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EFFECTIVE RESULTS ON PICARD BUNDLES VIA M-REGULARITY

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In this paper we study some properties, namely Global Generation and Strong Normal Presentation, of specific types of (twists of) Picard bundles over the Jacobian of a curve. Our main tool is the notion of M-regularity introduced by G. Pareschi and M.Popa.

1. Introduction

Let C be a smooth, irreducible, projective curve of genus $g \geq 2$, and consider its Jacobian J and a fixed theta-divisor Θ on it, so that the couple $(J, \mathscr{O}_J(\Theta))$, is a principally polarized abelian variety (ppav).

The purpose of this paper is to study some properties, namely Global Generation and Strong Normal Presentation, of particular types of (twists of) Picard bundles on J
Our main tool will be the notion of M-regularity introduced by G. Pareschi and M.Popa in [10], but let us first introduce the main objects of our investigation.

The first type of Picard bundles that we study are Fourier-Mukai transforms of the line bundle $\mathscr{O}_J(n\Theta_J)$, and its twists, that we call $E^{n,k}$. Such bundles could

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in fact be defined on any ppav.

Sticking to the case of the Jacobian of a curve, we consider the well known subvariety $W_d \subset J$. Shortly, we fix an isomorphism $\mathrm{Pic}^d(C) \simeq J$; composed with it the Abel-Jacobi map injects the Symmetric d-th product $C_d \hookrightarrow J$, and W_d is exactly the image of this map.
The second kind of Picard bundles that we work on are transforms of the sheaf $\mathscr{O}_{W_d}(n\Theta_J)$, for $n \geq 2$, possibly twisted again by some $\mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}})$. We call these $E^{n,k}_{W_d}$.

In the last part of this paper we extend our investigation to Picard bundles which are transforms of line bundles L of degree $n \geq 2g-1$ on the curve C. These are what where classically called Picard bundles, as for example in the original paper by Mukai [7].

All the above described bundles are in fact special cases of a more general notion of Picard Bundles, provided by Lazarsfeld in [6, 6.3]. It is worth noting that these bundles are known to be negative, *i.e.* they have ample dual bundle [6, Theorem 6.3.48]. One could say that the leitmotif of this work is trying to find effective bounds on the minimum integer k for which the bundles $E^{n,k}$ and $E^{n,k}_{W_d}$ have certain positivity properties.
The first one to become interested in this kind of results was Kempf: geometric properties of the Picard bundles are a very important tool in the study of the geometry of Jacobians of curves. In fact there is a wide literature on the subject, out of which let us quote the works by Kempf [4] and [5], [3], [1] and [2], and the original paper by Mukai [7].

More in detail, we start our investigation by studying Global Generation of the aforementioned bundles. In section 4 we prove that:

Theorem A.

- *If $n \geq 2$ the bundle $E^{n,k} = \widehat{\mathscr{O}_J(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}})$ is Globally Generated for every $k \geq 2$.*
- *The bundle $E^{2,k}_{W_d} = \widehat{\mathscr{O}_{W_d}(2\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}})$ is Globally Generated for every $k \geq 2$.*

As already mentioned, our main tool is the notion of M-regularity, which can be thought of as a natural strengthening of a Generic Vanishing condition. M-regularity has in fact significant geometric consequences, and M-regular sheaves enjoy generation properties, such as Continuous Global Generation. We clarify all these points in the introductory section of preliminaries.

In fact there is a proof of Proposition A by G. Kempf in [5, Theorem 6]. Our approach allows us to obtain a much simpler argument, showing the advantage of using M-regularity and all the related "machinery" set by Pareschi and Popa.

In section 5 we give a second proof of the results above. We think that this different approach is worth some attention, for it focuses on the links between this theory and the Kummer-Wirtinger duality.

Once the Global Generation is understood, we move on to studying the first syzygy step of Picard bundles of type $E^{2,k}$. In section 5 we recall the notion of Strong Normal Presentation, as introduced by Kempf in [4]. We prove that:

Theorem B. *The Picard bundle $E^{2,k}$ is Strongly Normally Presented with respect to $\mathscr{O}_J(h\Theta)$ for any $k \geq 3$ and $h \geq 4$.*

We remark that Proposition is true on any ppav, not only Jacobians of curves. In the particular case of Jacobians, we prove as a corollary that the same property holds for the bundles $E^{2,k}_{W_d}$, seen as quotient bundles of those of type $E^{2,k}$:

Proposition C.*The Picard bundle $E^{2,k}_{W_d}$ is Strongly Normally Presented with respect to $\mathscr{O}_J(h\Theta)$ for any $k \geq 3$ and $h \geq 4$.*

In the last part of this work we deal with the third aforementioned type of Picard bundles, the "classical" Picard bundles. We focus our attention on a result on regularity of Picard bundles and vanishing on symmetric products by Pareschi and Popa. In [13, Theorem 7.15] the authors give sufficient condition for the property of the Index Theorem with index 0.
We look for a weaker cohomological condition, namely M-regularity, and thus obtain a better bound on the degree n of the line bundle such that $\hat{L} = E$:

Proposition D.*Assume $n > 2g$. For every $1 \leq k \leq g-1$, the bundle $\otimes^k E \otimes \mathscr{O}_J(\Theta)$ is M-regular.*

The paper is organized as follows: in section 2 we give background material on Fourier-Mukai transform and vanishing properties. In section 3 we introduce with some detail the three types of Picard bundles that will be studied throughout the paper.
In section 4 we study Global Generation properties of the first two kinds of

Picard bundles, and in section 5 we provide an alternative proof of the same results, with a link to Kummer-Wirtinger duality.
In section 6 we recall the concept of Strong Normal Presentation and study the property on the Picard bundles $E^{2,k}$ and $E^{2,k}_{W_d}$.
Finally in section 5 we give some results on tensor products of Picard bundles of the third type.

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2. First preliminaries

2.1. The Fourier-Mukai transform on an abelian variety

We start this section of preliminaries by recalling some basic facts from Fourier-Mukai transform theory. We follow the original paper by Mukai [7], working on an abelian variety and using the Poincaré bundle, because this is the setting in which we work. We warn the reader though that some aspects of what follows hold in a much more general setting.
Let X be an abelian variety of dimension g, $\hat{X} = \mathrm{Pic}^0(X)$ its dual variety. We have natural projections

$$\begin{tikzcd} & X \times \hat{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \hat{X} \end{tikzcd}$$

Let P be a Poincaré bundle on the product $X \times \hat{X}$, normalized so that both $P|_{X\times\{0\}}$ and $P|_{\{0\times\hat{X}\}}$ are trivial. The functor Φ_P associates to any coherent sheaf $\mathscr{F}$ on X the sheaf $\Phi_P\mathscr{F} = p_{2*}(p_1^*\mathscr{F} \otimes P)$. Its derived functor is the well-known Fourier-Mukai functor

$$\mathbf{R}\Phi_P : D(X) \to D(\hat{X}).$$

In an analogous way one can define the functor going in the opposite direction

$$\mathbf{R}\Psi_P : D(\hat{X}) \to D(X),$$

and the main result is that the above functors are in fact equivalence of categories. In particular:

Theorem 2.1. *[7, Theorem 2.2] There are isomorphisms of functors:*

$$\mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P \simeq (-1_X)^*[-g]$$
$$\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P \simeq (-1_{\hat{X}})^*[-g],$$

where $[-g]$ denotes "shift the complex g places to the right"

2.2. IT and WIT properties, GV and M-regular sheaves

For a coherent sheaf $\mathscr{F}$ on an abelian variety X, the object $\mathbf{R}\Phi_P\mathscr{F}$ defined above is in general a complex, not a sheaf. If we want it to be a sheaf, *i.e.* a complex with only one nonzero term, we need to ask for $\mathscr{F}$ to have some specific properties, that we will now introduce.

