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ON THE NUMBER OF FINITE TOPOLOGICAL SPACES

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In this paper we deal with the problem of enumerating the finite topological spaces, studying the enumeration of a restrictive class of them. By employing simple techniques, we obtain a recursive lower bound for the number of topological spaces on a set of n elements. Besides we prove some collaterals results, among which we can bring a new proof (Cor. 1.5) of the fact that $p(n)$ – the number of partitions of the integer n – is the number of non-isomorphic Boolean algebras on a set of n elements.

Introduction.

Throughout this paper, we work with topological spaces on a set of n elements ($n \in \mathbb{N}$). In this sense, we denote $< X_n, \tau >$ a topological space, where X_n is a set such that $|X_n| = n$ and τ is a topology on X_n. The finite topological spaces (F.T.S.) $< X_n, \tau_1 >$, $< X_n, \tau_2 >$ are homeomorphic when there exist a continuous and open preserving 1-1 map from $< X_n, \tau_1 >$ onto $< X_n, \tau_2 >$. We denote this fact $\tau_1 \approx \tau_2$ or, if s design the above map, $\tau_1^s = \tau_2$.

We denote $\mathscr{T}(n)$ the set of F.T.S. on X_n, more generally we write with $\prod(n)$ the set of F.T.S. on X_n which satisfies a topological property π. For example, $\mathscr{C}(n)$ denotes the class of connected F.T.S. on X_n.

If $< X, \tau >$, $< Y, \sigma >$ are two topological spaces with $X \cap Y = \emptyset$, we call sum of them (cf. [1], [5]) to the space $< Z, \rho >$, where $Z = X \cup Y$ and $\rho = \{U \cup V : U \in \tau, V \in \sigma\}$. We also say that ρ is the *sum topology* of τ and

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σ, an we write $\rho = \tau \oplus \sigma$. For instance, if $\tau_d(m)$ is the discrete topology on $X_m (m \in \mathbb{N})$, we have $\tau_d(m) = \bigoplus_{i=1}^{m} \tau_d(1)$.

With $\tau_i(m)$ we indicate the trivial topology on $X_m (m \in \mathbb{N})$, and if it is unnecessary to specify the number m of points, we write simply τ_i (τ_d for the discrete topology).

A property π of a topological space $< X, \tau >$ is said hereditary (weakly hereditary) if all its subspaces (closed subspaces) have the property π too (cf. [1], [5]). The property π is said additive if the sum of two spaces which satisfy π also satisfies π.

If τ is a topology on X_n, we denote τ^c the topology whose open sets are the closed sets of $< X, \tau >$.

With $t(n)$ we symbolize the number of F.T.S. on X_n; i.e., $t(n) = |\mathscr{I}(n)|$. Furthermore, if π is a topological property, we put $\pi(n) = |\prod(n)|$ (for instance, $c(n) = |\mathscr{C}(n)|$).

The problem of enumerating $\mathscr{I}(n)$ and the related problem of enumerating all the $T(n)$ different topologies on X_n are rather complex (cf. [2], [4]). We only known the exact values of $t(n)$ for $n \leq 5$, and $T(n)$ for $n \leq 7$ (cf. [2], [6], [7], [9]). For some successful attempts of counting particular classes of topologies, the reader is referred to [6], [8] and [11]. In reference [10], the problem of the cardinality of a topology τ on X_n is considered. In this paper we reduce the problem of computing $|\mathscr{I}(n)|$ to the enumeration of a certain subset $H_0(n)$ of $\mathscr{C}(n)$. By ignoring in the calculations the topologies in the class $H_0(n)$, a recursive lower bound for $t(n)$ is derived. In passing, we will find a new way to count the Boolean algebras on a set of n elements.

