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SEMI-ORTHOGONAL PROPERTIES OF A CLASS OF GAUSS' HYPERGEOMETRIC POLYNOMIALS

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We present three semi-orthogonal properties of a class of Gauss' hypergeometric polynomials.

1. Introduction.

The object of this paper is to present three semi-orthogonal properties of a class of hypergeometric polynomials:

$$(1.1)\qquad {}_2F_1\left(\begin{matrix}-m; 1-b+m; x\\ 1-a-m\end{matrix}\right)=\sum_{n=0}^{m}\frac{(-m)_n(1-b+m)_n x^n}{(1-a-m)_n n!},$$

$m = 0, 1, 2, \cdots$. The Jacobi polynomials constitute an important, and a rather wide class of Gauss' hypergeometric polynomials, from which Chebyshev, Legendre, Laguerre and Gegenbauer polynomials follows as special cases. Their orthogonality, with the non-negative weight finction $(1-x)^a(1+x)^b$ on the interval $[-1, 1]$, for $a > -1$, $b > -1$, is usually derived by the use of the associated differential equation and Rodrigues' formula.
In this paper, we introduce a direct method of proof, which is much simpler and elegant to establish orthogonalities of the Gauss' hypergeometric polynomials. Our method could be employed to establish the orthogonalities of the Jacobi polynomials and other Gauss' hypergeometric polynomials.

The following three integrals are required for the proofs.

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(i) First Integral.

$$(1.2)\qquad \int_0^1 x^{h-1}(1-x)^{a-b+n}{}_2F_1\left[\begin{matrix}-n, 1-b+n; x\\ 1-a-n\end{matrix}\right]dx =$$

$$= \frac{\Gamma(1-a-n)\,\Gamma(h)\,\Gamma(a-b+2n+1)\,\Gamma(1-a-h)}{\Gamma(1-a)\,\Gamma(1-a-h-n)\,\Gamma(a-b+h+2n+1)}, \qquad n = 0, 1, 2, \ldots$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a-b) > -1$.

Proof. The integral (1.2) is established by expressing the hypergeometric polynomials in the intergrand as its series representation (1.1), interchanging the order of integration and summation, evaluating the resulting integral with the help of the Beta integral ([1], p. 9):

$$(1.3)\qquad \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, \qquad \operatorname{Re} p > 0, \ \operatorname{Re} q > 0$$

and simplifying, we get

$$(1.4)\qquad = \frac{\Gamma(a-b-n+1)\,\Gamma(h)}{\Gamma(1+h+a-b+n)}\,{}_3F_2\left[\begin{matrix}-n, 1-b+n, h & ;1\\ 1-a-n, 1+h+a-b+n\end{matrix}\right].$$

It can easily be verified that the generalized hypergeometric series (1.4) is Saalschutzian. Therefore, applying the Saalschutz's theorem ([1], p.188, (3)):

$$(1.5)\qquad {}_3F_2\left[\begin{matrix}-n, a, b & ;1\\ c, 1-c+a+b-n\end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

and simplifying with the help of the following form of the formula ([1], p.3, (4)):

$$(1.6)\qquad \Gamma(1+a-n) = \frac{(-1)^n\Gamma(1+a)}{(-a)_n}$$

the left hand side of (1.2) is obtained.

Note. The integral (1.2) can also be derived with the help of the integral ([2], p.398, (2)).

(ii) Second Integral.

$$(1.7)\qquad \int_1^\infty x^{h-1}(x-1)^{a-b+n}\,{}_2F_1\left(\begin{matrix}-n, 1-b+n; x\\ 1-a-n\end{matrix}\right)dx =$$

$$= (-1)^n\frac{\Gamma(1-a-h)\,\Gamma(1-a-n)\,\Gamma(b-a-h-2n)\,\Gamma(1+a-b+2n)}{\Gamma(1-h)\,\Gamma(1-a)\,\Gamma(1-a-n-b)},$$

$n = 0, 1, 2, \ldots$ *where* $\operatorname{Re}(a-b) > -1$, $\operatorname{Re}(a-b+h) < 0$, $\operatorname{Re}(a+h) < 0$.

