 + dp = 0 \, , \; u^2 - \frac{\lambda}{\mu} F_{x^2} = 0, \\ &u^3 - \frac{\lambda}{\mu} F_{x^3} = 0 \, , \; s = \nu p. \end{aligned} \tag{2.6}$$

Here $x^1 = F(t, x^2, x^3)$ is a small displacement of the discontinuity front,

$$\mu = -\frac{\lambda}{\widehat{\Gamma}[\hat{v}^1]} \, , \; d = \frac{\widehat{\Gamma}^2}{\hat{v}^1} \cdot \frac{a - \widehat{\beta}^2}{a} ,$$

$$\lambda = \frac{\hat{v}^1_\infty}{c_1 \hat{v}^1 \widehat{\Gamma}^2} (d\hat{v}^1 - \widehat{\Gamma}^2) \, , \; \nu = -\frac{\hat{h}\widehat{\Gamma}\widehat{\beta}^2[\hat{v}^1]}{\widehat{T}\hat{v}^1 c_1 a} \, , \; a = \hat{c}_s^2 a_1,$$

$$a_1 = 2 - \frac{\hat{v}^1[\hat{v}^1]}{c_1}(1 + \frac{(e_0)_{Vs}(\hat{\rho}, \hat{s})}{\widehat{T}\hat{c}_s^2 \hat{\rho}}) \, , \; c_1 = 1 - \hat{v}^1 \hat{v}^1_\infty .$$

We note again that boundary conditions (2.6) come from jump conditions (2.1). And since $\hat{v}^1_\infty > \hat{c}_{s\infty}$, small perturbations of the variables sought are assumed to be equal to zero to the left of the discontinuity $x^1 = 0$ (i.e., the perturbations do not propagate upstream of the flow). The procedure of linearization of relations (2.1) is detailed in [2] (see also [4], [5]).

As in classical gas dynamics (see [4], chapter III), mixed problem (1.8), (2.6) can be reduced to the mixed problem for wave equation (1.9) (or (1.9′)) with the boundary condition of the following form on $x^1 = 0$:

$$m(\tau')^2 p + n(\xi_1')^2 p - \frac{\hat{c}_s}{\hat{\upsilon}^1}\tau'\xi_1' p = 0, \tag{2.7}$$

where

$$n = -\left(\frac{\hat{c}_s}{\widehat{\Gamma}\widehat{\beta}}\right)^2 \lambda\,,\ m = \left(\frac{\hat{c}_s}{\widehat{\Gamma}\widehat{\beta}}\right)^2 \lambda + \frac{\hat{c}_s^2}{\hat{\upsilon}^1\widehat{\Gamma}^2} d.$$

At the end of this section we will study the question of construction of the Hadamard example for mixed problem (1.9), (2.7) (and, consequently, for initial problem (1.8), (2.6)). For problem (1.9), (2.7) we will seek exponential solutions of the form:

$$p = p^0 \exp\left\{i\left(-\frac{\hat{c}_s}{\widehat{\Gamma}^2\widehat{\Delta}}\omega t + l x^1 + \frac{1}{\widehat{\Gamma}\widehat{\Delta}^{1/2}} k x^2\right)\right\}, \tag{2.8}$$

where p^0, ω, l, k are some constants. Substituting (2.8) into (1.9), we obtain the algebraic expression:

$$(\omega - Ml)^2 = l^2 + k^k\,,\ M = \frac{\hat{\upsilon}^1(1-\hat{c}_s^2)}{\hat{c}_s}\widehat{\Gamma}^2 (< 1!), \tag{2.9}$$

all solutions of which are described by the parametrization:

$$\omega - Ml = k\frac{z^2+1}{2z}\,,\ l = k\frac{z^2-1}{2z}\,,$$

i.e.,

$$l = k\frac{z^2-1}{2z}\,,\ \omega = \frac{k}{2z}\{(1+M)z^2 + 1 - M\}.$$

It is known that if problem (1.9), (2.7) has solution of the form (2.8) with

$$\operatorname{Im}\omega > 0\,,\ \operatorname{Im} l > 0\,,\ \operatorname{Im} k = 0, \tag{2.10}$$

then problem (1.9), (2.7) (and, consequently, initial problem (1.8), (2.6)) is ill-posed (see [4]). Let $z = \hat{\rho}e^{i\varphi}$. Then condition (2.10) can be rewritten in the form

$$\begin{aligned} \operatorname{Im} l &= \tfrac{k}{2}\sin\hat{\varphi}(\hat{\rho} + \tfrac{1}{\hat{\rho}}) > 0,\\ \operatorname{Im}\omega &= \tfrac{K}{2}\sin\hat{\varphi}\left\{\hat{\rho}(1+M) - \tfrac{1-M}{\hat{\rho}}\right\} > 0. \end{aligned} \tag{2.10′}$$