1. Topologies which satisfy a weakly hereditary and additive property.

The central result of this section gives the number $\pi(n)$ of topologies which satisfy a weakly hereditary additive property Π as a function of the number $\pi_c(n)$ of connected topologies which satisfy Π. For this purpose, it will be useful to consider the set $\mathscr{P}_n$ of partitions of the integer n, i.e.,

$$\mathscr{P}_n = \{[n_1, n_2 \ldots, n_k] : n_1 \geq n_2 \geq \cdots \geq n_k, \quad n_k \in \mathbb{N}, \\ (1 \leq k \leq n), \quad n_1 + n_2 + \cdots + n_k = n\}.$$

For the sake of convenience, an alternative way to denote a partition

$$[n_1, n_2 \ldots, n_k] \in \mathscr{P}_n$$

is to write it as an n-tuple $p = (\rho_1, \rho_2 \dots, \rho_n)$, where ρ_k indicates the number of times that the part k appears in p.
For example, for the partition $p = [4, 2, 2, 1, 1, 1]$ of 11 we have

$$\rho_1[p] = 3\ ,\ \rho_2[p] = 2\ ,\ \rho_3[p] = 0\ ,\ \rho_4[p] = 1,$$

and

$$\rho_k[p] = 0 \quad \text{for any} \quad 5 \leq k \leq 11,$$

so that we can also write p in the form

$$p = (3, 2, 0, 1, 0, 0, 0, 0, 0, 0, 0).$$

We base our analysis on the following:

Lemma 1.1. *Let $\tau_1, \tau_2, \dots, \tau_k;\ \tau_1', \tau_2', \dots, \tau_j'$ be connected topologies, and $\tau = \bigoplus_{i=1}^{k} \tau_i$, $\tau' = \bigoplus_{i=1}^{j} \tau_i'$. Then $\tau \approx \tau'$ if and only if $k = j$ and, except a permutation of indexes, $\tau_i \approx \tau_i'$ holds for all $i = 1, 2, \dots, k$.*

Proof. Since $\tau_1, \tau_2, \dots, \tau_k$ are connected each one of them is the topology induced for τ on the respective component X_{n_i}. Thus, k is the number of components of τ.

Analogously, j is the number of components of τ' and, because this number is a topological invariant, we obtain $k = j$. Moreover, for each $i = 1, 2 \dots, k$ we have

$$\tau_i = \tau\ |_{X_{n_i}} \approx \tau'\ |_{X_{n_i}} = \tau_j' \quad \text{for any} \quad 1 \leq j \leq k.$$

The converse is immediate. $\square$

Let π be a weakly hereditary and additive property. If Π_c denote the class of connected topologies which satisfy π, and $p = [n_1, n_2, \dots, n_k]$ is a partition of $n \in \mathbb{N}$ then, we obtain the following result.

Lemma 1.2. *The number $\pi[n_1, n_2, \dots, n_k]$ of topologies that verify π and have components $X_{n_1}, X_{n_2}, \dots, X_{n_k}$ is*

$$\pi[n_1, n_2, \dots, n_k] = \binom{\pi_c(1) + \rho_1[p] - 1}{\pi_c(1) - 1}\binom{\pi_c(2) + \rho_2[p] - 1}{\pi_c(2) - 1}\cdots$$
$$\cdots\binom{\pi_c(n) + \rho_n[p] - 1}{\pi_c(n) - 1},$$

where $\pi_c(k)$ indicates the number of connected topologies on X_k which satisfy Π.

Proof. Define the sets

$$\Pi[n_1, n_2, \ldots, n_k] = \{\tau \in \Pi(n) : \tau \text{ has components } X_{n_1}, X_{n_2}, \ldots, X_{n_k}\},$$

and

$$H = \{(\tau_1, \tau_2, \ldots, \tau_k) : \tau_i \in \mathscr{C}(n_i) \cap \Pi(n_i),\ 1 \leq i \leq k\}.$$

In view of the hypothesis we can show that the maps

$$|| : \Pi[n_1, n_2 \ldots, n_k] \mapsto H$$
$$\tau \to (\tau\, |_{X_{n_1}}, \tau\, |_{X_{n_2}}, \ldots, \tau\, |_{X_{n_k}}),$$

and

$$\oplus : H \to \Pi[n_1, n_2, \ldots, n_k]$$
$$\tau \mapsto \bigoplus_{i=1}^{k} \tau_i,$$

are well-defined. In fact, since π is a weakly hereditary property and the components are closed subspaces, $\pi\, |_{X_{n_i}}$ satisfies τ for all $i = 1, 2, \ldots, k$; i.e., "$||$" is well-defined. Furthermore, π is an additive property, which means that if $(\tau_1, \tau_2, \ldots, \tau_k) \in H$ then $\bigoplus_{i=1}^{k} \tau_i \in \Pi[n_1, n_2, \ldots, n_k]$. This proves the well-definition of $\oplus$.

