LE MATEMATICHE
Vol. XLVIII (1993) – Fasc. I, pp. 77-86

REPRESENTATION FORMULAS FOR THE MOMENTS OF THE DENSITY OF ZEROS OF ORTHOGONAL POLYNOMIAL SETS

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The moments of the density of zeros of orthogonal polynomial systems generated by a three-term recurrence relation are represented by Lucas polynomials of the first kind and by Bell polynomials.

1. The problem.

Suppose the coefficients $\{\alpha_n\}$ and $\{\beta_{n-1}\}$ $(n \geq 1)$ of the three-term recurrence relation:

$$(1.1)\qquad \begin{cases} P_{-1} = 0 \quad , \quad P_0(x) = 1 \\ P_n(x) = (x - \alpha_n)P_{n-1}(x) - \beta_{n-1}^2 P_{n-2}(x), \quad (n \geq 1) \end{cases}$$

be given. By a well known Favard theorem (see [5]) there exists a measure $d\phi(x)$ with a suitable support $[a, b]$, such that the system $\{P_k(x)\}_{k \in \mathbb{N}_0}$ is an O.P.S. with respect to this measure in the interval $[a, b]$.

For any fixed $n \in \mathbb{N}$ consider:

I. - the density of the zeros $x_{k,n}$ $(k = 1, \dots, n)$ of $P_n(x)$ defined by:

$$(1.2)\qquad \rho_n(x) := \frac{1}{n} \sum_{k=1}^{n} \delta(x - x_{k,n}) \qquad (\delta = \text{Dirac delta})$$

Entrato in Redazione il 19 gennaio 1993.

and the related moments:

$$\mu_r^{(n)} := \frac{1}{n}\sum_{k=1}^{n} x_{k,n}^r; \tag{1.3}$$

II. - the asymptotic density of zeros of the system $\{P_k(x)\}_{k\in\mathbb{N}_0}$ defined by:

$$\rho(x) := \lim_{n\to\infty} \rho_n(x) \tag{1.4}$$

and the related moments:

$$\mu_r := \lim_{n\to\infty} \mu_r^{(n)}. \tag{1.5}$$

J.S. Dehesa [2] (see also [3]) proved the following theorem:

Theorem 1. (Dehesa). *The moments of the density of zeros $\rho_n(x)$ of $P_n(x)$ are given by the following formulae:*

$$\mu_r^{(n)} := \frac{1}{n}\sum_{(r)} F_r(r'_1, r_1, r'_2, r_2, \dots, r'_j, r_j, r'_{j+1}) \cdot \tag{1.6}$$

$$\cdot \sum_{i=1}^{n-t} \alpha_i^{r'_1}\beta_i^{2r_1}\alpha_{i+1}^{r'_2}\dots\alpha_{i+j-1}^{r'_j}\beta_{i+j-1}^{2r_j}\alpha_{i+j}^{r'_{j+1}} \qquad (r = 1, 2, \dots, n).$$

Where
$j := \left[\frac{r}{2}\right]$ is the greater integer contained in $r/2$;
$\sum_{(r)}$ denotes that the sum runs over all partitions of the integer r, say it $(r'_1, r_1, r'_2, r_2, \dots, r'_j, r_j, r'_{j+1})$, subject to the following conditions:

$$\begin{cases} r'_1 + r'_2 + \dots + r'_{j+1} + 2(r_1 + r_2 + \dots + r_j) = r \\ \text{if } r_s = 0,\ 1 < s < j,\ \text{then } r_k = r'_k = 0,\ \forall k > s; \end{cases}$$

t denotes the number of non vanishing r_i involved in the corresponding partition of r;
and lastly:

$$F_r(r'_1, r_1, r'_2, r_2, \dots, r'_j, r_j, r'_{j+1}) :=$$

$$:= r\,\frac{(r'_1 + r_1 - 1)!(r_{j-1} + r'_j - 1)!}{r'_1!r_1!(r_{j-1} - 1)!r'_j!}\prod_{i=1}^{j-2}\frac{(r_i + r'_{i+1} + r_{i+1} - 1)!}{(r_i - 1)!r'_{i+1}!r_{i+1}!}.$$

