A CLASS OF ELLIPTIC OPERATORS IN $\mathbb{R}^3$
IN NON DIVERGENCE FORM
WITH MEASURABLE COEFFICIENTS

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In an open cylinder of $\mathbb{R}^3$ a linear uniformly elliptic operator in non-
divergence form, with coefficients time independent but measurable only, is
investigated.
Existence and uniqueness results in suitable Sobolev spaces for the
Dirichlet problem are obtained.

1. Introduction.

A linear uniformly elliptic operator in non-divergence form in $\mathbb{R}^3$ is studied,
with coefficients measurable but depending on two variables only, and existence
and uniqueness results for the Dirichlet problem are obtained in suitable Sobolev
spaces.

The interest of our investigation comes from the fact that, while the theory
of smooth nonlinear elliptic operators had great advances in recent times, the
theory of linear non-smooth elliptic operators in non-divergence form still does
not have a natural class of solutions for which the uniqueness theorem holds.

Actually, "viscosity" solutions (see [5] where a large bibliography can be
found), "good" solutions (see [1], [2]) seem to be good candidates for natural
solutions of these equations.

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In the latter sense, M.C. Cerutti, L. Escauriaza and E.B. Fabes [1], [2] proved a uniqueness theorem for elliptic equations with coefficients continuous except for a countable set having at most one cluster point. The case of a finite number of discontinuities was due to L. Caffarelli (see [2]). In the same context we point out the work by N. V. Krylov [8] on the so-called weak uniqueness problem for elliptic equations.

L. Escauriaza [6] has obtained a uniqueness theorem for elliptic operators in $\mathbb{R}^n$ with coefficients depending on $(n - 1)$ variables, continuous in these variables except by a countable set of points with countable closure.

Our results are partially overlapping with these above mentioned: the coefficients of our operators in $\mathbb{R}^3$ depend on two variables but no continuity assumption is required.

The plan of the paper is the following. First we look for solutions periodic with respect to the variable which does not appear in the operator’s coefficients. Then such a requirement is removed and the Dirichlet problem is studied in the general setting.

Let us now introduce a few notations.

Let $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$ to be the unit ball in $\mathbb{R}^2$. Let $a^{ij}$, $b^j$ measurable functions in $B$ and consider the operators:

$$S = a^{ij} D_{ij} , \quad L_1 = b^j D_j$$

where $D_j = \partial / \partial x_j$, $D_{ij} = \partial^2 / \partial x_i \partial x_j$ $(i, j = 1, 2)$ and summation convention is assumed.

Let now $R = \{(x, t) \in \mathbb{R}^3 : x \in B, \ 0 < t < 1\}$ an open cylinder in $\mathbb{R}^3$ and $\partial_0 R = \partial B \times [0, 1]$ its lateral boundary.

In $R$ we will study a second order linear uniformly elliptic operator of the form:

$$(1) \quad Lu = a^{ij}(x) D_{ij} u(x, t) + b^j(x) D_j u_t(x, t) + u_{tt}(x, t) = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qc
where $D^2 u = (D_{ij} u)$ and
\[
W^{2, p}_{r_0}(B) = \{u : u \in W^{2, p}(B), \ u|_{\partial B} = 0\}.
\]

Analogously in $R$ will be used $W^{2, p}(R)$ and $W^{2, p}_{r_0}(R)$; we will need also
\[
W^{2, p}_{\text{loc}}(R \cup \partial_0 R) = \{u \in W^{2, p}(B \times (a, b)), \ 0 < a < b < 1\}.\]
Moreover $W^{2, p}_{#}(R)$ will be the closure in $W^{2, p}(R)$ norm of
\[
\{u \in C^\infty(\bar{B} \times \mathbb{R}), \ u \quad \text{periodic of period 1}, \quad u|_{\partial_B \times \mathbb{R}} = 0\}.
\]
In all the inequalities throughout the paper $C_\alpha$ will be a constant, not necessarily the same, depending on $\alpha$ only, $C_{\alpha, p}$ will depend on $\alpha$, $p$ only and so on.