Since it verifies

$$\oplus \circ || = 1_{\Pi[n_1, n_2, \ldots, n_k]}, \quad || \circ \oplus = 1_H,$$

we conclude

$$\pi[n_1, n_2, \ldots, n_k] = |\Pi[n_1, n_2, \ldots, n_k]| = |H|,$$

and from Lemma 1.1 we obtain

$$|H| = \binom{\pi_c(1) + \rho_1[p] - 1}{\pi_c(1) - 1}\binom{\pi_c(2) + \rho_2(p) - 1}{\pi_c(2) - 1} \cdots$$
$$\cdots \binom{\pi_c(n) + \rho_n[p] - 1}{\pi_c(n) - 1}. \qquad \square$$

Theorem 1.3. *Under the same assumptions of Lemma 1.2, the number $\pi(n)$ of topologies which satisfy Π can be expressed in the form*

$$\pi(n) = \sum_{p \in \mathscr{P}_n} \binom{\pi_c(1) + \rho_1[p] - 1}{\pi_c(1) - 1}\binom{\pi_c(2) + \rho_2[p] - 1}{\pi_c(2) - 1} \cdots$$
$$\cdots \binom{\pi_c(n) + \rho_n[p] - 1}{\pi_c(n) - 1}.$$

Proof. Obviously, we have $\pi(n) = \sum_{p \in \mathscr{P}_n} \pi_c[p]$. Lemma 1.2 applies. □

Corollary 1.4. *For each* $n \in \mathbb{N}$,

$$t(n) = \sum_{p \in \mathscr{P}_n} \binom{c(1) + \rho_1[p] - 1}{c(1) - 1} \binom{c(2) + \rho_2[p] - 1}{c(2) - 1} \cdots \binom{c(n) + \rho_n[p] - 1}{c(n) - 1}$$

holds.

Examples of application (separation properties).

It is well known that the separation properties T_0, $\mathscr{R}$ (regularity) and $\mathscr{N}$ (normality) are weakly hereditary properties of the topological spaces. Furthermore, T_0 and $\mathscr{R}$ are hereditary, (cf. [5]). One can easily verify that this properties are also additive, and therefore Theorem 1.3 holds.

Particularly interesting is the application of Theorem 1.3 to $\mathscr{R}(n)$, because it enables us to directly enumerate this class. In fact, if $\tau \in \mathscr{R}(n) \cap \mathscr{C}(n)$ it must be $\tau = \tau_i(n)$; thus, for each $n \in \mathbb{N}$, $r_c(n) = 1$ and

$$r(n) = \sum_{p \in \mathscr{P}_n} \binom{1 + \rho_1[p] - 1}{1 - 1} \binom{1 + \rho_2[p] - 1}{1 - 1} \cdots \binom{1 + \rho_n[p] - 1}{1 - 1} = \sum_{p \in \mathscr{P}_n} 1 = p(n),$$

where $p(n)$ is the partition function of number theory.

In [3] it is shown that $\mathscr{R}(n)$ is isomorphic with the class of (non-isomorphic to each other) Boolean algebras on X_n, so that we find the following known result:

Corollary 1.5. *There exist* $p(n)$ *non-isomorphic to each other Boolean algebras of* n *elements.*

2. Connected topologies.

As a consequence of Corollary 1.4 we deduce that once known, for $1 \le k \le n$, $c(k)$ – the number of connected topologies on X_k – it is possible to compute the number $t(n)$ of topologies on X_n. By means of the following result, we advance in this way to determinate $t(n)$.