The first few moments are:

$$\mu_1^{(n)} = \frac{1}{n}\sum_{k=1}^{n}\alpha_k$$

$$\mu_2^{(n)} = \frac{1}{n}\left\{\sum_{k=1}^{n}\alpha_k^2 + 2\sum_{h=1}^{n-1}\beta_h^2\right\}$$

$$\mu_3^{(n)} = \frac{1}{n}\left\{\sum_{k=1}^{n}\alpha_k^3 + 3\sum_{h=1}^{n-1}\beta_h^2(\alpha_h + \alpha_{h+1})\right\}$$

$$\mu_4^{(n)} = \frac{1}{n}\left\{\sum_{k=1}^{n}\alpha_k^4 + 4\sum_{h=1}^{n-1}\beta_h^2\left(\alpha_h^2 + \alpha_h\alpha_{h+1} + \alpha_{h+1}^2 + \frac{1}{2}\beta_h^2\right) + \right.$$
$$\left. + 4\sum_{h=1}^{n-2}\beta_h^2\beta_{h+1}^2\right\}.$$

The purpose of the Dehesa's study is to obtain informations about the asymptotic density of zeros directly from the coefficients of the three-term recurrence relation. However, formula (1.6) is highly not linear with respect to the coefficients $\{\alpha_n, \beta_{n-1}\}$, so that it is not practical to find the asymptotic density of zeros, since it is necessary again to perform the following two steps:

– to evaluate the limit (1.5);

– to solve the corresponding moment problem.

J.S. Dehesa proposed to us to represent the moments $\mu_r^{(n)}$ in terms of some particular system of number theory polynomials, in order to use relative properties to investigate the limit (1.5).

In this paper a representation formula for the moments $\mu_r^{(n)}$ is given in terms of Lucas polynomials of the first kind or, alternatively, in terms of Bell polynomials, by means of some preceding papers of M. Bruschi - P.E. Ricci [1] and A. Di Cave - P.E. Ricci [4].

2. The characteristic polynomial of the Jacobi matrix.

For any integer n, consider the three-diagonal symmetrix Jacobi matrix:

$$J_n := \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & \dots & \beta_{n-1} & \alpha_n \end{vmatrix} \tag{2.1}$$

so that the characteristic polynomial

$$P_n(x) := (-1)^n \det(J_n - xI) = \tag{2.2}$$

$$= \left(x^n - u_{1,n}x^{n-1} + u_{2,n}x^{n-2} + \cdots + (-1)^n u_{n,n}\right) =$$

$$= \left(x^n - u_1 x^{n-1} + u_2 x^{n-2} + \cdots + (-1)^n u_n\right)$$

satisfies for any $n \geq 1$ the recurrence relation (1.1).

Set, for any $s : 1 \leq s \leq n$, and integers $k_1, k_2, \cdots, k_s : 1 \leq k_1 < k_2 < \cdots < k_s \leq n$:

$$J_{k_1,\dots,k_s} := \tag{2.3}$$

$$= \begin{vmatrix} \alpha_{k_1} & \beta_{k_1}\delta_{k_1,k_2-1} & 0 & 0 & \dots & 0 & 0 \\ \beta_{k_2-1}\delta_{k_1,k_2-1} & \alpha_{k_2} & \beta_{k_2}\delta_{k_2,k_3-1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \beta_{k_s-1}\delta_{k_{s-1},k_s-1} & \alpha_{k_s} \end{vmatrix}$$

where $\delta_{h,k}$ denotes the Kronecker delta, and $J_{1,2,\dots,n} = J_n$.

The coefficients $u_s := u_{s,n}$ $(1 \leq s \leq n)$ are given by the following formulae:

$$\begin{aligned} u_1 &= \operatorname{tr} J_n \\ u_2 &= \sum_{k_1<k_2} \det J_{k_1,k_2} \\ & \dots\dots\dots\dots \\ u_s &= \sum_{k_1<k_2<\cdots<k_s} \det J_{k_1,k_2,\dots,k_s} \\ & \dots\dots\dots\dots \\ u_n &= \det J_n\,. \end{aligned} \tag{2.4}$$

It is easy to see that all the preceding determinants can be computed by recurrence relations. We have, in fact:

$$\det J_{k_1,\dots,k_s} = \alpha_{k_s} \det J_{k_1,\dots,k_{s-1}} - \beta_{k_s-1}\beta_{k_{s-1}}\delta_{k_{s-1},k_s-1} \det J_{k_1,\dots,k_{s-2}}, \tag{2.5}$$

and in particular:

$$u_n = \det J_n = \alpha_n \det J_{n-1} - \beta_{n-1}^2 \det J_{n-2}. \tag{2.6}$$