Introducing the variable

$$x = \frac{1+M}{1-M} z^2,$$

instead of z, from (2.10′) we come to

(2.10″) $$|x| > 1$$

(if x is a real number, then $x < -1$).

Substituting (2.8) into boundary condition (2.7), we finally have the quadratic equation for x:

(2.11) $$(\hat{d}+1)x^2 + 2x(\hat{d}+2\hat{\lambda}) + \hat{d} - 1 = 0,$$

where

(2.12) $$\hat{d} = \frac{\hat{c}_s}{\hat{\nu}^1} - \frac{\hat{c}_s \widehat{\beta^2}}{\hat{\nu}^1 a}\ ,\ \hat{\lambda} = \frac{\hat{\nu}^1 \hat{c}_s}{\widehat{\Gamma^2}\widehat{\beta^2}} \lambda,$$

and

$$m = \frac{\hat{c}_s}{\hat{\nu}^1}(\hat{\lambda}+\hat{d})\ ,\ n = -\frac{\hat{c}_s}{\hat{\nu}^1}\hat{\lambda}.$$

Having studied the roots of equation (2.11), we can divide the plane $\hat{d}, \hat{\lambda}$ into domains:

1) the domain I:
   a) $\hat{\lambda} < 0\ ,\ \hat{d} < -1\ ,\ |x_1| < 1\ ,\ x_2 < -1$;
   b) $\hat{\lambda} > 0\ ,\ \hat{d} > -1\ ,\ \hat{d}+\hat{\lambda} > 0\ ,\ |x_1| < 1\ ,\ x_2 < -1$;
   c) $\hat{\lambda} > 0\ ,\ \hat{d} > -1\ ,\ \hat{d}+\hat{\lambda} < 0\ ,\ x_1 > 1\ ,\ x_2 < -1$;
   d) $0 < \hat{\lambda} < \frac{1}{2}\ ,\ \hat{d} < -1\ ,\ 4\hat{\lambda}(\hat{\lambda}+\hat{d})+1 > 0\ ,\ x_{1,2} < -1$;
   e) $\hat{\lambda} > 0\ ,\ \hat{d} < -1\ ,\ 4\hat{\lambda}(\hat{\lambda}+\hat{d})+1 < 0$,
   $|x_{1,2}| > 1$ ($x_{1,2}$ are complex);
2) the domain II:
   $\hat{\lambda} < 0\ ,\ \hat{d}+\hat{\lambda} > 0\ ,\ |x_{1,2}| < 1$ ($x_{1,2}$ are either complex or real);
3) the domain III:
   a) $\hat{\lambda} < 0\ ,\ \hat{d} > -1\ ,\ \hat{d}+\hat{\lambda} < 0\ ,\ |x_1| < 1, x_2 > 1$;
   b) $\hat{\lambda} > 0\ ,\ \hat{d} < -1\ ,\ \hat{d}+\hat{\lambda} > 0\ ,\ |x_1| < 1\ ,\ x_2 > 1$;
   c) $\hat{\lambda} > \frac{1}{2}\ ,\ \hat{d}+\hat{\lambda} < 0\ ,\ \hat{d} < -1\ ,\ 4\hat{\lambda}(\hat{\lambda}+\hat{d})+1 > 0\ ,\ x_{1,2} > 1$.

Consequently, the domain I is the domain of the ill-posedness for problem (1.9), (2.7) (see (2.10″)); the domains II, III are the domains, where solution (2.8) does not grow in time. In the next section we will show that the domain II is the domain of the well-posedness for mixed problem (1.8), (2.6).