Definition 2.2.

- A coherent sheaf $\mathscr{F}$ on X is called IT_j if it satisfies the Index Theorem with index j, that is if $\mathrm{H}^i(\mathscr{F} \otimes \alpha) = 0$, for any $\alpha \in \mathrm{Pic}^0(X)$ and for any $i \neq j$.
- $\mathscr{F}$ is called WIT_j if it satisfies the Weak Index Theorem with index j, that is if $R^i\Phi_P\mathscr{F} = 0$ for any $i \neq j$.

Clearly if $\mathscr{F}$ is WIT_j then $\mathbf{R}\Phi_P\mathscr{F} = R^j\Phi_P\mathscr{F}$ is a sheaf, and we indicate it by $\hat{\mathscr{F}}$, calling it "the" Fourier-Mukai transform of $\mathscr{F}$.
Because of the equivalence of category that we have explained above, properties are transmitted from a sheaf $\mathscr{F}$ to its transform $\hat{\mathscr{F}}$. In particular if $\mathscr{F}$ is WIT_j, then $\hat{\mathscr{F}}$ is WIT_{g-j} and $\hat{\hat{\mathscr{F}}}$ is isomorphic to $(-1)^*\mathscr{F}$.
Moreover, by virtue of the base change theorem IT_j implies WIT_j and in this case the only nonzero $R^j\Phi_P\mathscr{F}$ is an (algebraic) vector bundle, that is a locally free sheaf [7, Corollary 2.4].

For the reader's convenience, we now list some important properties of the Fourier-Mukai transform that we will be using all throughout the paper.

Let X be an abelian variety and $\hat{X}$ its dual; let $\mathbf{R}\Phi_P$ and $\mathbf{R}\Psi_P$ be the corresponding Fourier-Mukai functors. Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent sheaves on X. The following properties hold:

Proposition 2.3. *[7, Corollary 2.5] If $\mathscr{F}$ and $\mathscr{G}$ are WIT_j for some indices $j(\mathscr{F})$ and $j(\mathscr{G})$ respectively, then*

$$Ext^i_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \simeq Ext^{i+\mu}_{\mathscr{O}_{\hat{X}}}(\hat{\mathscr{F}},\hat{\mathscr{G}}),$$

where $\mu = j(\mathscr{F}) - j(\mathscr{G})$.

Proposition 2.4. *[7, (3.1)] Let $a \in \hat{X}$, and denote the line bundle $P_a = \alpha$. Then we have an isomorphism*

$$\mathbf{R}\Phi_P(\mathscr{F} \otimes \alpha) \simeq t_a^* \mathbf{R}\Phi_P(\mathscr{F}),$$

where t_a is the translation by a.

Proposition 2.5 (Exchange of tensor and Pontrjagin product)**.** *[7, (3.7)] Let $m : X \times X \to X$ be the group law on X, and define the* Pontrjagin product *by $\mathscr{F} * \mathscr{G} = m_*(\pi_1^* \mathscr{F} \otimes \pi_2^* \mathscr{G})$. Then we have the following isomorphisms:*

$$\mathbf{R}\Phi_P(\mathscr{F} \underline{*} \mathscr{G}) \simeq \mathbf{R}\Phi_P(\mathscr{F}) \underline{\otimes} \mathbf{R}\Phi_P(\mathscr{G}),$$
$$\mathbf{R}\Phi_P(\mathscr{F} \underline{\otimes} \mathscr{G}) \simeq \mathbf{R}\Phi_P(\mathscr{F}) \underline{*} \mathbf{R}\Phi_P(\mathscr{G})[g].$$

Proposition 2.6 (Grothendieck duality)**.** *[7, (3.8)] Let $\mathbf{R}\Delta\mathscr{F} := \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathscr{O}_X)$. Then the following skew commutativity holds:*

$$\mathbf{R}\Delta \circ \mathbf{R}\Phi_P \simeq (-1_X)^* \circ \mathbf{R}\Phi_P \circ \mathbf{R}\Delta[g].$$

Proposition 2.7. *[7, Proposition 3.11 (i)] Let A be an ample line bundle on an abelian variety X. Then $\varphi_N : X \to \hat{X}$ is an isogeny and $\varphi_N^* \hat{N} = V \otimes N^{-1}$ for some vector space V.*

We end this section with the notion of Generic Vanishing and the (stronger) one of M-regularity. We refer to the works of Pareschi and Popa, like [13] and [11].

Definition 2.8.

- A coherent sheaf $\mathscr{F}$ on X is called a GV sheaf if $\operatorname{codim}_{\hat{X}} \operatorname{supp} R^i\Phi_P(\mathscr{F}) \geq i$ for any $i \geq 0$.
- $\mathscr{F}$ is called $M-regular$ if $\operatorname{codim}_{\hat{X}} \operatorname{supp} R^i\Phi_P(\mathscr{F}) > i$ for any $i > 0$.

Remark that IT_0 objects are trivially $M-regular$, and that by definition, every M-regular sheaf is a GV sheaf.

Also *GV* and M-regular objects imply nice properties on their Fourier-Mukai transforms. For any coherent sheaf $\mathscr{F}$ on X, define for each $i \geq 0$ its *i-th cohomological support locus*

$$V^i(\mathscr{F}) = \{\alpha \in \mathrm{Pic}^0(X) = \hat{X} \mid \mathrm{h}^i(X, \mathscr{F} \otimes \alpha) > 0\}.$$

As an easy consequence of the Base Change Theorem we have the following:

Proposition 2.9. *[11, Lemma 3.8] Given a coherent sheaf $\mathscr{F}$ on X then*

1. *$\mathscr{F}$ is GV if and only if $\operatorname{codim} V^i(\mathscr{F}) \geq i$ for any $i > 0$;*
2. *$\mathscr{F}$ is $M-$regular if and only if $\operatorname{codim} V^i(\mathscr{F}) > i$ for any $i > 0$.*

Proposition 2.10. *[13, Proposition 2.8] A GV sheaf $\mathscr{F}$ on X is M-regular if and only if $\widehat{\mathbf{R}\Delta\mathscr{F}} = \mathbf{R}\Phi_P(\mathbf{R}\Delta\mathscr{F})[g]$ is a torsion-free sheaf.*

We end this subsection by quoting the following "preservation of vanishing" statement:

Proposition 2.11. *[10, Proposition 2.9] Let $\mathscr{F}$ be an M-regular coherent sheaf on X and H a locally free sheaf satisfying IT_0. Then $\mathscr{F} \otimes H$ is IT_0.*

2.3. Global Generation and Continuous Global Generation

The well-known property for a sheaf of being Globally Generated has a very useful modified version.

Definition 2.12. A sheaf $\mathscr{F}$ on an abelian variety X is said to be Continuously Globally Generated if for any non-empty open subset $U \in \mathrm{Pic}^0(X) = \hat{X}$ the sum of all the evaluation maps

$$\bigoplus_{\alpha \in U} \mathrm{H}^0(\mathscr{F} \otimes \alpha) \otimes \alpha^\vee \to \mathscr{F}$$

is surjective.

