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ON THE SPECIALITY OF A CURVE

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Let $C \subset \mathbf{P}^r_k$, k algebraically closed field of characteristic 0, be a curve and let $e(C) = \{\max n \mid H^1(\mathcal{O}_C(n)) \neq 0\}$ its speciality. Let Γ be the generic hyperplane section and $\varepsilon = \{\max n \mid H^1(\mathcal{J}_\Gamma(n)) \neq 0\}$. We prove that, if Γ is generated in degree $\leq \varepsilon$, then $e(C) = \varepsilon - 1$. In the case $r = 3$ we discuss some relations between $e(C)$ and the Hilbert function of Γ.

0. Introduction.

Let $C \subset \mathbf{P}^3_k$, k algebraically closed field of characteristic 0, be a curve (i.e. a locally C.M., equidimensional subscheme of dimension 1) and let $\Gamma = C \cap H$ be the generic plane section. In [7], [8] we studied some relations between properties of C and of Γ. More precisely we proved the following result ([8] Teorema 4).

0.1. *Let $J = H^0_*(\mathcal{J}_\Gamma) \subset R = H^0_*(\mathcal{O}_H)$ be the homogeneous ideal of Γ in H and let t be an integer. Assume that, for $n \leq t+2$ is $\mathrm{Tor}_1^R(J,k)_n = 0$. Then the restriction map $H^0(\mathcal{J}_C(t)) \to H^0(\mathcal{J}_\Gamma(t))$ is surjective.*

From this we deduced the following corollary ([8] Corollario 1.).

0.2. *Assume that* $\deg(C) > 4$ *and C does not lie on a quadric. If Γ is a complete intersection, then C is a complete intersection.*

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This two results have been extended to the higher dimensional case (see [6]).

The proof of the above results is based on the study of $\operatorname{Ker}\varphi_{H^i}(n)$, where $\varphi_{H^i}(n)\colon H^1(\mathscr{I}_C(n-i)) \to H^1(\mathscr{I}_C(n))$ is the multiplication by H^i.

In the present paper we obtain a dual result, by studing the $\operatorname{Coker}\varphi_{H^i}(n)$ and precisely we get the following.

0.3. *Let $C \subset \mathbf{P}^r_k$, k algebraically closed field of characteristic 0, be a curve and let $\Gamma = C \cap H$ be the generic hyperplane section. Let $J = H^0_*(\mathscr{I}_\Gamma) \subset R = H^0_*(\mathscr{O}_H)$ be the homogeneous ideal of Γ in H and let t be an integer. Assume that, for $n \geq t+1$, is $\operatorname{Tor}^R_0(J,k)_n = 0$. Then the induced map $H^1(\mathscr{I}_\Gamma(t)) \to H^2(\mathscr{I}_C(t-1))$ is injective.*

As a corollary we obtain the following.

0.4. *Let $e(C) = \max\{n \mid H^1(\mathscr{O}_C(n)) \neq 0\}$ and $\varepsilon(\Gamma) = \max\{n \mid H^1(\mathscr{I}_\Gamma(n)) \neq 0\}$. Assume that J_Γ is generated in degree $\leq \varepsilon$. Then $e(C) = \varepsilon(\Gamma) - 1$.*

We note that it is possible to give an alternate proof of 0.3, by linking C to a curve C' and using 0.1. In the same way 0.1 can be deduced from 0.3.

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1. Preliminaries.

Let k be an algebraically closed field of characteristic 0, $S = k[x_0, \dots, x_r]$ and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded S-module. Define (see also [1], 0.1.7) a graded S-module M^* as follows: as a k-vector space it is $(M^*)_n = \operatorname{Hom}_k(M_n, k)$ and the S-module structure is defined by $s \cdot f(m) = f(sm)$ for $s \in S$, $m \in M$ and $f \in M^*$.

Proposition 1.1. *Let S, M, M^* as before. Then we have:*

1) *If $f : M \to N$ is a graded S-module homomorphism of degree 0 then f induces a graded S-module homomorphism of degree 0, $f^* : N^* \to M^*$.*
2) *The map $M \mapsto M^*$ is a contravariant exact functor.*
3) *If M is a graded S-module of finite type, then $M^* \simeq \operatorname{Hom}_S(M, S^*)$.*
4) *If L is free of finite type, then $L^* \simeq L^\vee \otimes_S S^*$, where $L^\vee = \operatorname{Hom}_S(L, S)$.*
5) *If L, L' are free of finite type and $f : L \to L'$, then $f^* = f^\vee \otimes_S S^*$. More generally for every M, $(f \otimes_S \operatorname{id}_M)^* = f^\vee \otimes_S \operatorname{id}_{M^*}$.*

Proof. 1) It is straightforward to see that the dual f^* of f as k-vector spaces is a homogeneous, degree 0, graded S-module homomorphism.
2) Is trivial.
For 3) see [1], 0.1.10.