Remark 2.1. On the plane $\hat{d}$, $\hat{\lambda}$ we draw the line

$$\hat{d} = \frac{\hat{c}_s}{\hat{\upsilon}^1} + \frac{\widehat{\Gamma}^2 \widehat{\beta}^2 c_1}{\hat{\upsilon}^1_\infty \hat{\upsilon}^1} \hat{\lambda}, \tag{2.13}$$

corresponding to the gas dynamics case.
Since

$$\frac{\hat{c}_s}{\hat{\upsilon}^1} > 0\,, \quad \frac{\widehat{\Gamma}^2 \widehat{\beta}^2 c_1}{\hat{\upsilon}^1_\infty \hat{\upsilon}^1} > 0,$$

then line (2.13) intersects the domain II.
The conditions

$$\hat{\lambda} < 0\,, \; \hat{d} + \hat{\lambda} > 0$$

result in requirements on the state equation $e_0 = e_0(\rho, s)$ which are additional to the ones given in section 1. These requirements coincide totally with conditionsgiven in [2].

Remark 2.2. As far as the domain III is concerned, we have a situation analogous to the one which takes place in usual gas dynamics (see [4], chapter III). We can show that there exists a problem, close to initial problem (1.9), (2.7), which is ill-posed. The latter means that in this case the question of shock waves stability can be solved only for the quasilinear formulation of the problem, i.e., we must consider initial quasilinear system (1.1) and relations (2.1).

3. Well-posedness of mixed problem for linear system of relativistic gas dynamics equations.

For the further consideration we give a symmetric representation of system (1.7′). It can be easily obtained from (1.7). It suffices to multiply (1.7) from the left by $\widehat{J}^*$ and to recall that $\delta \mathbf{Q} = \widehat{J} \delta \mathbf{U}$. Thus, we have

$$A\mathbf{U}_t + \sum_{k=1}^{3} A_k \mathbf{U}_{x^k} = 0. \tag{3.1}$$

Here

$$A = \begin{pmatrix} \bar{A} & 0 \\ 0 & \widehat{\Gamma} \hat{h} \hat{\rho} I_2 \end{pmatrix}\,, \quad A_1 = \begin{pmatrix} \bar{A}_1 & 0 \\ 0 & \hat{\upsilon}^1 \widehat{\Gamma} \hat{h} \hat{\rho} I_2 \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} \frac{\widehat{\Gamma}}{\hat{\rho}\hat{c}^2} & 0 & \hat{\upsilon}^1 \\ 0 & \widehat{\Gamma}\hat{\rho}\frac{\widehat{\Delta}}{\hat{c}^2} & 0 \\ \hat{\upsilon}^1 & 0 & \frac{\hat{h}\hat{\rho}}{\widehat{\Gamma}} \end{pmatrix} , \quad \bar{A}_1 = \begin{pmatrix} \frac{\hat{\upsilon}^1\widehat{\Gamma}}{\hat{\rho}\hat{c}^2} & 0 & 1 \\ 0 & \hat{\upsilon}^1\widehat{\Gamma}\hat{\rho}\frac{\widehat{\Delta}}{\hat{c}^2} & 0 \\ 1 & 0 & \hat{h}\hat{\rho}\hat{\upsilon}^1 \end{pmatrix},$$

$$\widetilde{\Delta} = \Delta(\hat{\rho}, \hat{s}),$$

I_2 is the unit matrix of order 2; in the matrices $A_2 = (a_{ij})$, $A_3 = (b_{ij})$, $i, j = 1, \ldots, 5$, $a_{14} = a_{41} = b_{15} = b_{51} = 1$, and the other elements of these matrices are equal to zero.

Excluding the function $F(t, x^1, x^2, x^3)$ from the boundary conditions, we present (2.6) in the form:

$$\begin{gathered} u^1 + dp = 0 \, , \ u^2_t - \lambda p_{x^2} = 0, \\ u^3_t - \lambda p_{x^3} = 0 \, , \ s = \nu p \, , \ u^2_{x^3} = u^3_{x^2}. \end{gathered} \tag{3.2}$$

It is apparent that on $x^1 = 0$ the following condition holds:

$$\xi_2 u^2 + \xi_3 u^3 = (\beta_1 \tau + \beta_2 \xi_1) p, \tag{3.3}$$

where

$$\beta_1 = \left(\frac{d}{\widehat{\Gamma}^2 \hat{\upsilon}^1} - 1 - \frac{\widehat{\Gamma}^2 \widehat{\Delta}}{\hat{c}^2_s} \right), \quad \beta_2 = \frac{-1}{\widehat{\Gamma}^2 \hat{\upsilon}^1} \left(1 + \frac{\widehat{\Gamma}^4 (\hat{\upsilon}^1)^2 \widehat{\Delta}}{\hat{c}^2_s} \right).$$