By (2.5) and induction method the following expression for the coefficients of the characteristic polynomial easily follows:

$$(2.7)_1 \qquad u_1 = \sum_{k=1}^{n} \alpha_k$$

$$(2.7)_2 \qquad u_2 = \sum_{k_1<k_2}^{1,n} \alpha_{k_1}\alpha_{k_2} - \sum_{h=1}^{n-1} \beta_h^2$$

$$(2.7)_3 \qquad u_3 = \sum_{k_1<k_2<k_3}^{1,n} \alpha_{k_1}\alpha_{k_2}\alpha_{k_3} - \sum_{h=1}^{n-1} \beta_h^2 \sum_{k=1}^{n}{}^{(h,h+1)} \alpha_k$$

where the symbol $\sum_{k=1}^{n}{}^{(h,h+1)}$ denotes that the sum runs over all indexes k different from h and $h+1$.

With similar notations we have:

$$(2.7)_4 \qquad u_4 = \sum_{k_1<k_2<k_3<k_4}^{1,n} \alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\alpha_{k_4} - \sum_{h=1}^{n-1} \beta_h^2 \sum_{k_1<k_2}^{1,n}{}^{(h,h+1)} \alpha_{k_1}\alpha_{k_2} + \\ + \sum_{\substack{h_1<h_2 \\ h_2-h_1\geq 2}}^{1,n-1} \beta_{h_1}^2 \beta_{h_2}^2$$

and in general, putting:

$$\sigma := \left[\frac{s}{2}\right] = \begin{cases} s/2 & \text{if } s \text{ is even} \\ (s-1)/2 & \text{if } s \text{ is odd} \end{cases}$$

we can write:

$$(2.7)_s \qquad u_s = \sum_{k_1<k_2<\ldots<k_s}^{1,n} \alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_s} + \\ - \sum_{h=1}^{n-1} \beta_h^2 \sum_{k_1<k_2<\ldots<k_{s-2}}^{1,n}{}^{(h,h+1)} \alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_{s-2}} + \\ + \sum_{h_1<h_2}^{1,n-1} \beta_{h_1}^2 \beta_{h_2}^2 \sum_{k_1<k_2<\ldots<k_{s-4}}^{1,n}{}^{(h_1,h_1+1;h_2,h_2+1)} \alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_{s-4}} +$$

$$+\begin{cases} (-1)^\sigma \sum\limits_{\substack{h_1<\ldots<h_\sigma \\ h_i-h_j\geq 2}}^{1,n-1} \beta_{h_1}^2 \cdots \beta_{h_\sigma}^2 \sum\limits_{k=1}^{n} {}^{(h_1,h_1+1;\ldots;h_\sigma,h_\sigma+1)}\alpha_k, & (s \text{ odd}) \\ (-1)^\sigma \sum\limits_{\substack{h_1<\ldots<h_\sigma \\ h_i-h_j\geq 2}}^{1,n-1} \beta_{h_1}^2 \cdots \beta_{h_\sigma}^2, & (s \text{ even}) \end{cases}$$

Note that the first term in the sum for obtaining u_s represents the elementary symmetric function of order s relative to the numbers $\alpha_1, \ldots, \alpha_n$ (sum of products of s elements chosen in any way among these numbers); the second term, with negative sign, represents the sum of products of each β_h^2 times the elementary symmetric function of order $s-2$ of the numbers $\alpha_1, \ldots, \alpha_{h-1}, \alpha_{h+2}, \ldots, \alpha_n$ (these are the elements that belong to the matrix obtained from J_n erasing the rows and columns containing β_h); the third term, with positive sign, represents the sum of products of each $\beta_{h_1}^2 \beta_{h_2}^2$ times the elementary symmetric function of order $s-4$ of the numbers $\alpha_1, \ldots, \alpha_{h_1-1}, \alpha_{h_1+2}, \ldots, \alpha_{h_2-1}, \alpha_{h_2+2}, \ldots, \alpha_n$ (belonging to the matrix obtained from J_n erasing the rows and columns containing β_{h_1} and β_{h_2}); and so on.