Lemma 2.1. *For each* $n \ge 1$ *is*

$$c(n) = 1 + t(1) + \cdots + t(n-1) + u(n),$$

where

$$u(n) = |\, \{\tau \in \mathscr{C}(n) : \cup\{V \in \tau : V \neq X_n\} = X_n\} \,|.$$

Proof. Let $n \ge 1$ and, for each $\tau \in \mathscr{C}(n)$, $X_\tau = \cup\{V \in \tau : V \neq X_n\}$ be; we define in $\mathscr{C}(n)$ the following relation:

$$\tau_1, \tau_2 \in \mathscr{C}(n), \ \ \tau_1 \sim \tau_2 \quad \text{if and only if} \quad |X_{\tau_1}| = |X_{\tau_2}|.$$

It is easy to prove that "$\sim$" is an equivalence relation which splits $\mathscr{C}(n)$ in $(n+1)$ classes $H(k) = \{\tau \in \mathscr{C}(n) : |X_\tau| = k\}$ $(k = 0, 1, \ldots, n)$. Next we will enumerate these classes. Obviously we have

$$H(0) = \{\tau \in \mathscr{C}(n) : |X_\tau| = 0\} = \{\tau_i(n)\},$$

so that $|H(0)| = 1$. We affirm that, for each $1 \le k \le n-1$, the maps

$$\alpha_k : H(k) \to \mathscr{I}(k)$$
$$\tau \mapsto \alpha_k(\tau) = \tau - \{X_n\},$$

are bijective.

In fact, let $1 \le k \le n-1$ be fixed and $\tau \in \mathscr{I}(k)$. By defining $\tau' = \tau \cup \{X_n\}$ we have that $\tau' \in H(k)$ and $\alpha_k(\tau') = \tau$; therefore α_k is surjective.

To prove the injectivity we chose $\tau_1, \tau_2 \in H(k)$ such that $\alpha_k(\tau_1) \approx \alpha_k(\tau_2)$; i.e., there exist a bijection $s : X_{\tau_1} \to X_{\tau_2}$ such that $(\alpha_k(\tau_1))^s = \alpha_k(\tau_2)$. There obviously exist a bijective map $\tilde{s} : X_n \to X_n$ such that $\tilde{s}\,|_{X_{\tau_1}} = s$; we will prove that $\tau_1^{\tilde{s}} = \tau_2$. In fact, if $V \in \tau_1$ there are two possibilities:

$$V \subseteq X_{\tau_1} \quad \text{or} \quad V = X_n.$$

In the first case $\tilde{s}(V) = (\tilde{s}\,|_{X_{\tau_1}})(V) = s(V) \in \tau_2$ (because $(\alpha_k(\tau_1))^s = \alpha_k(\tau_2)$), while if $V = X_n$ we have $\tilde{s}(V) = \tilde{s}(X_n) = X_n \in \tau_2$. Thus, $\tau_1^{\tilde{s}} \subseteq \tau_2$. Analogously we can prove that $\tau_1^{\tilde{s}} \supseteq \tau_2$, and therefore $\tau_1 \approx \tau_2$.

To complete the proof it is enough to see that

$$u(n) = |\Pi(n)| = |\{\tau \in \mathscr{C}(n) : X_\tau = X_n\}|. \qquad \square$$

Note that $H(n)$ is the class of connected topological spaces on X_n such that there exist a nontrivial open covering of X_n. Until now we can not compute the last term $u(n) = |H(n)|$ in the expression of $c(n)$ in Lemma 2.1. However, we can simplify this problem by splitting $H(n)$ by means of the "dual" equivalence relation "$\overset{d}{\sim}$" of "$\sim$"; that is

$$\tau_1, \tau_2 \in H(n), \ \tau_1 \overset{d}{\sim} \tau_2 \quad \text{if and only if} \quad |Y_{\tau_1}| = |Y_{\tau_2}|,$$

where, for each $\tau \in H(n)$, we define $Y_\tau = \cap\{V \in \tau : V \neq \emptyset\}$.

In this manner, if we denote with $H_k(n)$ the class

$$\{\tau \in H(n) : |Y_\tau| = k\} \quad (k = 0, 1, \ldots, n),$$

we get the following result.

Lemma 2.2. *It verifies* $u(1) = u(2) = 0$ *and, for* $n \geq 2$,

$$u(n) = \sum_{k=0}^{n-2} u_k(n);$$

where we have written $u_k(n) = |H_k(n)|$.

