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ON THE INJECTIVE TENSOR PRODUCT OF M-EMBEDDED SPACES

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We exhibit classes of Banach spaces X which are M-embedded i.e., when X is canonically embedded in X^{**}, X is an M-ideal in X^{**}, for which the injective tensor product is again an M-embedded space.

Introduction.

A Banach space X is said to be an M-embedded space if X under the canonical embedding in X^{**} is an M-ideal. Equivalently the natural decomposition $X^{***} = X^* \oplus X^{\perp}$ is an ℓ^1-direct sum.

These Banach spaces enjoy some remarkable topological and geometric properties. For instance, for any such X, X^* has the Radon-Nikodym property, has the Pélczyński property (V^*) and X has the property $(\mathscr{U})$. If X has the metric approximation property then so does X^* (see Chapter 3 of [3] for the definitions of these properties and for the proofs of these results and the Notes and Remarks section for their authorship). In this note we are interested in the question when is the injective tensor product $X \otimes_\varepsilon Y$ of two M-embedded spaces X and Y is again an M-embedded space? Part of our motivation in this is that, when $X \otimes_\varepsilon Y$ is M-embedded, it would automatically possess all the properties mentioned above. We refer the reader to the survey article [5] for the stability of the properties (V), (V^*) and $(\mathscr{U})$ under tensor products.

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It is well known in this theory that for $p \neq 2$, the space of compact operators $\mathcal{K}(L^p[0,1])$ is not an M-ideal in its bidual, which is the space of bounded operators $\mathcal{L}(L^p[0,1])$ (see [3]). Since any reflexive Banach space is trivially M-embedded, we have that, injective tensor product of two reflexive spaces need not be an M-embedded space. Since being an M-embedded space is a hereditary property (see [3], Chapter 3) in order that $X \otimes_\varepsilon Y$ be M-embedded clearly X and Y should be M-embedded spaces.

Suppose Y satisfies the stronger hypothesis that for all Banach spaces X, $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$ (i.e., Y is the so called M_∞-space, see [3], Chapter 6) then the author has observed in [4] that for all reflexive Banach spaces X, $X \otimes_\varepsilon Y$ is an M-embedded space and asks the question whether $X \otimes_\varepsilon Y$ is an M-embedded space for all M-embedded spaces X? In this short note we give the positive answer under the extra assumption that Y has the metric approximation property.

For standard concepts and results of tensor product theory we refer to the monograph [1].

Theorem. *Let X be an M-embedded space. Suppose Y is in the class M_∞ and Y has the M.A.P, then $X \otimes_\varepsilon Y$ is an M-embedded space.*

Proof. Crucial to our proof is a recent observation of G. Emmanuele [2], that $X^{**} \otimes_\varepsilon Y$ is a subspace of $(X \otimes_\varepsilon Y)^{**}$.

Note that since Y has the M.A.P,

$$X^{**} \otimes_\varepsilon Y = \mathcal{K}(X^*, Y).$$

Since X and Y are M-embedded spaces, both X^* and Y^* have the R.N.P and again by our assumption of the M.A.P,

$$\begin{aligned}(X \otimes_\varepsilon Y)^{**} &= (X^* \otimes_\pi Y^*)^* \\ &= \mathcal{L}(X^*, Y^{**}).\end{aligned}$$

Since Y is in the class M_∞, one actually has that $\mathcal{K}(X^*, Y)$ is a M-ideal in $\mathcal{L}(X^*, Y^{**})$ (see [3], Chapter 6). Therefore $X^{**} \otimes_\varepsilon Y$ is an M-ideal in $(X \otimes_\varepsilon Y)^{**}$.

Now since X is an M-ideal in X^{**}, $X \otimes_\varepsilon Y$ is an M-ideal in $X^{**} \otimes_\varepsilon Y$ (see [3]). Therefore by the transitivity of this property we have that $X \otimes_\varepsilon Y$ is an M-ideal in $(X \otimes_\varepsilon Y)^{**}$.

Remark. If X and Y satisfy the hypothesis of the above theorem and X', Y' are closed subspaces of X and Y respectively then since $X' \otimes_\varepsilon Y'$ is a closed subspace of $X \otimes_\varepsilon Y$ we have that $X' \otimes_\varepsilon Y'$ is an M-embedded space. The point here is that neither of the properties i.e., M.A.P and M_∞ are hereditary.

Added Note (13-10-93): It has been recently proved by N. Kalton and D. Werner (Property M, M-ideals, and almost isometric structure of Banach spaces) that any separable Banach space in the class M_∞ is almost isometric to a subspace of c_0.

Since for any M-embedded space X, the injective tensor product with c_0 is again a M-embedded space, it can be proved now, using the methods employed in [4], that for any space X (or a closed subspace of X) in the class M_∞, and for any M-embedded space Y, the injective tensor product is again a M-embedded space. As far as I know the converse of this is still an open question.

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