As a consequence from formulae (3.2), (3.3), we obtain:

$$\begin{gathered} (\xi_2^2 + \xi_3^2) u^2 = (\beta_1 \tau + \beta_2 \xi_1) \xi_2 p, \\ (\xi_2^2 + \xi_3^2) u^3 = (\beta_1 \tau + \beta_2 \xi_1) \xi_3 p. \end{gathered} \tag{3.4}$$

Let us consider the following problem, named Problem 1: we seek the solution to system (3.1) which satisfies condition (3.2) on $x^1 = 0$ and the initial data

$$\mathbf{U}(x^1, x^2, x^3, 0) = \mathbf{U}_0(x^1, x^2, x^3)$$

in the domain

$$\Omega(0) = \left\{ 0 < x_1 < 1 \, , \ -\frac{1}{2} < x_i < \frac{1}{2} \, , \ i = 2, 3 \right\}.$$

Let us introduce some notations and definitions. Let ω and $\Omega(\tau_0)$ be domains,

$$\omega \subset \{x_1 \geq 0\,,\ -\infty < x_i < \infty\,,\ i = 2, 3\,,\ 0 \leq t \leq \infty\};$$

$$\Omega(\tau_0) = \omega \cap \{t = \tau_0\}, \qquad 0 \leq \tau_0 \leq T < \infty.$$

Let $A_X = A_X^* > 0$, A_{kX}, $k = 1, 2, 3$, be expansions of the matrices A, A_k, correspondingly, and U_X be an expansion of the vector U, satisfying an expanded system:

$$\{A_X\tau + A_{1X}\xi_1 + A_{2X}\xi_2 + A_{3X}\xi_3\}\mathbf{U}_X = 0.$$

We define $J(\tau_0)$ as

$$J(\tau_0) = \int_{\Omega(\tau_0)} (A_X\mathbf{U}_X, \overline{\mathbf{U}}_X)dx.$$

Let us prove the following assertion (the proof is based on the technique, suggested in [4]).

Assertion. Let conditions $m > 0, n > 0$ hold. Then for Problem 1 there exist constants N, T, a domain ω, matrices $A_X = A_X^* > 0$, A_{kX}, $k = 1, 2, 3$ and a vector U_X such that for any τ_0, $0 < \tau_0 \leq T$, the inequality is fulfilled:

$$J(\tau_0) \leq J(0)\exp(NT).$$

Proof. First, we note that if p satisfies equation (1.9′), then the vector $\mathbf{Y} = (\tau' p, \xi_1' p, \xi_2 p, \xi_3 p)^*$ satisfies a symmetric system:

$$(E\tau' + Q\xi_1' + R_2\xi_2 + R_3\xi_3)\mathbf{Y} = 0,$$

$$E = \begin{pmatrix} 1 & -m_1 & -l_2 & -l_3 \\ -m_1 & 1 & 0 & 0 \\ -l_2 & 0 & 1 & 0 \\ -l_3 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} m_1 & -1 & 0 & 0 \\ -1 & m_1 & l_2 & l_3 \\ 0 & l_2 & -m_1 & 0 \\ 0 & l_3 & 0 & -m_1 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} l_2 & 0 & -1 & 0 \\ 0 & -l_2 & m_1 & 0 \\ -1 & m_1 & l_2 & l_3 \\ 0 & 0 & l_3 & -l_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} l_3 & 0 & 0 & -1 \\ 0 & -l_3 & 0 & m_1 \\ 0 & 0 & -l_3 & l_2 \\ -1 & m_1 & l_2 & l_3 \end{pmatrix},$$

where m_1, l_2, l_3 are some constants, and $E > 0$ if $1 > m_1^2 + l_2^2 + l_3^2$.