Proof. By examining the topologies of $\mathscr{I}(1)$ and $\mathscr{I}(2)$ we can prove that $u(1) = u(2) = 0$. If $n \geq 2$ we only must complete the above discussion by showing that $u_{n-1}(n) = u_n(n) = 0$ or, equivalently, that $H_{n-1}(n) = H_n(n) = \emptyset$. To this end, we realize that if $\tau \in H_n(n)$ it is $Y_\tau = X_n$, and therefore it is $V = X_n$ for each open set $V \in \tau$, $V \neq \emptyset$. Then, $X_\tau = \cup\{V \in \tau : V \neq X_n\} = \emptyset$ and this is a contradiction because $X_\tau = X_n \neq \emptyset$. Now, if we suppose that there exists $\tau \in H_{n-1}(n)$, Y_τ would be the unique atomic element of the lattice $< \tau, \subseteq >$, and $|Y_\tau| = n - 1$. Since $\tau \in \mathscr{C}(n)$ it must necessarily be $\tau = \{\emptyset, Y_\tau, X_n\}$; then $X_\tau = Y_\tau \neq X_n$, which is a contradiction again. $\square$

Next lemma shows that the computations of $u_k(n)$ $(k \geq 2)$ can be reduced to computing $u_1(m)$ with an appropriate m.

Lemma 2.3. *If* $n \geq 3$ *and* $k \geq 2$, *then* $u_k(n) = u_1(n - k + 1)$.

Proof. Assume that $n \geq 3,\ k \geq 2$. We define the map

$$\beta : H_k(n) \to H_1(n-k+1)$$
$$\tau \mapsto \beta(\tau) = \{V - Y'_\tau : V \in \tau\},$$

where $Y'_\tau = Y_\tau - \{x\}$ with $x \in Y_\tau$ fixed. A similar argument to the one used in the proof of Lemma 2.1 shows that β is bijective. □

Remark 2.4. From Lemma 2.3 we derive that $u_n(n) = u_1(1) = 0,\ u_{n-1}(n) = u_1(2) = 0$ as we have directly proved in Lemma 2.2. We can also obtain that $u_{n-2}(n) = u_1(3) = 1$ and $u_{n-3}(n) = u_1(4) = 4$ (see the diagrams of topologies of $\mathscr{I}(3)$ and $\mathscr{I}(4)$ in [3]).

At this point, the problem of calculating $t(n)$ has been reduced to obtain $u_0(k), u_1(k)$ for $1 \leq k \leq n$. As regards $u_1(k)$ we have the following result.

Lemma 2.5. $u_1(1) = u_1(2) = 0$ *and, for* $n \geq 3$,

$$u_1(n) = t(n-1) - [1 + t(1) + \cdots + t(n-2)],$$

holds.

Proof. In view of the remark above is $u_1(1) = u_1(2) = 0$. For $n \geq 3$ we will consider two cases depending on whether the lattice $< \tau - \{\emptyset\}, \subseteq>$ has one or more than one atomic elements.

1°) $< \tau - \{\emptyset\}; \subseteq>$ has a unique atomic element which we denote A_τ.

This case can only occur when $n \geq 4$ (see the diagram below). According to the equivalence relation

$$\tau_1 \equiv \tau_2 \quad \text{if and only if} \quad |A_{\tau_1}| = |A_{\tau_2}|,$$

we split the class

$$A = \{\tau \in H_1(n) :< \tau - \{\emptyset\}, \subseteq> \quad \text{admits a unique atom}\}$$

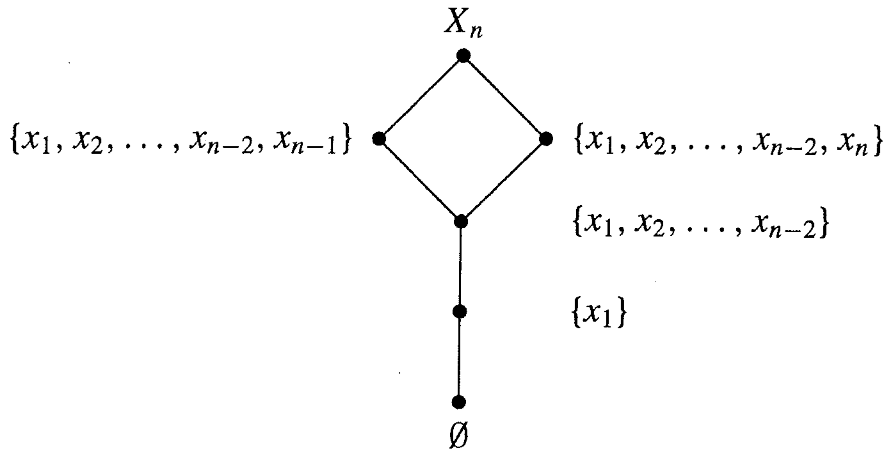
in the classes $A_k = \{\tau \in A : |A_\tau| = k\}$, $(2 \leq k \leq n-2)$. Next we put A_k in bijective correspondence with $H_{k-1}(n-1)$ through the map