4) We reduce to the case $M = S(a)$. In this case $M^* \simeq S^*(-a)$.
5) We can assume $f : S(a) \to S(b)$ and $f \in S$ homogeneous of degree $b - a$. Then $f^* : S^*(-b) \to S^*(-a)$ is given by $f^*(s^*) = f \cdot s^*$. □

Lemma 1.2. *Let S as before. Then we have:*

1) $$\operatorname{Tor}_i^S(S^*, k) = \begin{cases} 0 & \text{if } \; i \neq r+1 \\ k(-r-1) & \text{if } \; i = r+1 \end{cases}$$

2) For every graded S-module M it is

$$\operatorname{Tor}_{r+1}^S(M^*, k) = (M \otimes_S k)^*(-r-1).$$

Proof. 1) Let

(1) $$0 \to S(-r-1) \xrightarrow{g} S(-r)^{r+1} \to \cdots \to S(-1) \xrightarrow{f} S \to k \to 0$$

be the free resolution of k given by the Koszul complex. Observe that this sequence is self-dual. In particular $g = f^\vee(-r-1)$.

If we apply * and shift by $-r-1$ we obtain an exact sequence:

$$0 \to k^*(-r-1) \to S^*(-r-1) \xrightarrow{f^*(-r-1)} S^*(-r)^r \to \cdots \to S^* \to 0.$$

If we compare this exact sequence with the complex obtained from

(2) $$0 \to S(-r-1) \to S(-r)^{r+1} \to \cdots \to S \to 0$$

by tensoring $\otimes_S S^*$ we obtain the result, by Proposition 1.1.5), since

$$g \otimes_S S^* = f^\vee(-r-1) \otimes_S S^* = f^*(-r-1).$$

2) As before start from the exact sequence (1), tensor $\otimes_S M$ and apply *. We obtain a sequence

$$0 \to (M \otimes_S k)^* \to M^* \to M^*(1)^{r+1} \to \cdots$$

which is exact in $(M \otimes_S k)^*$ and M^* . On the other hand if we start with the complex (2) and tensor $\otimes_S M^*$ we get the result. □

Lemma 1.3. *Let $J \subset S$ be an homogeneous ideal generated in degree ≥ 1. Then we have:*

1) $$\mathrm{Tor}^S_{r+1}((S/J)^*, k) = k(-r-1).$$

2) $$\mathrm{Tor}^S_{r+1}(J^*, k) = \mathrm{Tor}^S_r((S/J)^*, k).$$

Proof. 1) Follows from Lemma 1.2 since $(R/J) \otimes_S k = k$.
2) From the exact sequence

$$0 \to (S/J)^* \to S^* \to J^* \to 0$$

we have an exact sequence

$$0 \to \mathrm{Tor}^S_{r+1}((S/J)^*, k) \to \mathrm{Tor}^S_{r+1}(S^*, k) \to \\ \to \mathrm{Tor}^S_{r+1}(J^*, k) \to \mathrm{Tor}^S_r((S/J)^*, k) \to 0$$

and the result follows since, by 1), $\mathrm{Tor}^S_{r+1}((S/J)^*, k) = \mathrm{Tor}^S_{r+1}(S^*, k) = k(-r-1)$. □

Now we recall two known results we need in section 2.