The following result gives a link between the two notions of Global Generation and Continuous Global Generation:

Proposition 2.13. *[10, Proposition 2.12] Let Y be a subvariety of an abelian variety X, $\mathscr{F}$ a coherent sheaf and L a line bundle supported on Y, both Continuously Globally Generated as sheaves on X. Then $\mathscr{F} \otimes L \otimes \alpha$ is Globally Generated for every $\alpha \in Pic^0(X)$.*

The key point, using the vanishing properties above defined, is that M-regularity implies Continuous Global Generation, as proved in [10, M-regularity criterion, Proposition 2.13]. As an easy consequence we have that:

Corollary 2.14. *Let $\mathscr{F}$ be an M-regular coherent sheaf on an abelian variety X and let L be a line bundle supported on a subvariety of X (possibly X itself). If L is IT_0 as sheaf on X, then the product $\mathscr{F} \otimes L \otimes \alpha$ is Globally Generated for every $\alpha \in Pic^0(X)$.*

Proof. We have already seen that IT_0 implies M-regularity, and that M-regularity implies Continuous Global Generation. Applying Proposition 2.13 the statement follows. □

3. Picard bundles

Let C be a smooth, irreducible, projective curve of genus $g \geq 2$, and denote by J its Jacobian, $J = J(C) = \mathrm{Pic}^0(C)$. Fix a theta-divisor Θ on J, so that the pair $(J, \mathscr{O}_J(\Theta))$ (sometimes simply denoted by (J, Θ)) is a ppav, that is a *principally polarized abelian variety*.

Recall that more generally if (X, L) is ppav, $\hat{X}$ s identified with X via the isomorphism $\phi_L : X \to \hat{X}$, and in this case $\mathbf{R}\Phi_P$ is considered to be an automorphism of the category $D(X)$. In particular:

Proposition 3.1. *[7, Theorem 3.13 (5)] If (X, L) is a ppav, then $\hat{L} \simeq L^{-1}$ and $\widehat{L^{-1}} \simeq (-1)_X^* L$.*

In our particular situation, the polarization identifies J and $\hat{J}$, and Proposition 3.1 reads:

$$\widehat{\mathscr{O}_J(\Theta)} = (-1)^* \mathscr{O}_{\hat{J}}(-\Theta_{\hat{J}}).$$

We usually denote by $\mathscr{O}_J(\Theta_J)$ (respectively $\mathscr{O}_{\hat{J}}(\Theta_{\hat{J}})$) the principal polarization on J (respectively on $\hat{J}$). In fact there is an isomorphism $\mathscr{O}_J(\Theta_J) \simeq \mathscr{O}_{\hat{J}}(\Theta_{\hat{J}})$, via $\phi^*_{\mathscr{O}_J(\Theta_J)}$. We drop the subscripts whenever it is clear from the context whether we are working on J or $\hat{J}$, trying to avoid a cumbersome notation.

Let C^d and C_d denote respectively the Cartesian and the Symmetric d-th product of C.
The choice of any invertible sheaf $\mathscr{L}_d$ of degree d on C gives an isomorphism

$$\mathrm{Pic}^d(C) \to J$$

via $\otimes\mathscr{L}_d^{-1}$. Consider this choice made once and for all throughout the paper. Then -if composed with this isomorphism- the Abel-Jacobi map u_d sends

$$C_d \hookrightarrow J,$$

and we use the standard notation W_d to denote the image of u_d. Remark that W_d is irreducible, since so is C_d, and $\dim_J(W_d) = d$, for $1 \leq d \leq g$.
Equivalently one could define:

$$W_d = \{L \in \mathrm{Pic}^d(C) \mid \mathrm{h}^0(L) > 0\}.$$

A *Picard bundle* is roughly speaking the Fourier-Mukai transform of a positive line bundle on some special subvariety the Jacobian of a curve. In this paper we are going to study three particular kinds of Picard bundles.
Let us introduce them with a little more detail.
Consider first the sheaves $\mathscr{O}_J(n\Theta)$. The sheaf $\mathscr{O}_J(\Theta)$ is ample, hence so is $\mathscr{O}_J(n\Theta)$, hence in particular they satisfy IT_0 and their Fourier-Mukai transform is a vector bundle.
We will consider Picard bundles of type

$$\widehat{\mathscr{O}_J(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}}).$$

Next, consider sheaves of type $\mathscr{O}_{W_d}(n\Theta)$, $n \geq 1$. We have that:

Proposition 3.2. *[10, Proposition 4.4] $\mathscr{O}_{W_d}(\Theta)$ is M-regular.*

So let us consider the next twist, $\mathscr{O}_{W_d}(2\Theta)$. For it we have that $\mathscr{O}_{W_d}(2\Theta) = \mathscr{O}_{W_d}(\Theta) \otimes \mathscr{O}_J(\Theta)$ is a tensor product of type M-regular $\otimes IT_0$, hence it is IT_0 by Proposition 2.11.
Hence the Fourier-Mukai transform of these sheaves is again a vector bundle, and the second type of Picard bundles that we will consider will be

$$\widehat{\mathscr{O}_{W_d}(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}}).$$

For the sake of simplicity, we give the following:

Definition/Notation 1. For any $n \geq 2$ and $k \geq 1$ we define the following Picard bundles:

$$E^{n,k} := \widehat{\mathscr{O}_J(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}})$$

and

$$E^{n,k}_{W_d} := \widehat{\mathscr{O}_{W_d}(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}}).$$

In the last part of this paper we extend our investigation to Picard bundles that are more generally transforms of line bundles L of degree $n \geq 2g-1$ on the curve C (seen as a subvariety of the Jacobian via the Abel-Jacobi embedding). We will explain this more carefully in the last section.

4. Global Generation of the bundles $E^{n,k}$ and $E^{n,k}_{W_d}$

In this section we want to study the Global Generation properties of the Picard bundles $E^{n,k}$ and $E^{n,k}_{W_d}$, giving estimates on the indices n and k for these properties to be satisfied.

We start with the following:

Lemma 4.1. *The bundle $E^{n,1} = \widehat{\mathscr{O}_J(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(\Theta_{\hat{J}})$ is IT_0 for $n \geq 2$.*

Proof. For $\alpha \in \mathrm{Pic}_0(\hat{J}) = J$, we shall relate the groups $\mathrm{H}^i(E^{n,1} \otimes \alpha)$ with the groups $\mathrm{H}^i(\mathscr{O}_J(\Theta_J))$, for $i > 0$. We have that:

$$\begin{aligned}
\mathrm{H}^i(\widehat{\mathscr{O}_J(n\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(\Theta_{\hat{J}}) \otimes \alpha) &= \mathrm{Ext}^i(\mathscr{O}_{\hat{J}}(-\Theta_{\hat{J}}), \widehat{\mathscr{O}_J(n\Theta_J)} \otimes \alpha) \\
&= \mathrm{Ext}^i((-1)^*\mathscr{O}_J(\Theta_J)[-g], (-1)^* t_a^* \mathscr{O}_J(n\Theta_J)[-g]) \\
&= \mathrm{Ext}^i(\mathscr{O}_J(\Theta_J), \mathscr{O}_J(n\Theta_J) \otimes \beta) \\
&= \mathrm{H}^i(\mathscr{O}_J((n-1)\Theta_J) \otimes \beta).
\end{aligned}$$

We have already remarked that $\mathscr{O}_J(\Theta_J)$ is IT_0, hence by semicontinuity so are all the twists $\mathscr{O}_J((n-1)\Theta_J)$, for $n \geq 2$, hence the statement follows. $\square$

Corollary 4.2. *If $n \geq 2$ and $k \geq 2$ the bundle $E^{n,k}$ is Globally Generated.*

Proof. $E^{n,k} = E^{n,1} \otimes \mathscr{O}_{\hat{J}}((k-1)\Theta_{\hat{J}})$, hence by Lemma 4.1 above it is an IT_0 sheaf tensored by an IT_0 line bundle. But IT_0 objects are trivially M-regular, so the statement follows by Corollary 2.14. $\square$

Remark 4.3. Corollary 4.2 had already been proved by G. Kempf in [5, Theorem 6] with a different argument. We would like to point out that our approach of using the Fourier-Mukai transform and vanishing properties allows a conceptually simpler proof.