$$\gamma_1 : A_k \to H_{k-1}(n-1)$$
$$\tau \mapsto \gamma_1(\tau) = \{V - Y_\tau : V \in \tau - \{\emptyset\}\}.$$

Thus, the number of topologies in $H_1(n)$ such that $< \tau - \{\emptyset\}, \subseteq>$ has a unique atom is

$$(2.1) \qquad \sum_{k=2}^{n-2} |A_k| = \sum_{k=2}^{n-2} |H_{k-1}(n-1)| =$$

$$= u_1(n-1) + u_2(n-1) + \cdots + u_{n-3}(n-1).$$



2°) $< \tau - \{\emptyset\}, \subseteq>$ has more than one atomic element.

Denote $B = \{\tau \in H_1(n) :< \tau - \{\emptyset\}, \subseteq>$ admits more than one atomic element $\}$; if $\tau \in B$, $\gamma_1(\tau)$ is not generally connected but, if $\gamma_1(\tau)$ is connected then $\gamma_1(\tau) \in H_0(n-1)$. In the latter case we can define

$$\gamma_2 : B \to H_0(n-1) \cup [\mathscr{I}(n-1) - \mathscr{C}(n-1)]$$
$$\tau \mapsto \gamma_2(\tau) = \{V - Y_\tau : V \in \tau - \{\emptyset\}\}.$$

We can easily prove that γ_2 is a bijective map; furthermore

$$\gamma_2^{-1} : H_0(n-1) \cup [\mathscr{I}(n-1) - \mathscr{C}(n-1)] \to B$$

$$\tau \mapsto \gamma_2^{-1}(\tau) = \{\emptyset\} \cup \{\{x_n\} \cup V : V \in \tau\},$$

where $\{x_n\} = X_n - X_{n-1}$.

Since $H_0(n-1) \cap [\mathscr{I}(n-1) - \mathscr{C}(n-1)] = \emptyset$ we obtain

$$(2.2) \qquad |B| = H_0(n-1)| + |\mathscr{I}(n-1) - \mathscr{C}(n-1)| =$$

$$= u_0(n-1) + t(n-1) - c(n-1).$$

Finally from (2.1), (2.2) and Lemmas 2.1, 2.3 we obtain, for $n \geq 3$,

$$\begin{aligned} u_1(n) &= [t(n-1) - c(n-1) + u_0(n-1)] + [u_1(n-1) + \cdots + \\ &\qquad\qquad\qquad\qquad + u_{n-3}(n-1)] \\ &= t(n-1) - c(n-1) + u(n-1) \\ &= t(n-1) - [1 + t(1) + \cdots + t(n-2) + u(n-1)] + u(n-1) \\ &= t(n-1) - [1 + t(1) + \cdots + t(n-2)]. \qquad \square \end{aligned}$$

To count the class $H_0(n)$ for each $n \in \mathbb{N}$, i.e., to determinate $u_0(n)$, is still an open problem. We emphasize that

$$H_0(n) = \{\tau \in \mathscr{C}(n) : \cup\{V \in \tau : \ V \neq X_n\} = X_n, \cap\{V \in \tau : V \neq \emptyset\} = \emptyset\};$$

in words, $H_0(n)$ is the class of connected topological spaces on X_n such that there exist a non-trivial open covering $\{V_1, V_2, \ldots, V_m\}$ of X_n with the property $\cap\{V_i : i = 1, 2, \ldots, m\} = \emptyset$.