Lemma 1.4. *Let $S = k[x_0, \dots, x_r]$ and let M be a graded S-module of finite type. Let H be a generic linear form in S and denote by $\varphi_{H^t}(n)$ the map $M_{n-t} \to M_n$ given by multiplication by H^t. Assume that $m \in \mathrm{Ker}\,\varphi_{H^t}(n)$. Then $mF \in \mathrm{Im}\,\varphi_H(n)$ for every $F \in S_t$.*

Proof. See [7], proof of Theorem 6; see also [6], Lemma 1. □

Lemma 1.5. *Let $R = k[x_1, \dots, x_r]$ and let M be a graded R-module. Then the following are equivalent:*

1) *for every r-uple $m_1, \dots, m_r \in M_n$ satisfying $m_i x_j = m_j x_i$ for $i, j = 1, \dots, r$ there exists an $m \in M_{n-1}$ s.t. $m x_i = m_i$ for $i = 1, \dots, r$.*
2) $\mathrm{Tor}^R_{r-1}(M, k)_{n+r-1} = 0$

Proof. The proof is an easy generalization of [3], Lemma p.141; see also [6], Lemma 2. □

2. The main result.

Let $C \subset \mathbf{P}^r$ be a curve (i.e. a 1-dimensional locally C.M. equidimensional subscheme) and let $\Gamma = C \cap H$ be its generic hyperplane section.

Let $S = k[x_0, \ldots, x_r]$, $R = k[x_1, \ldots, x_r]$ be the homogeneous coordinate rings of $\mathbf{P}^r$ and H respectively and denote by $\mathscr{I}_C$, $\mathscr{I}_\Gamma$ the ideal sheaves of C and Γ in $\mathscr{O}_{\mathbf{P}^r}$, $\mathscr{O}_H$ respectively. Moreover let $M = H^1_*(\mathscr{I}_C)$ be the Hartshorne-Rao module of C.

Consider the graded S-modules K, Q given by the exact sequence

$$0 \to K \to M(-1) \to M \to Q \to 0 \tag{3}$$

where $\varphi_H : M(-1) \to M$ is given by the multiplication by H. From the exact sequence

$$0 \to \mathscr{I}_C(-1) \to \mathscr{I}_C \to \mathscr{I}_\Gamma \to 0 \tag{4}$$

we get a long exact sequence

$$\begin{aligned} 0 \to H^0_*(\mathscr{I}_C)(-1) \to H^0_*(\mathscr{I}_C) \to H^0_*(\mathscr{I}_\Gamma) \to \\ \to M(-1) \to M \to H^1_*(\mathscr{I}_\Gamma) \to H^2_*(\mathscr{I}_C)(-1) \to \cdots \end{aligned} \tag{5}$$

from this we see that Q is the kernel of the map $H^1_*(\mathscr{I}_\Gamma) \to H^2_*(\mathscr{I}_C)(-1)$.

Theorem 2.1. *Let C, Γ as before and let $J = H^0_*(\mathscr{I}_\Gamma) \subset R$ be the homogeneous ideal of Γ. Let $t \geq 0$ be an integer and assume that $(J \otimes_R k)_n = 0$ for $n \geq t+1$ i.e. J is generated in degree $\leq t$; then the map $H^1(\mathscr{I}_\Gamma(t)) \to H^2(\mathscr{I}_C(t-1))$ is injective.*

Proof. With the above notations we have to prove that $Q_t = 0$. If we apply the functor $*$ we obtain an exact sequence:

$$0 \to Q^* \to M^* \to M^*(1) \to K^* \to 0$$

and we prove that $(Q_t)^* = 0$.

Let $\alpha \in (Q_t)^* \subset H^1(\mathscr{I}_C(t))^*$, then $\alpha H = 0$ in $H^1(\mathscr{I}_C(t-1))^*$. By Lemma 1.3, if we denote by $\overline{\alpha}$ the image of α in $H^0(\mathscr{I}_\Gamma(t+1))^*$ we have $\overline{\alpha} x_i = 0$ for every $i = 1, \ldots, r$.

Now let $\beta \in H^0(\mathscr{O}_H(t+1))^*$ a preimage of $\overline{\alpha}$ in the map

$$\psi : (H^0_*(\mathscr{O}_H))^* \to (H^0_*(\mathscr{I}_\Gamma))^*$$

we have $\beta x_i \equiv 0 \mod I$ for $i = 1, \dots, r$, where $I = \operatorname{Ker}\psi$. From the exact sequence $0 \to J \to R \to R/J \to 0$ we see that $I \simeq (R/J)^*$.