We rewrite boundary conditions (2.7) in the form:

$$(\tau' - a\xi_1')\widehat{L}p = 0,$$

where

$$\widehat{L} \doteq a_1\tau + a_2\xi_1'.$$

The constants a, a_1, a_2 are derived from the system:

$$a_1 = m \ , \ aa_2 = -n \ , \ am - a_2 = \gamma = \frac{\hat{c}_s}{\hat{\nu}^1}.$$

Solving this system we obtain that the number a is either real or complex, depending on the sign of the expression $\gamma^2 - 4mn$.

Let $\gamma^2 - 4mn > 0$, then

$$a = \frac{\gamma + \sqrt{\gamma^2 - 4mn}}{2m} \ , \ \mathbf{Y}_X = (\tau'\mathbf{Y}^*, \xi_1'\mathbf{Y}^*, \xi_2\mathbf{Y}^*, \xi_3\mathbf{Y}^*, \widehat{L}\mathbf{Y}^*)^*.$$

The vector Y_X satisfies a system:

$$\{E_X\tau + Q_X\xi_1' + R_{2X}\xi_2 + R_{3X}\xi_3\}\mathbf{Y}_X = 0, \tag{3.5}$$

where E_X, Q_X, R_{2X}, R_{3X} are block-diagonal matrices of order 20,

$$E_X = \begin{pmatrix} \sigma_1 E & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 E & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 E & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 E & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 E \end{pmatrix},$$

$E_X > 0$, if $\sigma_i > 0$, $i = 1, \ldots, 5$. The matrices Q_X, R_{2X}, R_{3X} are determined by Q, R_2, R_3 in a similar way. We choose $\sigma_i, m_{1i}, l_{2i}, l_{3i}, i = 1, \ldots, 5$ (m_{1i}, l_{2i}, l_{3i} are elements of the blocks, corresponding to different σ_i) such that the condition

$$-(Q_X Y_X, Y_X)|_{x^1=0} > 0$$

is fulfilled.

We choose

$$l_{2i} = l_{3i} = 0, \quad i = 1, \ldots, 5;$$

$$m_{11} = m_{14} = 0 \ , \ m_{15} = \frac{2a}{1+a^2} \ , \ -1 < m_{12} < 0,$$

$$0 < m_{13} < \min\left\{1, \frac{n}{ma}\right\}, \quad \sigma_1 = \frac{m}{n}\sigma_2,$$

σ_5 is an arbitrary positive number; the constants $\sigma_2, \sigma_3, \sigma_4$ are chosen to satisfy

$$\sigma_3, \sigma_4 < \frac{2mn}{a} m_{15}\sigma_5, \quad \sigma_2 < \min\left\{\frac{-n\sigma_4}{mam_{12}}, \frac{-n\sigma_3}{mam_{12}}\right\}.$$

By this choice,

$$-(Q_X Y_X, Y_X)|_{x^1=0} = \sum_{i=2}^{3}\left\{(L\xi_i p)^2 k_{1i} + (\tau'\xi_i p)^2 k_{2i} + \right.$$
$$\left. + (\xi_1'\xi_i p)^2 k_{3i} + (\xi_2\xi_i p)^2 k_{4i}\right\} + (\tau'\xi_1' p)^2\left(\frac{2\gamma}{n} - m_{12}\right) +$$
$$+ ((\xi_1')^2 p)^2(-\sigma_2 m_{12}),$$

where k_{ij}, $i = 1, \dots, 4$, $j = 2, 3$, are some positive constants. Thus,

$$(Q_X \mathbf{Y}_X, \mathbf{Y}_X)|_{x^1=0} > 0.$$

If $\gamma^2 - 4mn < 0$, then a is a complex number, and $\widehat{L}p$ is a complex function. Let us turn back to system (3.5). Provided that $m_{14} = 0$, $l_{2i} = l_{3i} = 0$, $i = 1, \dots, 5$, we obtain:

$$-(Q_X \mathbf{Y}_X, \overline{\mathbf{Y}}_X)|_{x^1=0} = |L\xi_1 p|^2 k_{11} +$$
$$+ \sum_{i=2}^{3}\left\{|L\xi_i p|^2 k_{1i} + (\tau'\xi_i p)^2 k_{2i} + (\xi_1'\xi_1 p)^2 k_{3i} + (\xi_2\xi_i p)^2 k_{4i}\right\} +$$
$$+ (\tau'\xi_1' p)^2 k_5 + ((\xi_1')^2 p)^2 k_6 + (\tau'\xi_1' p)((\xi_1')^2 p) k_7.$$