Proof. The integral (1.7) is established on following the technique employed to establish (1.2) except instead of (1.3), using the integral ([2], p. 201, (7)):

$$\int_1^\infty x^{-v}(x-1)^{w-1}dx = \frac{\Gamma(v-w)\,\Gamma(w)}{\Gamma(v)}, \quad \operatorname{Re} v > \operatorname{Re} w > 0. \tag{1.8}$$

(iii) Third Integral.

$$\int_0^\infty x^{h-1}(1+x)^{a-b+n}\,{}_2F_1\left(\begin{matrix}-n, 1-b+n; -x\\ 1-a-n\end{matrix}\right)dx = \tag{1.9}$$

$$= \frac{\Gamma(h)\,\Gamma(1-a-h)\,\Gamma(1-a-n)\,\Gamma(b-a-h-2n)}{\Gamma(b-a-2n)\,\Gamma(1-a)\,\Gamma(1-a-n-h)}, \quad n = 0, 1, 2, \ldots$$

where $\operatorname{Re} h > 0$, $\operatorname{Re}(a-b+h) < 0$, $\operatorname{Re}(a+h) < 0$.

Proof. The integral (1.9) is established on following the tecnique employed to established (1.2) except instead of (1.3), using the integral ([2], p.233, (8)):

$$\int_0^\infty x^{v-1}(1+x)^{-w}\,dx = \frac{\Gamma(v)\,\Gamma(w-v)}{\Gamma(w)}, \quad \operatorname{Re} w > \operatorname{Re} v > 0 \tag{1.10}$$

2. Semi-orthogonal relations.

The semi-orthogonal relations to be established are

$$\int_0^1 x^{-a-n}(1-x)^{a-b+n}\,{}_2F_1\left(\begin{matrix}-m, 1-b+m; x\\ 1-a-m\end{matrix}\right)\cdot {}_2F_1\left(\begin{matrix}-n, 1-b+n; x\\ 1-a-n\end{matrix}\right)dx \tag{2.1}$$

$$= 0, \quad \text{if } m < n \tag{2.1a}$$

$$= \frac{n!(1-b+n)_n\Gamma(1-a-n)\,\Gamma(1+a-b+2n)}{(1-a-n)_n\,\Gamma(2-b+2n)}, \quad \text{if } m = n \tag{2.1b}$$

$$= -\frac{(n+1)!\Gamma(1-a-n)\,\Gamma(-a-n)\,\Gamma(1+a-b+2n)}{\Gamma(1-a)\,\Gamma(2-b+n)}, \quad \text{if } m = n+1 \tag{2.1c}$$

where $\operatorname{Re} a < 1$, $\operatorname{Re}(a-b) > -1$.

$$\int_1^\infty x^{-a-n}(x-1)^{a-b+n}\,{}_2F_1\left(\begin{matrix}-m, 1-b+m; x\\ 1-a-m\end{matrix}\right)\cdot \tag{2.2}$$

$$\cdot {}_2F_1\begin{pmatrix} -n, 1-b+n; x \\ 1-a-n \end{pmatrix} dx$$

(2.2a) $= 0, \quad \text{if } m < n$

(2.2b) $$= \frac{n!(1-b+n)_n \Gamma(1+a-b+2n)\,\Gamma(b-1)\,\Gamma(2-b)}{(1-a-n)_n\,\Gamma(2-b+2n)\,\Gamma(a+n)}, \quad \text{if } m = n$$

(2.2c) $$= \frac{(n+1)!\Gamma(-a-n)\,\Gamma(1+a-b+2n)\,\Gamma(b-1)\,\Gamma(2-b)}{\Gamma(1-a)\,\Gamma(a+n)\,\Gamma(2-b+n)}, \quad \text{if } m = n+1$$

$\mathrm{Re}\,(a-b) > -1, \mathrm{Re}\,b > 1+m+n, \mathrm{Re}\,b < 1+m+n \Longrightarrow \mathrm{Re}\,b \neq 1+m+n.$

(2.3) $$\int_0^\infty x^{-a-n}(1+x)^{a-b+n}\, {}_2F_1\begin{pmatrix} -m, 1-b+m; -x \\ 1-a-m \end{pmatrix} \cdot {}_2F_1\begin{pmatrix} -n, 1-b+n; -x \\ 1-a-n \end{pmatrix} dx.$$

(2.3a) $= 0, \quad \text{if } m < n$

(2.3b) $$= (-1)^n \frac{n!(1-b+n)_n \Gamma(1-a-n)\,\Gamma(b-1)\,\Gamma(2-b)}{(1-a-n)_n\,\Gamma(2-b+2n)\,\Gamma(b-a-2n)}, \quad \text{if } m = n$$