By the means of the preceding formulae $(2.7)_s$ $(s = 1, \ldots, n)$, all the coefficients of the characteristic polynomial $P_n(x)$ can be theoretically constructed from the given coefficients of the recurrence relation $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}$.

A representation formula for the moments of the density of zeros of the O.P.S. in terms of Lucas polynomials of the first kind in several variables can be now easily deduced.

3. A representation formula by means of Lucas polynomials of the first kind.

We recall the definition of Lucas polynomials of the first kind in several variables $u_1, \ldots, u_n$.

For any integer n, set:

$$(3.1) \qquad \begin{cases} \Psi_{n-1}(u_1, \ldots, u_n) = u_1 = \sum\limits_{k=1}^{n} x_{k,n} \\ \Psi_n(u_1, \ldots, u_n) = u_1^2 - 2u_2 = \sum\limits_{k=1}^{n} x_{k,n}^2 \\ \Psi_{n+1}(u_1, \ldots, u_n) = u_1^3 - 3u_1u_2 + 3u_3 = \sum\limits_{k=1}^{n} x_{k,n}^3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Psi_{2n-2}(u_1, \ldots, u_n) = u_1\Psi_{2n-3} - u_2\Psi_{2n-4} + \ldots + \\ \qquad + (-1)^{n-2}u_{n-1}\Psi_{n-1} + (-1)^{n-1} n\, u_n = \sum\limits_{k=1}^{n} x_{k,n}^n \end{cases}$$

and, for $N > 2n-2$:

$$\Psi_N(u_1,\dots,u_n) = u_1\Psi_{N-1} - u_2\Psi_{N-2} + \dots + (-1)^{n-1}u_n\Psi_{N-n} = \tag{3.2}$$

$$= \sum_{k=1}^{n} x_{k,n}^{N-n+2}.$$

The choice of indexes is justified in [1], in order to find, in case $n = 2$, the classical Lucas polynomials of the first kind in two variables (see Lucas [6]).

From preceding considerations the following theorem follows:

Theorem 2. *For any integers $n \geq 1$ and $r \in \mathbb{N}_0$, the following representation formula for the moments of the density of zeros of orthogonal polynomials is true:*

$$\mu_r^{(n)} := \frac{1}{n}\sum_{k=1}^{n} x_{k,n}^r = \frac{1}{n}\Psi_{r+n-2}(u_1,\dots,u_n) \tag{3.3}$$

where the variables $u_1,\dots,u_n$ are given by the preceding formulae $(2.7)_s$ $(s = 1,\dots,n)$.

Remark I. If all $x_{k,n}$ does not vanish, the representation formula (3.3) is still true with $r \in \mathbb{Z}$, since the Lucas polynomials of the first kind in several variables are well defined.

Remark II. Observe that, by (3.2), only the first n moments are linearly independent, since the corresponding problem is an algebraic moment problem. All subsequent moments can be computed by formula (3.2).

4. A representation formula by means of Bell polynomials.

In [4] the following representation formula for Lucas polynomials of the first kind in several variables is shown:

Theorem 3. *For any integer $r \in \mathbb{N}$, the following representation formula is true:*

$$\begin{cases} \Psi_{r+n-2}(u_1,\dots,u_n) = \dfrac{1}{(r-1)!}Y_r(f_1,g_1;\dots;f_r,g_r) \\ f_h = (h-1)! \\ g_h = (-1)^{h-1}h!u_h & (h = 1,2,\dots,n) \\ g_{n+\ell} = 0 & (\ell \geq 1) \end{cases} \tag{4.1}$$

From Theorem 3 the following representation formula follows:

Theorem 4. *For any integers n and r, the moments of the density of zeros of orthogonal polynomials are represented in the following way by Bell polynomials:*

$$\left\{\begin{array}{lr} \mu_r^{(n)} = \dfrac{1}{n(r-1)!} Y_r(f_1, g_1; \ldots; f_r, g_r) & \\ f_h = (h-1)! & \\ g_h = (-1)^{h-1} h! u_h & (h = 1, 2, \ldots, n) \\ g_{n+\ell} = 0 & (\ell \geq 1) \end{array}\right. \tag{4.2}$$

with the same meaning $(2.7)_s$ $(s = 1, \ldots, n)$ *for the* u_s.