We now direct our attention to the Picard bundles $E^{2,n}_{W_d}$.

Lemma 4.4. *The bundle $E^{2,1}_{W_d} = \widehat{\mathscr{O}_{W_d}(2\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(\Theta_{\hat{J}})$ is $M-$regular.*

Proof. The proof is almost identical to that of Lemma 4.1, but we repeat it for the reader's convenience. We have the following equalities:

$$\begin{aligned}
\mathrm{H}^i(\widehat{\mathscr{O}_{W_d}(2\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(\Theta_{\hat{J}}) \otimes \alpha) &= \mathrm{Ext}^i(\mathscr{O}_{\hat{J}}(-\Theta_{\hat{J}}), \widehat{\mathscr{O}_{W_d}(2\Theta_J)} \otimes \alpha) \\
&= \mathrm{Ext}^i((-1)^*\mathscr{O}_J(\Theta_J)[-g], (-1)^* t_a^* \mathscr{O}_{W_d}(2\Theta_J)[-g]) \\
&= \mathrm{Ext}^i(\mathscr{O}_J(\Theta_J), \mathscr{O}_{W_d}(2\Theta_J) \otimes \beta)) \\
&= \mathrm{H}^i(\mathscr{O}_{W_d}(\Theta_J) \otimes \beta).
\end{aligned}$$

This means that the spaces $V^i(E^{2,1}_{W_d})$ are just translates of the spaces $V^i(\mathscr{O}_{W_d}(\Theta_J))$. Since we know from Proposition 3.2 that the latter sheaf is M-regular, so is $E^{2,1}_{W_d}$. □

Corollary 4.5. *The bundle $E^{2,k}_{W_d} = \widehat{\mathscr{O}_{W_d}(2\Theta_J)} \otimes \mathscr{O}_{\hat{J}}(k\Theta_{\hat{J}})$ is Globally Generated for every $k \geq 2$.*

Proof. Again, this is a simple consequence of Lemma 4.4 above and Corollary 2.14, because $E^{2,k}_{W_d}$ is the tensor product of the M-regular sheaf $E^{2,1}_{W_d}$ and the IT_0 line bundle $\mathscr{O}_{\hat{J}}((k-1)\Theta_{\hat{J}})$. □

5. The bundles $E^{2,1}$ and $E^{2,1}_{W_d}$ and the Kummer-Wirtinger duality

We would like now to show a different approach to prove Lemma 4.4, using some constructions which are interesting on their own sake. We work in the exact same setting as in the previous section.
We start by quoting some results on the ideal sheaf $\mathscr{I}_{W_d}$.

Theorem 5.1. *[10, Theorem 4.1] For any degree $1 \leq d \leq g-1$ the twisted ideal sheaf $\mathscr{I}_{W_d}(2\Theta)$ is IT_0.*

The result above tells us that the Fourier-Mukai transform $\widehat{\mathscr{I}_{W_d}(2\Theta)}$ is a vector bundle on the dual Jacobian $\hat{J}$. In particular the standard short exact sequence

$$0 \to \mathscr{I}_{W_d}(2\Theta) \to \mathscr{O}_J(2\Theta) \to \mathscr{O}_{W_d}(2\Theta) \to 0 \tag{1}$$

entails other exact sequences, for $k \geq 1$:

$$0 \to \widehat{\mathscr{I}_{W_d}(2\Theta)} \to E^{2,k} \to E^{2,k}_{W_d} \to 0 \tag{2}$$

Definition 5.2. Given a subvariety X of a ppav (A,Θ), we denote by $V(X)$ its *theta-dual*:

$$V(X) := \{\alpha \in \hat{A} \mid \mathrm{h}^0(\mathscr{I}_X(\Theta) \otimes \alpha) > 0\} \subset \hat{A}.$$

Under the identification $A \leftrightarrow \hat{A}$ induced by the principal polarization Θ, $V(X)$ is identified with the theta-translates containing X. More precisely:

$$V(X) \simeq \{a \in A : | X \subseteq \Theta_a\} \subset A.$$

In particular we are interested in the subvariety $X = W_d$ of our usual Jacobian J of a curve of genus g. It is easy to see that set-theoretically and up to translate:

$$V(W_d) = -W_{g-d-1}.$$

In fact the above equality also holds scheme-theoretically, as shown in [12, 8,1].

Proposition 5.3. *[12, Theorem 5.2 and Corollary 5.4] The ideal sheaf $\mathscr{I}_{W_d}(\Theta)$ is GV, the canonical sheaf $\omega_{W_d}(-\Theta)$ is WIT_d and we have that:*

$$\widehat{\omega_{W_d}(-\Theta)} \simeq (-1_{\hat{J}})^* \mathscr{I}_{V(W_d)}(\Theta).$$

We now prove some formulas that we will need later on. They are in some sense versions of the Kummer-Wirtinger duality, as we explain in detail in Remark 5.7.

Lemma 5.4. *Let (A, Θ_A) be a ppav of dimension g, $(\hat{A}, \Theta_{\hat{A}})$ its dual. Then:*

$$(-1_{\hat{A}})^* \widehat{\mathscr{O}_A(2\Theta_A)} \otimes \mathscr{O}_{\hat{A}}(\Theta_{\hat{A}})[-g] \cong \widehat{\mathscr{O}_A(-2\Theta_A)} \tag{3}$$

Proof. The formula above is an easy corollary of [7, (3.10)], that applied to our setting looks like this:

$$\mathscr{F} * \mathscr{O}_A(\Theta_A) = (-1_A)^*((\mathscr{F} \otimes \mathscr{O}_{\hat{A}}(\Theta_{\hat{A}}))\widehat{\ } \otimes \mathscr{O}_A(\Theta_A)),$$

for any coherent sheaf $\mathscr{F}$ on A, where in the RHS $\mathscr{F}$ is viewed as a sheaf on $\hat{A}$ via the isomorphism $\phi_{\mathscr{O}(\Theta)} : A \to \hat{A}$.
The RHS in formula (3) is:

$$\begin{aligned}\widehat{\mathscr{O}_A(-2\Theta_A)} &= (\mathscr{O}_A(-\Theta_A) \otimes \mathscr{O}_A(-\Theta_A))\widehat{\ } \\ &= (\mathscr{O}_{\hat{A}}(\Theta_{\hat{A}}) * \mathscr{O}_{\hat{A}}(\Theta_{\hat{A}}))[-g] \\ &= (-1_{\hat{A}})^*(\mathscr{O}_A(\Theta_A) \otimes \mathscr{O}_A(\Theta_A))\widehat{\ } \otimes \mathscr{O}_{\hat{A}}(\Theta_{\hat{A}})[-g]\end{aligned}$$

□

Let us now focus again our attention on the case $A = J$ is the Jacobian of a curve. Given our subvariety $W_d \subset J$, we want to study the connections between the sheaves $\widehat{\mathscr{I}_{W_d}(2\Theta)}$ and $\widehat{\mathscr{O}_{W_d}(2\Theta)}$.