2. A priori bounds and existence theorems in $W^{2, p}_{#}(R)$.

**Lemma 1.** Let $a^{ij}, b^j$ ($i, j = 1, 2$) satisfying (2), $S = a^{ij} D_{ij}, L_1 = b^j D_j$. Let $v \in \mathcal{L}^2, U \in W^{2, 2}_{r_0}(B)$ (complex valued) and let
\[
(SU + ivL_1 U - v^2 U = f + ig.
\]

Let $z \in W^{2, 2}_{r_0}(B)$ be the solution of
\[
Sz - av^2 z = -(f^2 + g^2)^{1/2}.
\]
Then:
\[
|U| \leq z \quad \text{in } B.
\]

**Proof.** Let $u^{(1)} = \text{Re } U, u^{(2)} = \text{Im } U$. Then $u^{(j)} \in W^{2, 2}_{r_0}(B)$ ($j = 1, 2$) and
\[
Su^{(1)} - vL_1 u^{(2)} - v^2 u^{(1)} = f,
\]
\[
Su^{(2)} + vL_1 u^{(1)} - v^2 u^{(2)} = g.
\]

Let now $w = |U|$. By evaluating $D_i w$ and $D_{ij} w$ in the open set $w > 0$ we have a.e. in $B$:
\[
w^3 Sw = w^2 \left[ u^{(1)} Su^{(1)} + u^{(2)} Su^{(2)} + a^{ij} D_i u^{(1)} D_j u^{(1)} + a^{ij} D_i u^{(2)} D_j u^{(2)} \right] +
- \left[ (u^{(1)})^2 a^{ij} D_i u^{(1)} D_j u^{(1)} + 2(u^{(1)}) u^{(2)} a^{ij} D_i u^{(1)} D_j u^{(2)} +
+ (u^{(2)})^2 a^{ij} D_i u^{(2)} D_j u^{(2)} \right].
\]
Therefore:

\[ w^3 Sw = w^2 \left[ u^{(1)} Su^{(1)} + u^{(2)} Su^{(2)} \right] + \left( u^{(2)} \right)^2 a^{ij} D_i u^{(1)} D_j u^{(1)} + \]

\[ + (u^{(1)})^2 a^{ij} D_i u^{(2)} D_j u^{(2)} - 2 u^{(1)} u^{(2)} a^{ij} D_i u^{(1)} D_j u^{(2)}. \]

Namely we have that:

\[ w^3 Sw = w^2 \left[ u^{(1)} Su^{(1)} + u^{(2)} Su^{(2)} \right] + \]

\[ + a^{ij} \left[ u^{(2)} D_i u^{(1)} - u^{(1)} D_i u^{(2)} \right] \left[ u^{(2)} D_j u^{(1)} - u^{(1)} D_j u^{(2)} \right]. \]

Now using (6), (7) and writing for short \( \xi_i = u^{(2)} D_i u^{(1)} - u^{(1)} D_i u^{(2)} \), we get:

\[ w^3 Sw = w^2 \left[ u^{(1)} f + u^{(2)} g \right] + v^2 w^4 + \]

\[ + v w^2 b^j \left[ u^{(1)} D_j u^{(2)} - u^{(2)} D_j u^{(1)} \right] + a^{ij} \xi_i \xi_j. \]

Let \( \eta = v w^2 \). Then:

\[ w^3 Sw = w^2 \left[ u^{(1)} f + u^{(2)} g \right] + \eta^2 + b^j \xi_j \eta + a^{ij} \xi_i \xi_j. \]

Taking into account (2), it follows that:

\[ w^3 Sw \geq w^2 \left[ u^{(1)} f + u^{(2)} g \right] + v^2 w^4. \]

Thus:

\[ Sw - \alpha v^2 w \geq u^{(1)} f + u^{(2)} g \geq \left( f^2 + g^2 \right)^{1/2}. \]

Let us now show that \( w < z \). First observe that \( z > 0 \) in \( B \), by Alexandrov-Pucci’s maximum principle. On the other hand the function \( w - z \) is zero on the boundary of \( B \) and not greater than zero where \( w = 0 \). Since in the region where \( w > 0 \) one has:

\[ S(w - z) - \alpha^2 v^2 (w - z) \geq 0, \]

again by the maximum principle the function \( w - z \) cannot have a positive maximum there. Therefore \( w \leq z \), i.e. (5) holds.