If

$$m_{11} = m_{13} = 0, \quad m_{15} = \frac{2\,\mathrm{Re}\,a}{1 + |a|^2}, \quad -1 < m_{12} < 0,$$

$\sigma_1 = \frac{\sigma_2 m}{n}$, σ_3, σ_4 are arbitrary positive numbers,

$$\sigma_5 m_{15} > \frac{\max(\sigma_3, \sigma_4)}{-a_1 \mathrm{Re}\, a_2}, \quad \sigma_2 m_{12} > \min\left\{\frac{a_1\sigma_4}{\mathrm{Re}\, a_2}, \frac{a_1\sigma_3}{\mathrm{Re}\, a_2}\right\},$$

then $k_7 = 0$, $k_5, k_6 > 0$, $k_{ij} > 0$, $i = 1, \dots, 4$, $j = 2, 3$,, and, consequently, $-(Q_X \mathbf{Y}_X, \overline{\mathbf{Y}}_X)|_{x^1=0} > 0$.

Now we begin to construct the expanded system:

$$\{A_X\tau + A_{1X}\xi_1 + A_{2X}\xi_2 + A_{3X}\xi_3\}\mathbf{U}_X = 0, \tag{3.6}$$

$$\mathbf{U}_X = (\mathbf{U}^*, \mathbf{U}_t^*, \mathbf{U}_{x^2}^*, \mathbf{U}_{x^3}^*, \mathbf{U}_{tt}^*, \mathbf{U}_{tx^2}^*, \mathbf{U}_{tx^3}^*, \mathbf{U}_{x^2x^2}^*, \mathbf{U}_{x^2x^3}^*, \mathbf{U}_{x^3x^3}^*, \mathbf{Y}_X^*)^*,$$

$$A_X = \begin{pmatrix} I_4 \otimes A & 0 & 0 \\ 0 & \varepsilon(I_6 \otimes A) & 0 \\ 0 & 0 & \frac{E_X}{\hat{\Gamma}\hat{\beta}} - \frac{\widehat{\nu\Gamma}(1-\hat{c}_s^2)}{\hat{\beta}\hat{c}_s} Q_X \end{pmatrix},$$

$$A_{1X} = \begin{pmatrix} I_4 \otimes A_1 & 0 & 0 \\ 0 & \varepsilon(I_6 \otimes A_1) & 0 \\ 0 & 0 & \frac{\hat{c}_s}{\hat{\Gamma}\hat{\beta}} Q_X \end{pmatrix},$$

$$A_{iX} = \begin{pmatrix} I_4 \otimes A_i & 0 & 0 \\ 0 & \varepsilon(I_6 \otimes A_i) & 0 \\ 0 & 0 & R_{iX} \end{pmatrix}, \quad i = 2, 3,$$

where $I_n \otimes A = \text{diag}\,\underbrace{(A, \dots, A)}_{n}$.

Remark 3.1. System (3.6) is the symmetric t-hyperbolic (by Friedrichs) system.

Remark 3.2. We note that on $x_1 = 0$ the components of the vector U_X meet, in particular, conditions (3.2), (2.7).

Remark 3.3. Preparatory to the estimating, we note that all the derivatives of $\mathbf{U}_{x^1}$ can be estimated with the help of the initial system because A_1 is not degenerated, and

$$\mathbf{U}_{x^1} = -A_1^{-1}(A\mathbf{U}_t + A_2\mathbf{U}_{x^2} + A_3\mathbf{U}_{x^3}).$$

Let ω be a domain, restricted by the hyperplanes $t = 0$, $t = T$, $x^1 = 0$ and by a surface S_b, such that

$$\int\cdots\int_{S_b} \left\{(A_X\mathbf{U}_X, \overline{\mathbf{U}}_X)\tilde{\tau} + \sum_{i=1}^{3}(A_{iX}\mathbf{U}_X, \overline{\mathbf{U}}_X)\tilde{\xi}_i\right\} dS \geq 0,$$

where $(\tilde{\tau}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)$ is the unit normal to the surface S_b.