Lemma 5.5. *[12, Corollary 4.3] In the above setting:*

$$(-1)^*(\mathbf{R}\Delta(\mathscr{I}_{V(W_d)}(\Theta)))\widehat{\ } = \mathscr{O}_{W_d}(\Theta)[-g]$$

Corollary 5.6. *In the above setting:*

$$(-1)^*(\widehat{\mathscr{O}_{W_d}(2\Theta)} \otimes \mathscr{O}_{\hat{J}}(\Theta))[-g] = (\mathbf{R}\Delta(\mathscr{I}_{V(W_d)}(2\Theta)))\widehat{\ }. \tag{4}$$

Proof.

$$
\begin{aligned}
(\mathbf{R}\Delta(\mathscr{I}_{V(W_d)}(2\Theta)))\widehat{\ } &= (\mathbf{R}\Delta(\mathscr{I}_{V(W_d)}(\Theta))\otimes\mathscr{O}_J(-\Theta))\widehat{\ }\\
&= (\mathbf{R}\Delta(\mathscr{I}_{V(W_d)}(\Theta)))\widehat{\ } * (\mathscr{O}_J(-\Theta))\widehat{\ }[g]\\
&= (-1)^*\mathscr{O}_{W_d}(\Theta)[-g] * (-1)^*\mathscr{O}_{\hat{J}}(\Theta)[-g][g]\\
&= (-1)^*\mathscr{O}_{W_d}(\Theta) * (-1)^*\mathscr{O}_{\hat{J}}(\Theta)[-g]\\
&= (-1)^*(\widehat{\mathscr{O}_{W_d}(2\Theta)}\otimes\mathscr{O}_{\hat{J}}(\Theta))[-g]
\end{aligned}
$$

□

Remark 5.7. As we have quickly mentioned above, Lemma 3 and Corollary 5.6 are related to the well-known Kummer-Wirtinger duality. Let's see this with some more detail.
Since by Grothendieck duality $\widehat{\mathscr{O}_A(-2\Theta)}$ is the dual of $\widehat{\mathscr{O}_A(2\Theta)}$, formula (3) becomes:

$$(-1_{\hat{A}})^*\widehat{\mathscr{O}_A(2\Theta_A)}\otimes\mathscr{O}_{\hat{A}}(\Theta_{\hat{A}})[-g]\cong\widehat{\mathscr{O}_A(2\Theta)}^{\vee}. \tag{5}$$

The formula above is a global version of the Kummer-Wirtinger duality [8, p. 335]

$$\mathrm{H}^0(\mathscr{O}_A(2\Theta))\xrightarrow{\sim}(\mathrm{H}^0(\mathscr{O}_{\widehat{A}}(2\Theta))^{\vee}.$$

Precisely, the central fiber of the formula (5) is the Kummer-Wirtinger duality.
In a very similar way, formula (4) gives a global description of the isomorphism between the spaces $H^0(\mathscr{O}_{W_d}(2\Theta))$ and $H^0(\mathscr{I}_{V(W_d)}(2\Theta)^{\vee}$. This isomorphism is obtained using the same idea of the proof of the Kummer-Wirtinger duality. We report it for the sake of completeness (recall that $V(W_d)=(-)W_{g-d-1}$, and that $\widehat{\omega_{W_d}(-\Theta)}\simeq\mathscr{I}_{V(W_d)}(\Theta)[-d-1]$):

$$
\begin{aligned}
\mathrm{H}^0(\mathscr{I}_{W_d}(2\Theta)) &= \mathrm{Hom}(\mathscr{O}_J(-\Theta),\mathscr{I}_{W_d}(\Theta))\\
&= \mathrm{Hom}(\widehat{\mathscr{O}_J(-\Theta)},\widehat{\mathscr{I}_{W_d}(\Theta)})\\
&= \mathrm{Hom}((-1)^*\mathscr{O}_{\hat{J}}(\Theta)[-g],(-1)^*\omega_{V(W_d)}(-\Theta)[-d-1])\\
&= \mathrm{Ext}^{g-d-1}(\mathscr{O}_{\hat{J}}(\Theta),\omega_{V(W_d)}(-\Theta))\\
&= \mathrm{H}^{g-d-1}(\omega_{V(W_d)}(-2\Theta)).
\end{aligned}
$$

Applying Serre duality we get exactly that

$$\mathrm{H}^0(\mathscr{I}_{W_d}(2\Theta))=\mathrm{H}^{g-d-1}(\omega_{W_{g-d-1}}(-2\Theta))=\mathrm{H}^0(\mathscr{O}_{W_{g-d-1}}(2\Theta))^{\vee}.$$

We are now ready to give a second proof of Lemma 4.4.

Second proof of Lemma 4.4. We start by giving an explicit description of the spaces $R^i\Phi_P E^{2,1}_{W_d}$, that we obtain thanks to formula (4):

$$\begin{aligned} R^i\Phi_P E^{2,1}_{W_d} = R^i\Phi_P(\widehat{\mathscr{O}_{W_d}(2\Theta)} \otimes \mathscr{O}_{\hat{J}}(\Theta)) &= \mathscr{E}xt^i(\mathscr{I}_{V(W_d)}(2\Theta), \mathscr{O}_J) \\ &= \mathscr{E}xt^i(\mathscr{I}_{V(W_d)}, \mathscr{O}_J) \otimes \mathscr{O}_J(-2\Theta). \end{aligned} \tag{6}$$

It is well-known that $V(W_d) = W_{g-d-1}$ is Cohen-Macaulay. Hence we have that:

$$\mathscr{E}xt^i(\mathscr{O}_{W_{g-d-1}}, \mathscr{O}_J) = \begin{cases} 0 & i \neq \operatorname{codim}(W_{g-d-1}) = d+1 \\ \omega_{W_{g-d-1}} & i = d+1 \end{cases}$$

Consider now the short exact sequence

$$0 \to \mathscr{I}_{W_{g-d-1}} \to \mathscr{O}_J \to \mathscr{O}_{W_{g-d-1}} \to 0, \tag{7}$$

and apply to it the functor $Hom(-, \mathscr{O}_J)$.
Then:

$$\mathscr{E}xt^i(\mathscr{I}_{W_{g-d-1}}, \mathscr{O}_J) = \begin{cases} \mathscr{O}_J & i = 0 \\ \omega_{W_{g-d-1}} & i = d \\ 0 & \text{otherwise} \end{cases}$$

¿From this and (6) it is easy to see that the condition

$$\operatorname{codim}(\operatorname{Supp}(R^i\Phi_P(\widehat{\mathscr{O}_{W_d}(2\Theta)} \otimes \mathscr{O}_J(\Theta)))) > i$$

is satisfied for all $i > 0$, and this concludes the proof of the M-regularity of $E^{2,1}_{W_d}$. $\square$

Remark 5.8. Consider in $\operatorname{Pic}^0(J) = \hat{J}$ the open subset U of all α such that the groups $\mathrm{H}^1(E^{2,1}_{W_d} \otimes \alpha) = 0$, that is all α such that $\mathrm{h}^0(E^{2,1}_{W_d} \otimes \alpha)$ is minimal (equal to 1).