We are now able to prove the following a priori estimate.
Theorem 1. Let \( u \in W^{2,2}_\#(\mathbb{R}) \). Then:

\[
\|u\|_{W^{2,2}_\#(\mathbb{R})} \leq C_\alpha \|Lu\|_{L^2(\mathbb{R})}.
\]

Proof. Let \( f = Lu \). Let us expand \( u \) and \( f \) in Fourier series with respect to the \( t \) variable:

\[
u(x, t) = \sum_{\nu=-\infty}^{+\infty} u_\nu(x) e^{i2\pi \nu t},
\]

\[
f(x, t) = \sum_{\nu=-\infty}^{+\infty} f_\nu(x) e^{i2\pi \nu t}.
\]

The functions \( u_\nu \) (\( \nu \in \mathbb{R} \)) are in \( W^{2,2}_{\gamma_0}(B) \) and we have:

\[
Su_\nu + i\nu L_1 u_\nu - \nu^2 u_\nu = f_\nu \quad \text{a.e. in } B.
\]

As \( u_\nu \) satisfies the hypothesis of Lemma 1, then

\[
|u_\nu| \leq |z_\nu|,
\]

where \( z_\nu \in W^{2,2}_{\gamma_0}(B) \) is the unique solution of the equation:

\[
Sz_\nu - \alpha^2 \nu^2 z = -|f_\nu|.
\]

Let us recall (see e.g. M. Chicco [3]) that:

\[
|\nu^2 z_\nu|_{L^2(B)} \leq C_\alpha \|f_\nu\|_{L^2(B)} \quad \nu \in \mathcal{F}.
\]

Thus:

\[
|\nu^2 u_\nu|_{L^2(B)} \leq C_\alpha \|f_\nu\|_{L^2(B)} \quad \nu \in \mathcal{F}.
\]

Then, owing to the Plancherel theorem:

\[
\|u_{tt}\|_{L^2(\mathbb{R})} \leq C_\alpha \|f\|_{L^2(\mathbb{R})}.
\]

From Talenti's [11] a priori bound, we derive:

\[
\|D^2 u_\nu\|_{L^2(B)} \leq C_\alpha \left( \|f_\nu\|_{L^2(B)} + |\nu| \|L_1 u_\nu\|_{L^2(B)} + \nu^2 \|u_\nu\|_{L^2(B)} \right).
\]
By interpolation on $|v| \|L_1 u_v\|_{L^2(B)}$ and using again (10), one has:

$$\|D^2 u_v\|_{L^2(B)} \leq C_\alpha \|f_v\|_{L^2(B)}$$

and by Plancherel theorem:

$$\|D_{ij} u\|_{L^2(R)} \leq C_\alpha \|f\|_{L^2(R)}.$$ 

This inequality and (11) give:

(12) $$\|\Delta u\|_{L^2(R)} \leq C_\alpha \|f\|_{L^2(R)}.$$ 

As $u$ is periodic in the $t$ variable, then:

$$\|u\|_{L^2(R)} \leq C_\alpha \|u_{tt}\|_{L^2(R)}.$$ 

From the last two inequalities and (11) the thesis of the Theorem follows.

Let us now prove a similar apriori bound in $L^p$, for $p$ in a neighborhood of 2.

**Theorem 2.** There exist $p_1(\alpha) < 2 < p_2(\alpha)$, such that for any $u$ of class $W^{2,p}_\#(R)$ with $p \in [p_1(\alpha), p_2(\alpha)]$ the following estimate holds:

(13) $$\|u\|_{W^{2,p}_\#(R)} \leq C_\alpha \|Lu\|_{L^p(R)}.$$ 

**Proof.** We will use the complex interpolation result of R. Coifman and others [4].

(i) Let $1 < p < \infty$. There exists $\eta_0(\alpha, p)$ such that if we take $|\eta| \leq \eta_0(\alpha, p)$, $p_1$ between 2 and $p$, then:

(14) $$\|\Delta u\|_{L^{p_1}(R)} \leq C_{\alpha,p} \|(L + i\eta(L - \Delta))u\|_{L^{p_1}(R)},$$

for every $u \in W^{2,p_1}(R)$.

This fact follows from the inequality (see e.g. [7]):

$$\|D^2 u\|_{L^{p_1}(R)} \leq C_p \|\Delta u\|_{L^{p_1}(R)}$$

and small perturbation techniques.
(ii) There exists $\eta_0^{(2)}(\alpha)$ such that if $\eta \leq \eta_0^{(2)}(\alpha)$ and $0 \leq \xi \leq 1$ then:

$$\|\Delta u\|_{L^2(R)} \leq C_{\alpha} \|\{[1 - (\xi + i \eta)]\Delta + (\xi + i \eta)L\}u\|_{L^2(R)}$$

for every $u \in W^{2,2}_#(R)$.