Next lemma, whose proof is immediate, establishes that $H_0(n)$ is a closed-under-complementation class of connected topologies.

Lemma 2.6. $\tau \in H_0(n)$ *if and only if* $\tau^c \in H_0(n)$.

3. A lower bound for $t(n)$.

The tools employed above fail when we attempt to apply them to the class $H_0(n)$. Table 1 shows the known values of $u_0(n)$. It would not be too much tedious to calculate $u_0(6)$ by hand, which would permit us to know exactly $t(6)$. Instead of this, we will derive a recursive lower bound for $t(n)$.

Because the just proved lemmas, we have

$$t(n) = \sum_{p \in \mathscr{P}_n} \binom{c(1) + \rho_1[p] - 1}{c(1) - 1} \binom{c(2) + \rho_2[p] - 1}{c(2) - 1} \cdots \\ \cdots \binom{c(n) + \rho_n[p] - 1}{c(n) - 1},$$

and

$$\begin{aligned} c(n) &= 1 + t(1) + \cdots + t(n-1) + u(n), \\ u(n) &= u_0(n) + u_1(n) + \cdots + u_{n-2}(n)\ , \ (n \geq 2), \\ u_k(n) &= u_1(n-k+1)\ , \ (n \geq 3,\ k \geq 2), \\ u_1(n) &= t(n-1) - (1 + t(1) + \cdots + t(n-2))\ , \ (n \geq 3), \\ u_1(1) &= u_1(2) = u_0(1) = u_0(2) = u_0(3) = 0. \end{aligned}$$

By assuming $u_0 \equiv 0$ in the previous equations we can recursively generate sequences $\bar{u}_1(n), \bar{u}(n), \bar{c}(n), \bar{t}(n)$, $(n \in \mathbb{N})$, which respectively bounds from below $u_1(n), u(n), c(n), t(n)$. After some simplifications in the previous equations, we obtain

$$\bar{t}(n) = \sum_{p \in \mathscr{P}_n} \binom{\bar{c}(1) + \rho_1[\rho] - 1}{\bar{c}(1) - 1} \binom{\bar{c}(2) + \rho_2[p] - 1}{\bar{c}(2) - 1} \cdots \cdots \binom{\bar{c}(n) + \rho_n[\rho] - 1}{\bar{c}(n) - 1},$$

$$\begin{aligned} \bar{c}(n) &= 1 + \bar{t}(1) + \cdots + \bar{t}(n-1) + \bar{u}(n), \\ \bar{u}(n) &= \bar{u}_1(n) + \bar{u}_1(n-1) + \cdots + \bar{u}_1(3), \\ \bar{u}_1(n) &= \bar{t}(n-1) - (1 + \bar{t}(1) + \cdots + \bar{t}(n-2)), \\ \bar{u}_1(1) &= \bar{u}_1(2) = 0, \end{aligned}$$

and as we said, for each $n \in \mathbb{N}$, it verifies

$$\bar{u}_1(n) \leq u_1(n)\ ,\ \bar{u}(n) \leq u(n)\ ,\ \bar{c}(n) \leq c(n)\ ,\ \bar{t}(n) \leq t(n).$$

Table 2 shows the values of $\bar{c}(n)$ and $\bar{t}(n)$ for $n \leq 26$. Obviously, the approximation to the real values $c(n), t(n)$ become worse as n increases.

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n	$u_0(n)$
1	0
2	0
3	0
4	2
5	23

Table 1

n	$\bar{c}(n)$	$\bar{t}(n)$	n	$\bar{c}(n)$	$\bar{t}(n)$
1	1	1	14	6094264	9919325
2	2	3	15	22142446	34762196
3	3	9	16	77957045	126685043
4	19	31	17	282855142	443457430
5	67	104	18	994612970	1620190932
6	230	378	19	3616380372	5669289741
7	837	1307	20	12716154460	20666163712
8	2924	4763	21	46140386749	72436845175
9	10616	16661	22	162439818252	264275295237
10	37341	60823	23	589979304704	924944167448
11	135729	212698	24	2074581240341	3375757685294
12	477044	776600	25	7535866044222	11814289003965
13	1733465	2717089	26	26498455800151	43112847789841

Table 2

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