(2.3c) $$= (-1)^n \frac{(n+1)!\,\Gamma(1-a-n)\,\Gamma(-a-n)\,\Gamma(b-2)\,\Gamma(3-b)}{\Gamma(1-a)\,\Gamma(2-b+n)\,\Gamma(b-a-2n)}, \quad \text{if } m = n+1$$

$\mathrm{Re}\,a < 1, \mathrm{Re}\,b > 1+m+n, \mathrm{Re}\,b < 1+m+n \Longrightarrow \mathrm{Re}\,b \neq 1+m+n.$

Proof. To prove (2.1), we write its left hand side in the form:

(2.4) $$\sum_{r=0}^{\infty} \frac{(-m)_r(1-b+m)_r}{r!(1-a-m)_r} \int_0^1 x^{r-a-n}(1-x)^{a-b+n} \cdot {}_2F_1\begin{pmatrix} -n, 1-b+n; x \\ 1-a-n \end{pmatrix} dx$$

On evaluating the integral in (2.4) with help of (1.2), it reduces to the form

(2.5) $$\sum_{r=0}^{\infty} \frac{(-m)_r(1-b+m)_r}{r!(1-a-m)_r} \cdot \frac{\Gamma(1-a-n)\,\Gamma(r-a-n+1)\,\Gamma(1+a-b+2n)(-r)_n}{\Gamma(1-a)\,\Gamma(2-b+n+r)}.$$

If $r < n$, the numerator of (2.5) vanishes, and since r runs from 0 to m, it follows that (2.5) also vanishes, when $m < n$. Now, it is clear that for $m < n$ all terms of (2.5) vanish, which proves (2.1a).

When $m = n$, using the standard result

$$(2.6) \qquad (-r)_n = \begin{cases} \dfrac{(-1)^n r!}{(r-n)!} & \text{if} \quad 0 \le n \le r \\ 0 & \text{if} \quad n > r \end{cases}$$

and simplifying the right hand side of (2.1b) follows from (2.5).

In (2.5), putting $m = n+1$, using (2.6) and additing the resulting two terms $(r = n, n+1)$, we obtain (2.1c).

To prove (2.2), first we reduce its left hand side to the form similar to (2.4), then evaluate the last integral with the help of the integral (1.7) to obtain

$$(2.7) \qquad \sum_{r=0}^{\infty} \frac{(-m)_r (1-b+m)_r}{r!(1-a-m)_r} \cdot \\ \cdot (-1)^n \frac{(-r)_n \Gamma(1-a-n)\,\Gamma(b-n-r-1)\,\Gamma(1+a-b+2n)}{\Gamma(a+n-r)\,\Gamma(1-a)}.$$

We see that (2.7) is of the same form as (2.5). Therefore, for $m < n$ all terms of (2.7) vanish, which proves (2.2a).

When $m = n$, using the standard result (2.6) and simplifying, the right hand side of (2.2b) follows from (2.7).

To prove (2.3), we first reduce its left hand side to the form similar to (2.4), then evaluate the last integral with the help of the integral (1.9) to get

$$(2.8) \qquad \sum_{r=0}^{\infty} (-1)^r \frac{(-m)_r (1-b+m)_r}{r!(1-a-m)_r} \cdot \\ \cdot \frac{(-r)_n \Gamma(1+r-a-n)\,\Gamma(1-a-n)\,\Gamma(b-n-r-1)}{\Gamma(b-a-2n)\,\Gamma(1-a)}.$$

Since (2.8) is of the same form as (2.5). Therefore, for $m < n$ all terms of (2.8) vanish, which proves (2.3c).

When $m = n$, using the standard result (2.6) and simplifying, the result (2.3b) follows from (2.8).

In (2.8), putting $m = n+1$, using (2.6) and adding the resulting two terms $(r = n, n+1)$, we obtain (2.3c).

Note. On continuing as above ,we can find the values of the integrals (2.1), (2.2) and (2.3) for $m = n+2, n+3, n+4, \ldots$.

3. Approximate Fourier expansions involving hypergeometric polynomials.

Based on the results of this paper, in a future communication, we propose to generate a theory concerning the expansion of arbitrary polynomilas, or functions in general, in series of the hypergeometric polynomials (1.1). Specially if $F(x)$ is a suitable function defined for all x.

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