Let $F_i \in (R/J)^*$ of degree $-t$ such that $\beta x_i = F_i$ in $(R_t)^*$ for $i = 1, \dots, r$. We have $F_i x_j = F_j x_i$ for every $i, j = 1, \dots, r$. By hypothesis we have $(J \otimes_R k)_{t+1} = 0$, hence by Lemma 1.5

$$\operatorname{Tor}^R_{r-1}((R/J)^*, k)_{-t+r-1} \simeq \operatorname{Tor}^R_r((J)^*, k)_{-t+r-1} \simeq$$
$$\simeq (J \otimes_R k)^*(-r)_{-t+r-1} = 0$$

and by Lemma 1.4 there exists $F \in (R/J)^*$ of degree $-t-1$ such that $Fx_i = \beta x_i$ in R^* for $i = 1, \dots, r$. We want to show that this implies $\overline{\alpha} = 0$: in fact since $t + 1 \geq 1$ we deduce $\beta = F \in I = \operatorname{Ker}\psi$, hence $\overline{\alpha} = \psi(F) = 0$.

Since $\overline{\alpha} = 0$ from the exact sequence (4) we see that $\alpha = \gamma H$ with $\gamma \in H^1(\mathcal{J}_C(t+1))^*$ and $\gamma H^2 = 0$. We continue as above, with $\overline{\gamma} \in H^0(\mathcal{J}_\Gamma(t+2))^*$ and let δ be a preimage of $\overline{\gamma}$ in $H^0(\mathcal{O}_H(t+2))^*$. We have $\delta x_i x_j \equiv 0 \mod I$, for $i, j = 1, \dots, r$, hence there are elements $F_{ij} \in I$ of degree $-t$ such that $F_{ij} = \delta x_i x_j$ in $(R_t)^*$. Since $(J \otimes_R k)_{t+1} = (J \otimes_R k)_{t+2} = 0$ there exists $F \in I$ of degree $-t-2$ such that $\delta x_i x_j = F x_i x_j$ for every $i, j = 1, \dots, r$. Since $t + 2 \geq q2$ this implies $F = \delta$ and hence $\overline{\gamma} = 0$. Continuing in this way we get the result since $H^1(\mathcal{J}_C(n)) = 0$ for $n \gg 0$. □

Let C be a curve and Γ its generic hyperplane section; we set

$$e(C) = \max\{n \mid H^2(\mathcal{J}_C(n)) \neq 0\}$$

and

$$\varepsilon(\Gamma) = \max\{n \mid H^1(\mathcal{J}_\Gamma(n)) \neq 0\}.$$

From the exact sequence (5) we see easily that $e(C) \leq \varepsilon(\Gamma) - 1$.

Corollary 2.2. *Let C be a curve and assume that Γ is generated in degree $\leq \varepsilon(\Gamma)$. Then $e(C) = \varepsilon(\Gamma) - 1$.*

Proof. Follows from Theorem 2.1. □

3. An application to curves in $\mathbf{P}^3$.

In this section we consider the case of a reduced and irreducible curve $C \subset \mathbf{P}^3$ and we give conditions on the Hilbert function of Γ in order to apply Theorem 2.1.

We recall that Γ verify the uniform position property (U.P.P.) and this implies that the Hilbert function of Γ is of decreasing type (see [2]). If we consider the difference function

$$\Delta H(\Gamma, n) = H(\Gamma, n) - H(\Gamma, n-1) = h_n$$

it has the form

$$\{1, 2, \ldots, h_{a-1} = a, \ldots, a, h_b, h_{b+1}, \ldots\}$$

where $a > h_b > h_{b+1} > \ldots$. We note that $\varepsilon(\Gamma) = \max\{n \mid h_n \neq 0\} - 1$. On the other hand the following lemma gives bounds on the minimal number of generators of Γ.

Lemma 3.1. *Let $\Gamma \subset \mathbf{P}^2$ be a set of d points with the* U.P.P.*; denote by α_i the number of minimal generators of Γ in degree i, and let a, b be as before. Then we have*

$$\alpha_a = -\Delta^3 H(\Gamma, a) \quad , \quad \alpha_b = -\Delta^3 H(\Gamma, b)$$

$$\max\{-\Delta^3 H(\Gamma, i), 0\} \leq \alpha_i \leq -\Delta^2 H(\Gamma, i) - 1 \qquad \text{for} \qquad i > b.$$

Proof. See [4], Prop.1.4. □

Proposition 3.2. *Let $C \subset \mathbf{P}^3$ be a reduced and irreducible curve not lying on a quadric surface. Assume that the Hilbert function of the generic plane section Γ of C satisfy $h_\varepsilon = 2$, $h_{\varepsilon+1} = 1$. Then $e(C) = \varepsilon - 1$.*

Proof. Set $h_{\varepsilon-1} = c$. Two cases are possible:

1) $c \leq 2$. If $c = 1$ then $d = 4$ and C lies on a quadric. If $c = 2$, then since $H(\Gamma, n)$ is of decreasing type we see that d is even and $\Delta(\Gamma, n)$ is of the form $\{1, 2, 2, \ldots, 2, 1\}$. By 0.1 we see that C lies on a quadric.