To construct the surface S_b we use the idea from [9]. We find the largest root τ^* of the equation $\det(\tilde{\tau}A_X + \tilde{\xi}_1 A_{1X}) = 0$ and the largest roots τ^* of the equations $\det(\tilde{\tau} A_X + \tilde{\xi}_i A_{iX}) = 0$, $i = 2, 3$. Substituting the concrete matrices A_X, A_{iX}, $i = 1, 2, 3$, we obtain: for $\tilde{\xi}_1 \geq 0$

$$\tilde{\tau}^* = \tilde{\xi}_1 \frac{1 - \hat{c}_s \hat{\mathrm{v}}^1}{\hat{c}_s - \hat{\mathrm{v}}^1},$$

for $\tilde{\xi}_i \gtrless 0$, $i = 2, 3$,

$$\tilde{\tau}^* = \pm \frac{\tilde{\xi}_i}{(\hat{c}_s^2 - (\hat{\mathrm{v}}^1)^2)^{1/2}} .$$

Draw the hyperplanes

$$\frac{t}{x^1 - 1} = -\frac{\hat{c}_s - \hat{\mathrm{v}}^1}{1 - \hat{c}_s \hat{\mathrm{v}}^1}, \quad \frac{t}{x^i - 1/2} = -(\hat{c}_s^2 - (\hat{\mathrm{v}}^1)^2)^{1/2},$$

$$\frac{t}{x^i + 1/2} = (\hat{c}_s^2 - (\hat{\mathrm{v}}^1)^2)^{1/2}, \quad i = 2, 3,$$

through the faces of the unit cube $\Omega(0)$. An arbitrary number

$$\alpha(\hat{c}_s^2 - (\hat{\mathrm{v}}^1)^2)^{1/2}, \ 0 < \alpha < \frac{1}{2}$$

can be taken as T. The surface S_b is constructed. It is apparent that S_b possesses the desired property.

Let us obtain the integral estimation promised above.

$$\frac{1}{2} \int \cdots \int_{\omega} \int \left\{ (A_X \mathbf{U}_{Xt}, \overline{\mathbf{U}}_X) + \sum_{i=1}^{3} (A_{iX} U_{Xx^i}, \overline{\mathbf{U}}_X) \right\} d\omega =$$

$$= \int \cdots \int_{\omega} \int \{ (A_X \mathbf{U}_X, \overline{\mathbf{U}}_X)_t + (A_{1X} \mathbf{U}_X, \overline{\mathbf{U}}_X)_{x^1} + (A_{2X} U_X, \overline{\mathbf{U}}_X)_x^2 +$$

$$+ (A_{3X} \mathbf{U}_X, \overline{\mathbf{U}}_X)_{x^3} \} d\omega = J(T) - J(0) - \int \cdots \int_S \int (A_{1X} \mathbf{U}_X, \overline{\mathbf{U}}_X) \, dS +$$

$$+ \int \cdots_{S_b} \int \left\{ (A_X \mathbf{U}_X, \overline{\mathbf{U}}_X) \tilde{\tau} + \sum_{i=1}^{3} (A_{iX} \mathbf{U}_X, \overline{\mathbf{U}}_X) \tilde{\xi}_i \right\} dS = 0.$$

By the properties of the surface S_b, we have:

$$J(T) - J(0) - \int_0^T \left(\int_{\sigma(t)} (A_{1X} \mathbf{U}_X, \overline{\mathbf{U}}_X) dx^2 dx^3 \right) dt \leq 0.$$

We consider the form $(A_{1X}\mathbf{U}_X, \overline{\mathbf{U}}_X)$:

$$-(A_{1X}\mathbf{U}_X, \overline{\mathbf{U}}_X)|_{x^1=0} = -(A_1\mathbf{U}, \mathbf{U})|_{x^1=0} - (A_1\mathbf{U}_t, \mathbf{U}_t)|_{x^1=0}$$

$$-\sum_{i=2}^{3}(A_i\mathbf{U}_{x^i}, \mathbf{U}_{x^i})|_{x^1=0} + V;$$

$$V = -\varepsilon\Big\{(A_1\mathbf{U}_{tt}, \mathbf{U}_{tt}) + \sum_{i=2}^{3}(A_1\mathbf{U}_{tx^i}, \mathbf{U}_{tx^i}) +$$

$$+\sum_{i=2}^{3}(A_1\mathbf{U}_{x^2x^i}, \mathbf{U}_{x^2x^i}) + (A_1\mathbf{U}_{x^3x^3}, \mathbf{U}_{x^3x^3})\Big\}|_{x^1=0} -$$

$$-\frac{\hat{c}_s}{\widehat{\Gamma}\widehat{\beta}}(Q_X\mathbf{Y}_X, \overline{\mathbf{Y}}_X)|_{x^1=0}.$$

