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ON PROJECTIVE VARIETIES ADMITTING A BIELLIPTIC OR TRIGONAL CURVE-SECTION

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We study smooth projective varieties $X \subseteq \mathbf{P}^N$ of dimension $n \geq 3$, such that for some linear $(N-n+1)$-dimensional space $\Pi \subseteq \mathbf{P}^N$, $C = X \cap \Pi$ is either a bielliptic or a trigonal curve. We give a description of X when $\deg X \geq 18$ and when $\deg X \leq 8$.

Introduction.

$(*)$ Let $X \subseteq \mathbf{P}^N$ be a projective, algebraic, smooth, irreducible variety defined over an algebraically closed field of characteristic zero, having $\dim X = n \geq 3$. Let H be one of its hyperplane sections and C a smooth curve-section of X (i.e. C is the scheme-theoretic intersection of $n-1$ hyperplane sections of X).

The aim of this short note is to describe the varieties X as above which posses a trigonal or bielliptic curve-section C for $\deg X \geq 18$ or $\deg X \leq 8$, extending the results in [4] and [5] about surfaces to varieties of higher dimension.

This kind of problem has been completely solved in [15] when C is hyperelliptic, while the case of C trigonal or tetragonal has been studied in [2] under the additional hypothesis that $h^1(\mathcal{O}_X) = 0$ and $(H + K_X) \cdot K_X \leq -4, -5$, respectively .

Note that a bielliptic curve C cannot be hyperelliptic when $g \geq 4$, and C cannot be trigonal when $g(C) \geq 5$ (e.g. see [4]).

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1. Bielliptic case: $\deg X \geq 18$.

Theorem A. *Let* $(X, H) \subseteq \mathbf{P}^N$ *be as in* $(*)$ *with* $\deg X = H^n \geq 18$, *and suppose that* X *possesses a smooth bielliptic curve-section* C *of genus* $g \geq 3$. *Then either*

A.1) X *is a scroll on* C, *i.e.* X *is a* $\mathbf{P}^{n-1}$ *bundle over* C *and on every fiber* $F : \mathcal{O}_F(H) \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$.

A.2) X *is a* Q*-bundle on an elliptic curve* E, *where every fiber* $Q \subseteq \mathbf{P}^n$ *is a quadric hypersurface.*

A.3) $\deg X = 18$ *and* $-K_X \cong (n-2)H$.

Proof. Let us consider the case $\deg X \geq 19$ first. We will prove the theorem for $n = 3$, and that will be a starting point for an inductive procedure in order to get the result for varieties of any dimension.

Note that we can suppose that there are no (-1)-contractions on X, i.e. that X is not the projection of a variety $Y \subseteq \mathbf{P}^{N+1}$ from one of its points. In fact if (Y, M) is such a contraction of (X, H), we can work on it, and if (Y, M) is as in A.1 or in A.2, also (X, H) is.

Let S be a smooth surface in $|H|$ containing C (S exists by Bertini, since C is smooth); then $\mathcal{O}_X(K_X + H)|_S \cong \mathcal{O}_S(K_S) = \omega_S$, and by[4], Theorem 3.5, S is either a scroll over C or a conic bundle on an elliptic curve E, of which C is a double covering.

It is quite standard (using the adjunction mappings associated to $\mathcal{O}_X(K_X + iH)$, $i = 2, 3$) to extend the bundle structure of S to an analogous structure on X in order to show that that X is of type A.1 or A.2 (see e.g. [6], Prop 1.8 and Remark 1.12,*c*)). Hence the case $n = 3$ is proved.

Now let us suppose $\dim X = n \geq 4$ and let us work by induction on n. If the result is true for varieties of dimension $n-1$, consider a smooth hyperplane section H of X. In both cases A.1 and A.2, H is a bundle over a curve B (either $B = C$ or $B = E$), i.e. there is a morphism $\lambda : H \to B$. Since $\dim H - \dim B = n - 2 \geq 2$, we can apply a result of [14] to show that λ extends to a morphism $\mu : X \to B$, such that $\mu|_H = \lambda$. Then the fibers F of μ have the fibers of λ as hyperplane sections, hence they are either linear, and we are in case A.1, or they are quadrics, as in A.2.

Now let $\deg X = 18$ and let S be as before. Then, again by [4], Theorem 3.5, we have that S is minimal and is either a double plane in $\mathbf{P}^{10}$ (which is also a K3 surface), or the triple embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^{15}$.

In the first case we are in case $\deg X = 2\ (\operatorname{codim} X + 1)$, hence we are in case A.3, see [8] Theorem I, case *h)*; and in the second, using case *b)* of the same theorem, we have that X is a scroll on some surface S'; we will show that this case cannot occur.

We can consider $\dim X = 3$. If $p : X \to S'$ is the scroll structure on X, then let $X = \mathbf{P}(\mathscr{E})$ and we can assume $\mathscr{E} = p * \mathscr{O}_X(H)$.