In fact, for every $\xi \in [0, 1]$, the operator $(1 - \xi)\Delta + \xi L$ is of the form (1) with coefficients satisfying (2) and therefore, by Theorem 1, for any $u \in W^{2,2}_#(R)$ one has:

$$\|\Delta u\|_{L^2(R)} \leq C_{\alpha} \|\{(1 - \xi)\Delta + \xi L\}u\|_{L^2(R)}.$$ 

Thus a perturbation argument gives (15).

Now set $\eta_0(\alpha, p) = \min \{\eta_0^1(\alpha, p), \eta_0^2(\alpha)\}$ and let

$$Q = \{\xi + i \eta, \ 0 \leq \xi \leq 1, \ |\eta| \leq \eta_0(\alpha, p)\}.$$ 

If $z = \xi + i \eta \in Q$ and $f \in L^\infty(R)$ is a simple function, extended periodically in $t$ to $B \times \mathbb{R}$, then the equation:

$$L_z u = \{(1 - z)\Delta + z L\}u = f$$

has a unique solution in $W^{2,2}_#(R)$; notice that if $z = i \eta$ then $L_{i \eta}$ has a unique solution in $W^{2,p_1}_#(R)$, $p_1$ between 2 and $p$.

Let us define: $T_z f \equal: \Delta u$. Then $T_z f$ is a measurable function for every $z \in Q$. If $\|T_z\|_p$ denotes the norm of $T_z$ as an operator in $L^p$, then

$$\|T_z\|_2 \leq C_{\alpha, p}, \|T_{i \eta}\|_{p_1} \leq C_{\alpha, p}.$$ 

(iii) Let us prove now that, if $f$ and $g$ are simple functions, then:

$$\int_R (T_z f)g \, dx \, dt$$

is an holomorphic function inside $Q$.

Let $z_0$ an interior point in $Q$ and let $v = \sum_{k=0}^{\infty} u_k(z - z_0)^k$ where $u_k \in W^{2,2}_#(R)$ and $L_{z_0} u_0 = f$, $L_{z_0} u_k = (L - \Delta)u_{k-1}$ ($k > 1$). Notice that, by (15):

$$\|\Delta u_0\|_{L^2(R)} \leq C_{\alpha, p} \|f\|_{L^2(R)}$$

and

$$\|\Delta u_k\|_{L^2(R)} \leq C_{\alpha, p} \|(L - \Delta)u_{k-1}\|_{L^2(R)} \leq C_{\alpha, p} \|\Delta u_{k-1}\|_{L^2(R)};$$
thus:

$$||\Delta u_k||_{L^2(R)} \leq C_{\alpha, p}^{k+1} ||f||_{L^2(R)}.$$  

Therefore, if $|z - z_0| < C_{\alpha, p}^{-1}$, then $v \in W^{2, 2}_\#(R)$ and $L_z v = f$; moreover:

$$\int_R (T_z f) g \, dx \, dt = \int_R (\Delta v) g \, dx \, dt = \sum_{k=0}^\infty (z - z_0)^k \int_R (\Delta u_k) g \, dx \, dt.$$  

Hence $\int_R (T_z f) g \, dx \, dt$ is holomorphic in a neighborhood of $z_0$ and so everywhere inside $Q$.

Now we are in a position to apply the complex interpolation result [4]. Let $T_z$ be a linear operator, mapping simple functions, defined in $R$, into measurable functions defined in $R$. Moreover, let:

$$\omega = \omega(\alpha, p) = \pi (2\eta_0(\alpha, p))^{-1}$$

and

$$b(z) = \frac{1}{2} + \left( \frac{1}{p} - \frac{1}{2} \right) \frac{\sinh \omega (1 - \xi) \cos \omega \eta}{\sinh \omega}. $$

Notice, by the way, that $b(z)$ is harmonic in $Q$,

$$b(0, \eta) = \frac{1}{2} + \left( \frac{1}{p} - \frac{1}{2} \right) \cos \omega \eta, \quad b(1, \eta) = \frac{1}{2}, \quad b(\xi, \pm \eta_0) = \frac{1}{2}.$$  

Let $\mathcal{N}(z)$ denote the norm of $T_z$ as an operator from $L^{1/b(z)}(R)$ into $L^{1/b(z)}(R)$; then $\log \mathcal{N}(z)$ is subharmonic.

Observe that, if $z = i\eta$, then $1/b(z)$ is between 2 and $p$; if $z = 1 + i\eta$, $\xi \pm i\eta_0(\alpha, p)$ then $1/b(z) = 2$. As a consequence, $\log \mathcal{N}(z)$ is bounded on $\partial Q$ by $C_{\alpha, p}$; thus: $\log \mathcal{N}(\xi + i0) \leq C_{\alpha, p}$.