By base change, on such open subset the fibers of $R^0\Phi_P(E^{2,1}_{W_d})$ are:

$$R^0\Phi_P(E^{2,1}_{W_d}) \otimes k[\alpha] \simeq H^0(J, E^{2,1}_{W_d} \otimes \alpha).$$

Since $R^0\Phi_P(E^{2,1}_{W_d}) = \mathscr{O}_J(-2\Theta)$ is locally free, the fibers $H^0(J, E^{2,1}_{W_d} \otimes \alpha)$ have constant dimension. Now $h^0(J, E^{2,1}_{W_d} \otimes \alpha) = \operatorname{rk}(\mathscr{O}_J(-2\Theta)) = 1$ for all $\alpha \in U$.
In particular $E^{2,1}_{W_d} \otimes \alpha$ is not Globally Generated for α general in $Pic^0 J$.
On the other hand, from Corollary 2.14, it follows that $E^{2,2}_{W_d} \otimes \beta$ is Globally Generated for all β. Thus from this point of view our result is sharp.

6. (Strong) Normal Presentation

Once the Global Generation is understood, we want to try to put conditions also on the first syzygy step of our Picard bundles.
We have the following definition, according to Kempf [4]:

Definition 6.1. A coherent sheaf $\mathscr{F}$ on a projective variety X is said to be *Normally Presented* (with respect to some very ample sheaf $\mathscr{L}$) if we have an exact sequence

$$R \otimes \mathscr{L}^{-1} \to V \otimes \mathscr{O}_X \to \mathscr{F} \to 0$$

for some vector spaces R and V. Furthermore $\mathscr{F}$ is said to be *Strongly Normally Presented* if the homomorphism $G \to \mathrm{H}^0(X, \mathscr{F})$ is surjective.

A natural question arising now is thus: for which h, k the Picard bundle $E^{2,k}$ is (Strongly) Normally Presented with respect to $\mathscr{O}_J(h\Theta)$? Needless to say, a necessary condition for this to be true is that $E^{2,k}$ is Globally Generated.
In fact the computations that we are going to perform are true in a slightly more general context, where the couple $(J, \mathscr{O}(\Theta))$ is any ppav. We warn the reader that since there is no risk of confusion nor ambiguity, in this section we identify the variety J with its dual (via the principal polarization), and denote the latter with the letter J.

We start by defining:

Definition 6.2. $M_{E^{2,k}}$ is the kernel of the evaluation map for $E^{2,k}$.
In particular if $k \geq 2$ then we have seen in the previous section that the map is surjective and we have the exact sequence

$$0 \to M_{E^{2,k}} \to H^0(E^{2,k}) \otimes \mathscr{O}_J \to E^{2,k} \to 0. \tag{8}$$

Generation properties of $E^{2,k}$ and the kernel $M_{E^{2,k}}$ are in fact strictly related:

Lemma 6.3. *Fix $k \geq 2$. Then $E^{2,k}$ is Strongly Normally Presented with respect to $\mathscr{O}(h\Theta)$ if and only if $M_{E^{2,k}} \otimes \mathscr{O}(h\Theta)$ is Globally Generated.*

Proof. In this proof we don't even need the principal polarization; let E be any Globally Generated bundle on any abelian variety X, and let $M_{\mathscr{E}}$ be the kernel of the evaluation map, as in Definition 6.2. Then $\mathscr{E}$ is Strongly Normally Presented w.r.t. some very ample sheaf $\mathscr{L}$ if and only if $M_{\mathscr{E}} \otimes \mathscr{L}$ is Globally Generated. To see this it is enough to look at the diagram:

$$\begin{tikzcd}
V \otimes \mathscr{L}^{-1} \arrow[rr, dashed] \arrow[dr] & & \mathrm{H}^0(\mathscr{E}) \otimes \mathscr{O}_X \arrow[r] & \mathscr{E} \arrow[r] & 0 \\
& M_{\mathscr{E}} \arrow[ur, hook] & & &
\end{tikzcd} \tag{9}$$

Tensoring (9) by $\mathscr{L}$ it is immediate to see that if $\mathscr{E}$ is Strongly Normally Presented w.r.t. $\mathscr{L}$ (and the dotted arrow exists), then there is a surjection $V \otimes \mathscr{O}_X \twoheadrightarrow M_{\mathscr{E}} \otimes \mathscr{L}$, hence the latter is Globally Generated.
On the other hand taking $V = \mathrm{H}^0(M_{\mathscr{E}} \otimes \mathscr{L})$ one sees that the other implication holds. $\square$

Then we have the following:

Lemma 6.4. *Let $k \geq 3$ and $h \geq 4$, then $M_{E^{2,k}} \otimes \mathscr{O}(h\Theta)$ is Globally Generated.*

Assume for a moment that Lemma 6.4 is true. Then as an easy consequence of it together with Lemma 6.3 we get the result that we were aiming for:

Proposition 6.5. *The Picard bundle $E^{2,k}$ is Strongly Normally Presented with respect to $\mathscr{O}(h\Theta)$ for any $k \geq 3$ and $h \geq 4$.*

To conclude the section we only need to give a proof of Lemma 6.4, that we do using the technology of generic vanishing criterions. Before we do that we need some background results and definitions.

First we introduce a slight variation of the Pontrjagin product, called the *skew Pontrjagin product*.

Definition 6.6. Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent sheaves on an abelian variety X. Their skew Pontrjagin product is defined as:

$$\mathscr{F} \hat{*} \mathscr{G} = \pi_{1*}((\pi_1 + \pi_2)^* \mathscr{F} \otimes \pi_2^* \mathscr{G},$$

where $\pi_i : X \times X \to X$ are the natural projections, $i = 1, 2$.

Notice that [9, Remark 1.2] given two sheaves $\mathscr{F}$ and $\mathscr{G}$, their skew Pontrjagin product $\mathscr{F} \hat{*} \mathscr{G}$ is isomorphic to $\mathscr{F} * (-1)^* \mathscr{G}$, where $*$ is the usual Pontrjagin product that we had already defined in the first section.

Similarly to what happens for the usual Pontrjagin product, we have rules for exchanging the tensor and the skew Pontrjagin product:

Lemma 6.7. *[9, Lemma 3.2] Let L, M and E be vector bundles on X such that $\mathrm{h}^i((T_x^* L) \otimes E) = \mathrm{h}^i((T_x^* L) \otimes M) = 0$ for any $i > 0$ and for any $x \in X$. Then, for any $i \geq 0$:*

$$\mathrm{h}^i((L \hat{*} E) \otimes M) = \mathrm{h}^i((L \hat{*} M) \otimes E).$$

We will also need:

Lemma 6.8. *[9, Proposition 3.6] Let L be a line bundle on X, and let a and b be two integers such that both a and $a+b$ are positive. Then for any $\alpha \in Pic^0(X)$:*

$$(a+b)^*(A^a \hat{*} (A^b \otimes \alpha)) \simeq H^0(A^{a+b} \otimes \alpha) \otimes (a+b)^*(A^a) \otimes a^*(A^{-a-b}) \otimes \alpha^\vee.$$

In other words, the LHS $(a+b)^*(A^a \hat{*} (A^b \otimes \alpha))$ is isomorphic to a trivial bundle times a line bundle algebraically equivalent to $A^{ab(a+b)}$.