Then S corresponds to a general section of the tautological line bundle of $\mathscr{E}$ and so it must contain $c_2(\mathscr{E}) > 0$ fibers of p. Hence S cannot be minimal, in contradiction with the fact that S, being the triple embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^{15}$, does not contain any line.

2. Bielliptic case: $\deg X \leq 8$.

Theorem B. *Let (X, H) be as in $(*)$ with $\deg X \leq 8$, and suppose that X possesses a smooth bielliptic curve-section C of genus $g \geq 3$. Then if (X, H) is not as in A.1, A.2, it is one of the following:*

B.1) A quartic hypersurface in $\mathbf{P}^{n+1}$;

B.2) a threefold which is a scroll over $\mathbf{P}^2$, with $H^3 = d = 6, 7$ or 8 and $X \subseteq \mathbf{P}^N$ with $N = 5, 6$ or 7, respectively;

B.3) $(X, H) \subseteq \mathbf{P}^{n+2}$ is the complete intersection of a quadric and a cubic;

B.4) either $(X, H) \subseteq \mathbf{P}^{n+3}$ is the complete intersection of three quadric hypersurfaces or $(X, H) \subseteq \mathbf{P}^{n+2}$ is its projection from one of its points.

Proof. From [4], Theorem 4.1, we have a complete list of the smooth surfaces of degree ≤ 8 which possess a bielliptic curve-section. Such list tells us which are the possibilities for the surface-sections of the variety we want to classify.

The cases B.1, B.2 and B.3 have surface-sections as in cases 1,2 and 3 of Theorem 4.1 of [4], while case B.4 corresponds to case 7 there. Those are the only varieties with such surface-sections as one can check by using the classification of varieties of degree ≤ 8 made in [6] and [7]. Namely, to check that no variety of dimension ≥ 3 can have any surface-section as in 4, 5, 6 of [4],Thm. 4.1, one uses [7], as follows:

case 4 is excluded by [7], § 1, IV, *b* and [6], Table on page 148;
case 5 by [7], § 1, V, c_1;
case 6 by [7], § 1, VI.

Note that in cases B.1, B.3 and B.4 it is well known that such varieties do exist, while in case B.2 only the deg6 threefold is known to exist for certain (see [6], [7], [11]).

We want to observe that other examples of varieties with bielliptic curve sections can be found in degree d, $9 \leq d \leq 17$:

Example 1. Let $F \subseteq \mathbf{P}^{d+1}$ be a Fano threefold of degree d, $3 \leq d \leq 7$, whose generic hyperplane section is a Del Pezzo surface. Let $F \subseteq \mathbf{P}^{d+1} \subseteq \mathbf{P}^{d+2}$ and $P \in \mathbf{P}^{d+2} \setminus \mathbf{P}^{d+1}$. If Λ is the cone over F of vertex P and X is the intersection of Λ with a generic quadric (not containing P) of $\mathbf{P}^{d+2}$, then when we cut X with a hyperplane H passing through P we get that $S = X \cap H$ is a double covering of the Del Pezzo surface $F \cap H \subseteq \mathbf{P}^d$, hence if we intersect again with another hyperplane H' through P, $X \cap H \cap H'$ is a bielliptic curve (it is the double covering of the elliptic curve $F \cap H \cap H'$).

Here $X \subseteq \mathbf{P}^{d+2}$ is such that $\dim X = 3$ and $6 \leq \deg X = 2d \leq 14$: this example generalizes cases B.3 and B.4.

Example 2. Consider the embedding X of $\mathbf{P}^4$ into $\mathbf{P}^{14}$ given by $\mathcal{O}_{\mathbf{P}^4}(2)$. Consider an elliptic normal curve $E \subseteq \mathbf{P}^3 \subseteq \mathbf{P}^4$. E is the complete intersection of two (smooth) quadric in $\mathbf{P}^3$, hence the cone Λ on E from a point $P \in \mathbf{P}^4$ is the intersection of two (3-dimensional) quadric cones in $\mathbf{P}^4$. If we cut Λ with a generic quadric $Q \subseteq \mathbf{P}^4$ we get a bielliptic curve C (it is a canonical curve of degree 8 and genus 5) which is the double covering of E. The image of C will be a curve-section of X (here $\dim X = 4$, $\deg X = 16$). Of course we have also that the image of Q is a threefold with a bielliptic curve section.

3. Trigonal case: $\deg X \geq 18$.

Several results are known when X is a surface with a trigonal curve-section (see [3], [5], [13]). We are going to use the results in [5], to get the following, which will also extend Prop. (1,1) of [2]:

Theorem C. *Let $(X, H) \subseteq \mathbf{P}^N$ be as in $(*)$ with $\deg X \geq 18$, and suppose that X possesses a smooth curve-section C which is trigonal of genus $g \geq 4$. Then either*

C.1) X is a scroll, i.e. a $\mathbf{P}^{n-1}$-bundle over C and $\mathcal{O}_F(1) \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ for every fiber F.