So we have that:

$$\|\Delta u\|_{L^{1/b(\xi)}(R)} \leq C_{\alpha, p} \|L_{\xi} u\|_{L^{1/b(\xi)}(R)}, \quad 0 \leq \xi \leq 1.$$  

Notice that $1/b(\xi)$ is a monotonic function with $1/b(0) = p$, $1/b(1) = 2$.

Now let us take an operator $\Lambda$ of the form (1), with coefficients satisfying (2), and define:

$$L = \vartheta^{-1} \Lambda - \vartheta^{-1}(1 - \vartheta) \Delta, \quad 0 < \vartheta < 1.$$  

$L$ is of the form (1) and satisfies (2) with $\alpha$ replaced by $\alpha/2$, if $\vartheta$ is sufficiently near to 1.
Then equation (17) holds with $p = 3$ and $p = 3/2$, and $\alpha$ replaced by $\alpha/2$. If $\xi = \vartheta$, then $L_{\xi} = \Lambda$, so we get two exponents $p_1$ and $p_2$ (both of them depending on $\alpha/2$ only), $p_1 < 2 < p_2$, such that:

$$
\| \Delta u \|_{L^{p_1}(R)} \leq C_{\alpha/2,3/2} \| \Lambda u \|_{L^{p_1}(R)}
$$

$$
\| \Delta u \|_{L^{p_2}(R)} \leq C_{\alpha/2,3} \| \Lambda u \|_{L^{p_2}(R)}.
$$

The thesis follows.

**Remark 1.** Theorem 1 and Theorem 2 easily give us an existence and uniqueness theorem in $R$ for the equation $Lu = f$ of the type (1), with coefficients satisfying (2), with boundary conditions: $u|_{\partial_0 R} = 0$, $u|_{t=0} = u|_{t=1} = 0$ and $u|_{t=0} = u|_{t=1} = 0$.

Moreover, by Sobolev imbedding theorems, we have that, if the function $f$ is of class $L^p(R)$ ($p$ around 2), then:

(i) $u \in W^{1,q}(R)$, $1/q = 1/p - 1/3$ ($q$ is around 6);

(ii) for $p > 3/2$ $u \in C^{0,\sigma}(\bar{R})$, $\sigma = 2 - 3/p$ ($\sigma$ is around 1/2), and:

$$
\| Du \|_{L^p(R)} \leq C_{\alpha,p} \| f \|_{L^p(R)}
$$

(18)

$$
\| u \|_{C^{0,\sigma}(\bar{R})} \leq C_{\alpha,p} \| f \|_{L^p(R)}.
$$

(19)

**Remark 2.** If one changes $R$ into $B \times (a, a + 1)$ theorems 1 and 2 hold with the same constants and exponents. If we consider $B \times (a, b)$, then theorems 1 and 2 hold, but the constants and exponents may change; however, if $1/2 < b - a < 2$ (say) there exists $C_\alpha$ such that if $p_1(\alpha/4) < p < p_2(\alpha/4)$, then

$$
\| D^2 u \|_{L^p(B \times (a,b))} \leq C_\alpha \| Lu \|_{L^p(B \times (a,b))} \quad \forall u \in W^2_0(B \times (a,b)).
$$

From now on we will redefine: $p_1(\alpha) := \max\{p_1(\alpha/4), 3/2\}$, $p_2(\alpha) := p_2(\alpha/4)$. 
3. Uniqueness and existence theorems for the Dirichlet problem with homogeneous boundary conditions.

**Theorem 3.** Let \( p_1(\alpha) < p < p_2(\alpha) \). Let \( u \in W^{2,p}_\text{loc}(R) \cap C^0(\overline{R}) \), \( u|_{\bar{\partial}R} = 0 \), \( Lu \in L^p(R) \). Then the following estimate holds:

\[
\|u\|_{L^\infty(R)} \leq C_{\alpha,p} \|Lu\|_{L^p(R)}.
\]

**Proof.** Let \( w \in C^\infty(\overline{R}) \), \( w|_{\bar{\partial}R} = 0 \). Let us show that

\[
\|w\|_{L^\infty(R)} \leq C_{\alpha,p} \left( \|Lw\|_{L^p(R)} + \|w\|_{L^\infty(\partial R \setminus \bar{\partial}_0 R)} \right).
\]