Proposition 6.9. *[9, Proposition 1.1] Let L be an ample line bundle and E a vector bundle on an abelian variety. Assume that $L \otimes E$ is IT_0. Then the multiplication map*

$$H^0(E) \otimes H^0(L \otimes \alpha) \xrightarrow{m} H^0(E \otimes L \otimes \alpha)$$

is surjective for all α if and only if $L \hat{*} E$ is Globally Generated.

As pointed out by Pareschi, the advantage of the above result is that the condition of Global Generation of the bundle $L \hat{*} E$ can be investigated by global cohomological methods.
In particular we have the following statement, that can be thought of as a generalization of the well-known Castelnuovo-Mumford criterion for sheaves on projective spaces.

Theorem 6.10. *[9, Theorem 2.1] Let L be an ample line bundle and E a vector bundle on an abelian variety. If $\mathrm{h}^i(E \otimes L^\vee \otimes \alpha) = 0$ for any $i > 0$ and for any line bundle $\alpha \in Pic^0$, then E is Globally Generated.*

We are now ready to give the

Proof of Lemma 6.4. In fact we prove that $M_{E^{2,k}} \otimes \mathscr{O}((h-1)\Theta)$ is M-regular. Then using Corollary 2.14 we obtain that the next twist $M_{E^{2,k}} \otimes \mathscr{O}(h\Theta)$ is Globally Generated.

Tensoring the short exact sequence (8) by $\mathscr{O}((h-1)\Theta)$, we obtain:

$$0 \to M_{E^{2,k}} \otimes \mathscr{O}((h-1)\Theta) \to H^0(E^{2,k}) \otimes \mathscr{O}((h-1)\Theta) \to E^{2,k} \otimes \mathscr{O}((h-1)\Theta) \to 0. \tag{10}$$

Now $H^0(E^{2,k}) \otimes \mathscr{O}((h-1)\Theta)$ is IT_0, because $\mathscr{O}((h-1)\Theta)$ is ample.
Also $E^{2,k} \otimes \mathscr{O}((h-1)\Theta) = E^{2,k+h-1}$ is IT_0, by corollary 4.1. So we only have to check the condition

$$\operatorname{codim} V^1(M_{E^{2,k}} \otimes \mathscr{O}((h-1)\Theta)) > 1.$$

Taking the long exact cohomology sequence of (10):

$$0 \to H^0(M_{E^{2,k}} \otimes \mathscr{O}((h-1)\Theta) \otimes \alpha) \to H^0(E^{2,k}) \otimes H^0(\mathscr{O}((h-1)\Theta) \otimes \alpha)$$
$$\xrightarrow{m_\alpha} H^0(E^{2,k} \otimes \mathscr{O}((h-1)\Theta) \otimes \alpha) \to H^1(M_{E^{2,k}} \otimes \mathscr{O}((h-1)\Theta) \otimes \alpha) \to 0.$$

In other words all we have to show is that the dimension $\dim\{\alpha \mid m_\alpha$ is surjective$\}$ is strictly greater than 1.
In fact we have much more: we show that m_α is surjective for all $\alpha \in Pic^0$.
By corollary 4.1 $E^{2,k} \otimes \mathscr{O}((h-1)\Theta) = E^{2,k+h-1}$ is IT_0, so we can apply proposition 6.9, *i.e.* we only need to check that $E^{2,k} \hat{*} \mathscr{O}((h-1)\Theta)$ is Globally Generated.
By Lemma 2.14 it is sufficient to prove that $(E^{2,k} \hat{*} \mathscr{O}((h-1)\Theta)) \otimes \mathscr{O}(-\Theta)$ is IT_0 (recall that IT_0 trivially implies M-regularity).
The fact that both bundles $\mathscr{O}((h-1)\Theta) \otimes E^{2,k}$ and $\mathscr{O}((h-1)\Theta) \otimes \mathscr{O}(-\Theta)$ are IT_0 allows us to exchange skew Pontrjagin and tensor product, as in Lemma 6.7.
Hence instead of checking that $(E^{2,k} \hat{*} \mathscr{O}((h-1)\Theta)) \otimes \mathscr{O}(-\Theta)$ is IT_0 we can check the same property for $(\mathscr{O}(-\Theta) \hat{*} \mathscr{O}((h-1)\Theta)) \otimes E^{2,k}$.
Now we make the important remark that the cohomology groups H^i are invariant under isogenies, hence without loss of generality in our computation we can substitute the bundle $(\mathscr{O}(-\Theta) \hat{*} \mathscr{O}((h-1)\Theta)) \otimes E^{2,k}$ with its pull-back via the multiplication by $(h-2)$:

$$(h-2)^*(\mathscr{O}(-\Theta) \hat{*} \mathscr{O}((h-1)\Theta)) \otimes (h-2)^* E^{2,k}.$$

The final step of the proof consists in applying Lemma 6.8, in the special case $a = h-1$ and $b = -1$, and where the line bundle is $A = \mathscr{O}(\Theta)$. Thanks to Lemma 6.8 we are reduced to study the bundle:

$$\mathscr{O}((h-1)(h-2)^2\Theta) \otimes \mathscr{O}((2-h)(h-1)^2\Theta) \otimes (h-2)^* E^{2,k},$$

where we purposely leave out the vector space $\mathrm{H}^0(A^{a+b} \otimes \alpha)$, for its presence (or absence) obviously doesn't affect our computations.
Let us now consider the worst possible case $h = 4$. We apply Proposition 2.7 from the preliminaries section and get:

$$(h-2)^* E^{2,k} = V \otimes \mathscr{O}(-2\Theta) \otimes \mathscr{O}(4k\Theta).$$

All in all:

$$\mathscr{O}(12\Theta) \otimes \mathscr{O}(-18\Theta) \otimes V \otimes \mathscr{O}(-2\Theta) \otimes \mathscr{O}(4k\Theta) = V \otimes \mathscr{O}((4k-8)\Theta),$$

which is obviously IT_0 for any $k \geq 3$. $\square$

Let us now consider our particular case, where J is the Jacobian of a smooth projective curve. Then the property of Strong Normal Presentation of the Picard bundles $E^{2,k}$ is "inherited" by the bundles $E^{2,k}_{W_d}$, if we look at the latter as quotients of the former ones, as in the short exact sequence (2).
The result about quotients of Strongly Normally Presented sheaves that we need is the following:

Lemma 6.11. *[4, Lemma 5] Given an exact sequence of coherent sheaves*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

on an abelian variety X, and suppose that $\mathscr{F}_2$ is Strongly Normally Presented w.r.t. some very ample sheaf $\mathscr{L}$. If moreover:

1. *$\mathscr{F}_1$ is Globally Generated, and*
2. *$H^1(\mathscr{F}_1) = 0$,*

then $\mathscr{F}_3$ is Strongly Normally Presented (w.r.t. $\mathscr{L}$).

The corollary is then the following:

Corollary 6.12. *The Picard bundle $E^{2,k}_{W_d}$ is Strongly Normally Presented with respect to $\mathscr{O}_J(h\Theta)$ for any $k \geq 3$ and $h \geq 4$.*

Proof. According to Lemma 6.11 above, we only need to check conditions (1) and (2) on the bundle $\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(k\Theta)$, which is the one playing the role of $\mathscr{F}_1$.
In fact it is enough to prove that $\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(\Theta)$ is GV, for on a ppav the following holds:

Lemma 6.13. *[12, Lemma 3.1] Let $\mathscr{F}$ be a GV-sheaf on a ppav (A,Θ). Then $\mathscr{F}(\Theta)$ is IT_0.*

Hence if we prove that $\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(\Theta)$ is GV we also prove that $\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(2\Theta)$ is IT_0, and hence that:

1. $\mathrm{H}^1(\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(k\Theta)) = 0$ for all $k \geq 2$ and
2. $\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(k\Theta)$ is Globally Generated for all $k \geq 3$ (by Corollary 2.14).