Recalling Faa' Di Bruno's formula (see e.g. Riordan [7]), formula (4.2) can be written in the following way:

$$\mu_r^{(n)} = \frac{1}{n(r-1)!} \sum_{\pi(r)} \frac{r! f_k}{k_1! k_2! \ldots k_r!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_r}{r!}\right)^{k_r} =$$

$$= \frac{r}{n} \sum_{\pi(r)} \frac{(k-1)!}{k_1! k_2! \cdots k_r!} (u_1)^{k_1} (-u_2)^{k_2} \cdots \left((-1)^{r-1} u_r\right)^{k_r},$$

where $u_{n+\ell} = 0,\ \forall \ell \geq 1,\ k = k_1 + k_2 + \ldots + k_r$, and the sum $\sum\limits_{\pi(r)}$ is extended to all *partitions* of the integer r, that is to all non-negative integral solutions of the equation: $k_1 + 2k_2 + \ldots + rk_r = r$.

5. An integral representation formula.

In [1] a classical formula of the complex variable is interpreted in order to obtain integral representation formulas for the Lucas polynomials of the first kind.

The following theorem is valid:

Theorem 5. *Let* γ *be the boundary of a circular domain, with center at the origin, containing all the zeros* $x_{k,n}$ *of the polynomial* $P_n(x)$. *Let* $N > n - 2$ *be an integer. Then the following integral representation formulas are true:*

$$\Psi_N(u_1, \ldots, u_n) = \frac{1}{2\pi i} \oint_\gamma \frac{\lambda^{N-n+2} P_n'(\lambda)}{P_n(\lambda)} d\lambda = \tag{5.1}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{n\lambda^{N+1} - (n-1)u_1\lambda^N + \ldots + (-1)^{n-1} u_{n-1} \lambda^{N-n+2}}{P_n(\lambda)} d\lambda\,;$$

$$\Psi_N(u_1, \ldots, u_n) = \tag{5.2}$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{u_1\lambda^N - 2u_2\lambda^{N-1} + \ldots + (-1)^{n-1} n u_n \lambda^{N-n+1}}{P_n(\lambda)} d\lambda .$$

Putting $N = r + n - 2$ the integral representation formulas for the moments of the density of zeros of O.P.S. follow:

$$\mu_r^{(n)} = \tag{5.3}$$

$$= \frac{1}{2n\pi i} \oint_\gamma \frac{n\lambda^{r+n-1} - (n-1)u_1\lambda^{r+n-2} + \ldots + (-1)^{n-1} u_{n-1}\lambda^r}{P_n(\lambda)} d\lambda$$

$$\mu_r^{(n)} = \tag{5.4}$$

$$= \frac{1}{2n\pi i} \oint_\gamma \frac{u_1\lambda^{r+n-2} - 2u_2\gamma^{r+n-3} + \ldots + (-1)^{n-1} n u_n \lambda^{r-1}}{P_n(\lambda)} d\lambda .$$

Formulae (5.3)-(5.4) could be used in computing the limit (1.5) provided that a circular domain containing the support $[a, b]$ of the measure $d\phi(x)$ would be known.

REFERENCES

[1] M. Bruschi - P.E. Ricci, *I polinomi di Lucas e di Tchebycheff in più variabili,* Rend. Mat. (VI), 13 (1980).

[2] J.S. Dehesa, *On the conditions for a Hamiltonian matrix to have an eigenvalue density with some prescribed characteristics,* J. Comp. Appl. Math., 2 n. 4 (1976).

[3] J.S. Dehesa - F.J. Gálvez, *Global properties of zeros of orthogonal polynomials,* in: Marcellan (Ed.) Proc. Third Spanish Symposium on Orthogonal Polynomials and its Appl., Segovia, 1985.

[4] A. Di Cave - P.E. Ricci, *Sui polinomi di Bell ed i numeri di Fibonacci e di Bernoulli,* Le Matematiche, 35 (1980).

[5] J. Favard, *Sur les polynomes de Tchebicheff,* C.R. Acad. Sc. Paris, 200 (1935).

[6] E. Lucas, *Théorie des Nombres,* Gauthier Villars, Paris, 1891.

[7] J. Riordan, *An Introduction to Combinatorial Analysis,* J. Wiley & Sons, New York, 1958.

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