2) $c > 2$. In this case we have:

$$\Delta^2 H(\Gamma, \varepsilon) = 2 - c \quad , \quad \Delta^2 H(\Gamma, \varepsilon + 1) = -1,$$
$$\Delta^2 H(\Gamma, \varepsilon + 2) = -1 \quad , \quad \Delta^2 H(\Gamma, \varepsilon + 3) = 0$$
$$\Delta^3 H(\Gamma, \varepsilon + 1) = c - 3 \quad , \quad \Delta^2 H(\Gamma, \varepsilon + 2) = 0,$$
$$\Delta^3 H(\Gamma, \varepsilon + 3) = 1 \quad , \quad \Delta^3 H(\Gamma, \varepsilon + 4) = 0$$

from which it follows, by Lemma 3.1, that $\alpha_i = 0$ for $i > \varepsilon$. □

Remark 3.3. Proposition 3.2 gives some conditions on the Hilbert function of Γ for curves with low speciality. For example consider an integral curve C of degree 18: Proposition 3.2 implies that for every integral curve C, not lying on a quartic surface, with $e(C) \leq 4$ the Hilbert function of Γ has the form $h_0 = 1$, $h_1 = 2$, $h_2 = 3$, $h_3 = 4$, $h_4 = 5$, $h_5 = 3$.

In particular one is lead to conjecture that for integral non special curves i.e. with $e(C) < 1$, (in particular smooth rational curves), the Hilbert function of the generic plane section is maximal. More precisely we can conjecture the following.

Conjecture 3.4. Let C be an integral non special curve of degree d and let $s = \min\{n \mid H^0(\mathscr{I}_C(n)) \neq 0\}$. Then the Hilbert function of the generic plane section of C is the following:

$$H(\Gamma, i) = \begin{cases} \min\{\binom{i+2}{2}, d\} & \text{for} \quad i \leq s-1 \\ \min\{\binom{s+1}{2} + (i-s+1)s, d\} & \text{for} \quad i \geq s \end{cases}$$

We examine Conjecture 3.4 for low values of s: for $s = 1, 2$ it is trivial since the Hilbert function $H(\Gamma, n)$ has no choice.

$s = 3$. Let a as above; it is $a = 3$ for $d > 5$ by Laudal's Lemma: in this case $H(\Gamma, n)$ has the above form unless Γ is a complete intersection $(3, k)$ but, by 0.2, C itself is a complete intersection $(3, k)$, hence $e(C) = k - 1 > 2$. Hence for $s = 3$ the conjecture is true.

$s = 4$. It is $a = 4$ for $d > 10$. If $d \leq 10$ there is one open case when $\Delta H(\Gamma, n)$ has the form $\{1, 2, 3, 3, 1\}$. If $d > 10$ we have four cases to examine:

i) Γ is a complete intersection $(4, k)$, but in this case C is a complete intersection $(4, k)$, hence $e(C) = k > 3$.

ii) $\Delta H(\Gamma, n)$ has the form $\{1, 2, 3, 4, \ldots, 4, 2, 1\}$, but in this case, by Proposition 3.3, it is $e(C) = \varepsilon - 1 \geq 3$.

iii) $\Delta H(\Gamma, n)$ has the form $\{1, 2, 3, 4, \ldots, 4, 3, 2\}$. In this case $d = 4t + 3$, $t > 2$ and we want to prove that C is is aritmetically Cohen-Macaulay, hence $e(C) = \varepsilon(\Gamma) - 1 = t + 1$. Using Lemma 3.1 we see that J_Γ has a minimal free resolution :

$$0 \to R(-t-4)^2 \to R(-t-3) \oplus R(-t-1) \oplus R(-4) \to J_\Gamma \to 0$$

hence, by 0.1, C is contained in a complete intersection $(4, t+1)$. Thus C is linked to a line.

iv) $\Delta H(\Gamma, n)$ has the form $\{1, 2, 3, 4, \ldots, 4, 3, 1\}$ and this is an open case.

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