We proceed in the usual way; we have:

$$\begin{aligned}
\mathrm{H}^i(\widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \mathscr{O}(\Theta) \otimes \alpha) &= \mathrm{Ext}^i(\mathscr{O}(-\Theta), \widehat{\mathscr{I}_{W_d}(2\Theta)} \otimes \alpha) \\
&= \mathrm{Ext}^i((-1)^*\mathscr{O}(\Theta)[-g], (-1)^* t_a^* \mathscr{I}_{W_d}(2\Theta)[-g]) \\
&= \mathrm{Ext}^i(\mathscr{O}(\Theta), \mathscr{I}_{W_d}(2\Theta) \otimes \alpha)) \\
&= \mathrm{H}^i(\mathscr{I}_{W_d}(\Theta) \otimes \alpha).
\end{aligned}$$

¿From the IT_0 property of $\mathscr{O}_J(\Theta)$, together with the M-regularity of $\mathscr{O}_{W_d}(\Theta)$, and from the short exact sequence (1), it is immediate to see that the sheaf $\mathscr{I}_{W_d}(\Theta)$ is GV, and this concludes our proof. □

7. Regularity of classical Picard bundles

Going on with the intent of improving bounds on nice properties of our Picard bundles, we try to improve some results from [13].
Recall that we denote by C^n and C_n respectively the Cartesian and the Symmetric products of C. We have the following diagram

$$\begin{tikzcd}
C^n \arrow[r, "\pi_n"] \arrow[d] & J \\
C_n \arrow[ur, "u_n"'] &
\end{tikzcd}$$

where u_n is the Abel-Jacobi map, while π_n is the desymmetrized one.

Fix L a line bundle of degree $n \geq 2g-1$ on C, and denote $E = \widehat{L}$. In this section we will focus on such "classical" Picard bundles.
Via the isomorphism $\mathrm{Pic}^n(C) \xrightarrow{\sim} J$ that we have explained in detail in Section 2 the projectivization of E - seen as a vector bundle over $\mathrm{Pic}^n(C)$ - is the Symmetric product C_n (cf. [13, Section 7.2]).

We have the following

Lemma 7.1. *[13, Lemma 7.16] For every $k \geq 1$, $\pi_{k*}(L \boxtimes \ldots \boxtimes L)$ satisfies IT_0, and*

$$(\pi_{k*}(L \boxtimes \ldots \boxtimes L))\widehat{\ } = \bigotimes^k E.$$

The vanishing cohomological result obtained by Pareschi and Popa is:

Theorem 7.2. *[13, Theorem 7.15] Assume $n \geq 4g-4$. For every $1 \leq k \leq g-1$, $\bigotimes^k E \otimes \mathscr{O}_J(\Theta)$ is IT_0.*

We stress the fact that the above result is true when the degree n of the line bundle L is at least $4g-4$.
Now we have seen from the very beginning of this paper that there are weaker cohomological conditions than being IT_0, such as M-regularity. We show that if we ask for M-regularity we can in fact improve the bound on the degree, taking it down to $d > 2g$ (notice that for smaller values of the degree we can't even tell that the transform $\hat{L}$ is a vector bundle).

Proposition 7.3. *Assume $n > 2g$. For every $1 \leq k \leq g-1$, the bundle $\otimes^k E \otimes \mathscr{O}_J(\Theta)$ is M-regular.*

Proof. We need to check that for all $i \geq 0$:

$$\operatorname{codim} V^i(\otimes^k E \otimes \mathscr{O}_J(\Theta)) = \operatorname{codim} \operatorname{supp}\{\alpha \in \hat{J} \mid \mathrm{h}^i(\otimes^k E \otimes \mathscr{O}_J(\Theta) \otimes \alpha) \neq 0\} > i. \tag{11}$$

The idea is that since $\otimes^k E = (\pi_{k*}(L \boxtimes \ldots \boxtimes L))\widehat{\ }$ is a Fourier-Mukai transform of an IT_0 bundle (as shown in Lemma 7.1), it is WIT_g itself and moreover

$$\widehat{\otimes^k E} = (-1)^* \pi_{k*}(L \boxtimes \ldots \boxtimes L).$$

Hence we have:

$$\begin{aligned}
\mathrm{H}^i(\otimes^k E \otimes \mathscr{O}_J(\Theta_\alpha)) &= \mathrm{Ext}^i(\mathscr{O}(-\Theta_\alpha), \otimes^k E) \\
&= \mathrm{Ext}^i(\widehat{\mathscr{O}(-\Theta_\alpha)}, \widehat{\otimes^k E}) \\
&= \mathrm{Ext}^i(\mathscr{O}(\Theta_\alpha), (-1)^* \pi_{k*}(L \boxtimes \ldots \boxtimes L)) \\
&= \mathrm{H}^i((-1)^* \pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathscr{O}(-\Theta_\alpha)).
\end{aligned}$$

Now the multiplication by -1 does not affect our computation, and we need to show that for every $i \geq 0$

$$\operatorname{codim} \operatorname{supp}\{\alpha \mid \mathrm{h}^i(\pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathscr{O}(-\Theta_\alpha)) \neq 0\} > i.$$

In fact we can reduce to the computation of

$$\mathrm{h}^i(u_k^*(\pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathscr{O}(-\Theta_\alpha))) = \mathrm{h}^i(L \boxtimes \ldots \boxtimes L \otimes u_k^* \mathscr{O}(-\Theta_\alpha)).$$

The locus where such H^i groups vanish is exactly (cf. the proof of Theorem 7.5 in [13]) where

$$\mathrm{Sym}^i \mathrm{H}^1(C, \omega_C^{\otimes 2} \otimes L^{-1} \otimes A^{-1} \otimes \alpha) \otimes \bigwedge^{k-i} \mathrm{H}^0(C, \omega_C^{\otimes 2} \otimes L^{-1} \otimes A^{-1} \otimes \alpha) \tag{12}$$

vanishes, where A is an ample line bundle of degree g-d-1.
A necessary condition for (12) not to vanish is that $\mathrm{H}^1(C, \omega_C^{\otimes 2} \otimes L^{-1} \otimes A^{-1} \otimes$

$\alpha) \neq 0$.
Since the degree of $\omega_C^{\otimes 2} \otimes L^{-1} \otimes A^{-1}$ is $4g-4-(g-d-1)-n = 3g+d-n-3$, one has:

$$\{\alpha \mid \mathrm{h}^1(C, [\omega_C^{\otimes 2} \otimes L^{-1} \otimes A^{-1}] \otimes \alpha) \neq 0\} \subset W^{2g+d-n-2}_{3g+d-n-3}$$

Hence the left hand side in (11) is bigger than:

$$\operatorname{codim} W^{2g+d-n-2}_{3g+d-n-3} = 2g+d-n-1.$$

Since $n > 2g$ we have $2g+d-n-1 > i$ for all $i = 1, \ldots, d-1$, proving the claim. □

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