Using the form of the matrix A_1, boundary condition (3.2), we obtain that

$$V = -\varepsilon(\alpha_1(p_{tt}^2 + p_{tx^2}^2 + p_{tx^3}^2 + p_{x^2x^2}^2 + p_{x^2x^3}^2 + p_{x^3x^3}^2) +$$

$$+\alpha_2((u_{x^2x^2}^2)^2 + (u_{x^2x^3}^2)^2 + (u_{x^3x^3}^2)^2 + (u_{x^2x^2}^3)^2 + (u_{x^2x^3}^3)^2 + (u_{x^3x^3}^3)^2)),$$

α_i are some constants, depending on the coefficients of the matrix A_1. To estimate the quadratic form of the variables $u^i_{x^jx^k}$ we use the fact (see [10]) that:

$$\int_S \sum_{j,k=2}^{3}(u^i_{x^jx^k})^2 dS \le \text{const}\int_S (u^i_{x^2x^2} + u^i_{x^3x^3})^2\, dS$$

$$\le \text{const}\int_S ((\beta_1\tau + \beta_2\xi_1)\xi_2 p)^2\, dS,$$

where S is a part of the plane $x^1 = 0$, $S \subset \omega$. The inequality is continued with the help of boundary condition (3.4). Since

$$-Q_X\mathbf{Y}_X, \overline{\mathbf{Y}}_X)|_{x^1=0} > 0,$$

for Y_X which satisfies system (3.5) we have:

$$J(T) - J(0) \le \int_0^T \int_{\sigma(t)} \Big\{(A_1\mathbf{U}, \mathbf{U}) + (A_1\mathbf{U}_t, \mathbf{U}_t) +$$

$$+\sum_{i=2}^{3}(A_1\mathbf{U}_{x^1},\mathbf{U}_{x^i})\Big\}\,dx^2dx^3dt.$$

Applying the Sobolev's embedding theorem:

$$J(T)-J(0)<N\int_0^T J(t)\,dt,$$

$N>0$ is a constant, we finally have

$$J(T)\leq J(0)\exp(NT). \tag{3.7}$$

The existence of the estimation (3.7) allows to assert that Problem 1 has the unique solution U (see [9]), and $U\in W_2^2(\Omega(t))$, if $U_0\in W_2^2(\Omega(0))$. Thus, Problem 1 is well-posed for $m,n>0$, i.e., the domain II is a domain of the well-posedness for mixed problem (1.8), (2.6).

REFERENCES

[1] A.M. Anile - G. Russo, *Corrugation stability for plane relativistic shock waves,* Phys. Fluid, 29 n. 9 (1986), pp. 2847-2852.

[2] A.M. Anile - G. Russo, *Linear stability for plane relativistic shock waves,* Phys. Fluids, 30 n. 4 (1987), pp. 1045-1051.

[3] A.M. Anile - G. Russo, *Stability properties of relativistic shock waves: Basic results,* Phys. Fluids, 30 n. 8 (1987), pp. 2406-2413.

[4] A.M. Blokhin, *Energy integrals and their applications to problems of gas dynamics,* Novosibirsk, Nauka, 1986.

[5] L.D. Landau - E.M. Lifshitz, *Hydrodynamics,* Moscow, Nauka, 1986.

[6] A.M. Blokhin - E.V. Mishchenko, *Symmetrization of relativistic gas dynamics equations,* Continuum dynamics, Novosibirsk, 88 (1988), pp. 13-22.

[7] S.K. Godunov, *Fragments from continuum mechanics,* Moscow, Nauka, 1978.

[8] L.V. Ovsyannikov, *Lectures on foundations of gas dynamics,* Moscow, Nauka, 1981.

[9] S.K. Godunov, *Equations of mathematical physics,* Moscow, Nauka, 1979.

[10] O.A. Ladyzhenskaya, *Mathematical problems of viscous noncompressible fluid,* Moscow, Nauka, 1970.

Institute of Mathematics,
Universitetsky pr 4,
630090 Novosibirsk (Russia)