C.2) X is a R-bundle over $\mathbf{P}^1$ whose general fiber R is a rational cubic scroll of dimension $n-1$ in $\mathbf{P}^{n+1}$ (in this case $n \leq 4$).

C.3) X is an F-bundle on $\mathbf{P}^1$, where every fiber $F \subseteq \mathbf{P}^{n-1}$ is a cubic hypersurface and $\mathcal{O}_F(H) \cong \mathcal{O}_F(1)$.

Proof. Let $n = 3$ first. Let $(S, L \cong H|_S)$ be a smooth hyperplane section of (X, H) which contains C. If S is a scroll on C, we get case C.1 as we did for Theorem A. If S is a conic bundle then by [5], Prop.1.2, we have $8 \leq L^2 = H^3 \leq 12$, and so this case cannot occour.

The results in [5], Theorem 2.13 and in the *Note added in proof* give us that the other possibilities for S when $\deg S \geq 18$ are:

– S is rational and ruled by cubics;

– S is an elliptic bundle on $\mathbf{P}^1$ with $\kappa(S) = 1$ and plane cubics as fibers;

– S is a K3 surface;

– $S = D \times \mathbf{P}^1$, where D is a plane cubic, and $\mathcal{O}_S(L) = p_1^*(\mathcal{O}_D(3P)) \otimes p_2^*(\mathcal{O}_{\mathbf{P}^1}(b))$ where $b \geq 3$, $P \in D$ and p_1, p_2 are the two projections of S on its factors. In the first two cases we get, respectively, that X is as in cases C.2 and C.3, because one can easily extend the fibers of S to the threefold using a theorem by Serrano ([12], Thm. 5.2).

If S is a K3 surface, then $\deg S = 2g - 2$, where $g = g(C)$, and C is a canonical curve, so X is a Fano threefold, by [8], Thm I, h). In this case, by [9], Thm. 2.5, $g = 10$ and $X \cong F \times \mathbf{P}^1$, where F is a smooth cubic surface in $\mathbf{P}^3$ and

$$\mathcal{O}_X(H) = p_1^*(\mathcal{O}_F(1)) \otimes p_2^*(\mathcal{O}_{\mathbf{P}^1}(2)),$$

where p_1, p_2 are the two projections of X on its factors. Hence we are in case C.3.

Now we want to check that in the last case we have no threefold with such hyperplane section.

In fact, by [1], Theorem 2, if $S \cong D \times \mathbf{P}^1$, were a very ample divisor on X, then (X, H) would be a $\mathbf{P}^2$-scroll over D, contradicting the fact that $\mathcal{O}_S(L)$ induces $\mathcal{O}_{\mathbf{P}^1}(3)$ on the rational fibers of S, so that they should be embedded as rational normal cubics.

In order to get the case $n \geq 4$, one uses induction on n as in Theorem A, since all our threefolds are bundles over a curve.

4. Trigonal case: $\deg X \leq 8$.

Theorem D. *Let $(X, H) \subseteq \mathbf{P}$ be as in $(*)$, with $\deg X \leq 8$. Suppose that X possesses a smooth curve-section C which is trigonal of genus $g \geq 4$ and that X is not as in C.1 or in C.2. Let (X', H') be a minimal reduction of X. Then either:*

D.1) $\deg X' = 6$ *and $X' \subseteq \mathbf{P}^{2+n}$ is the complete intersection of a quadric and a cubic hypersurfaces.*

D.2) $\dim X = 3$ *and $X \subseteq \mathbf{P}^5$ is an F-bundle on $\mathbf{P}^1$, where every fiber F is a cubic surface in $\mathbf{P}^3$ with $\mathcal{O}_F(H) \cong \mathcal{O}_F(1)$.*

Proof. Let S be a generic surface-section of X through C. If $\deg S \leq 7$, by [5], Prop. 3.4, the possibilities are:

– $S \subseteq \mathbf{P}^4$ is a K3 surface of degree 6, the complete intersection of a quadric and a cubic hypersurface;

– $S \subseteq \mathbf{P}^4$, $\deg S = 7$ and S is an F-bundle on $\mathbf{P}^1$, where every fiber F is a plane cubic curve (S is a minimal elliptic surface with $q = 0$ and $\kappa(S') = 1$).

If S is as in the first case then X is as in D.1. If S is as in the second case, we have that X is as in D.2 (see the table in [6]).

Now let $\deg S = 8$. If S is not contained in $\mathbf{P}^4$ we have, by [5], that $S \subseteq \mathbf{P}^5$ is a K3 surface of degree 8 (not a complete intersection). In this case it can only be the hyperplane section of a Fano threefold of the principal series, but, by [9], we know that those threefold can have a trigonal curve-section only for degrees 10,12 or 18, hence this case cannot occur.

When $\deg S = 8$ and $S \subseteq \mathbf{P}^4$, the only possibilities for S are described by [10], Theorem 0.1. In this case either the surface S cannot have trigonal hyperplane sections, or it cannot be a section of a threefold (see [5] and [7], § 1, point VI) and so we are done.

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