For let a sequence of operators:

\[
L_n = a_n^{ij} D_{ij} + b_n^i D_j \partial / \partial t + \delta^2 / \partial t^2
\]

smooth with \( a_n^{ij} \to a^{ij}, b_n^i \to b^i \) (\( i, j = 1, 2 \)) a.e. in \( B \). We may assume that the coefficients of \( L_n \) satisfy (2). Let us solve the problem:

\[
L_n v_n = L_n w, \quad v_n \in W^{2,p}_\#(R) \cap C^2(R).
\]

If \( q = (p + p_1(\alpha))/2 \), by Remark 1:

\[
\|v_n\|_{L^\infty(R)} \leq C_{\alpha,p} \|L_n v_n\|_{L^q(R)} = C_{\alpha,p} \|L_n w\|_{L^q(R)}.
\]

As \( L_n(w - v_n) = 0 \) in \( R \), by the maximum principle:

\[
\|w - v_n\|_{L^\infty(R)} \leq \|w\|_{L^\infty(\partial R \setminus \bar{\partial}_0 R)} + \|v_n\|_{L^\infty(R)}.
\]

Then, from (22) and (23):

\[
\|w\|_{L^\infty(R)} \leq \|w\|_{L^\infty(\partial R \setminus \bar{\partial}_0 R)} + C_{\alpha,p} \|L_n w\|_{L^q(R)}.
\]

On the other hand:

\[
\|L_n w - L w\|_{L^q(R)} \leq \sum_{i,j=1}^{2} \|a_n^{ij} - a^{ij}\|_{L^{pq/p-q}(R)} + \sum_{j=1}^{2} \|b_n^i - b^i\|_{L^{pq/p-q}(R)} \|D^2 w\|_{L^p(R)}.
\]
Thus, letting \( n \to \infty \) yields \( L_n w \to Lw \) in \( L^q \) and so the estimate (24) easily implies (21), once we recall also that \( q < p \).

Now let \( 0 < \varepsilon < 1/2 \) and \( R_\varepsilon = B \times (\varepsilon, 1 - \varepsilon) \).

Notice that \( u \in W^{2,p}(R_\varepsilon) \); hence there exist \( u_n \in C^\infty(\bar{R}_\varepsilon) \), \( u_n|_{\partial R_\varepsilon} = 0 \) such that \( u_n \to u \) in \( W^{2,p}(R_\varepsilon) \). Then, for \( u_n \), the estimate (21) holds in \( R_\varepsilon \) (recall also Remark 2). Taking the limit as \( n \to \infty \), we have:

\[
\|u\|_{L^\infty(R_\varepsilon)} \leq C_{\alpha,p} \left( \|Lu\|_{L^p(R_\varepsilon)} + \|u\|_{L^\infty(\partial R_\varepsilon \setminus \partial R_\varepsilon)} \right).
\]

Now, letting \( \varepsilon \to 0 \) yields \( R_\varepsilon \to R \) and:

\[
\|u\|_{L^\infty(R_\varepsilon)} \to \|u\|_{L^\infty(R)},
\]

\[
\|Lu\|_{L^p(R_\varepsilon)} \to \|Lu\|_{L^p(R)},
\]

\[
\|u\|_{L^\infty(\partial R_\varepsilon \setminus \partial R_\varepsilon)} \to 0.
\]

Hence the estimate (20) follows and the Theorem is proved.

In what follows we need next Lemma 2, involving weighted norms. Let \( w(t) = t(1 - t) \). The following result is true.

**Lemma 2.** Let \( u \in C^2(\bar{R}) \), \( u|_{\partial R} = 0 \). Let \( p \in [p_1(\alpha), p_2(\alpha)] \). Then:

\[
(25) \quad \|w^2 D^2 u\|_{L^p(R)} \leq C_{\alpha,p} \left( \|w^2 Lu\|_{L^p(R)} + \|u\|_{L^p(R)} \right).
\]

Before giving the line of the proof, let us point out for our purposes that standard Sobolev spaces techniques allow to get the following weighted interpolation inequality: for any \( u \in C^2(\bar{R}) \), \( u|_{\partial R} = 0 \), and for any \( \varepsilon > 0 \):

\[
(26) \quad \|w Du\|_{L^p(R)} \leq \varepsilon \|w^2 D^2 u\|_{L^p(R)} + C_{\varepsilon,p} \|u\|_{L^p(R)}.
\]

**Proof of Lemma 2.** Set \( v = w^2 u \). The function \( v \) can be extended outside of \( R \) into \( B \times \mathbb{R} \) as a \( C^2 \) function periodic in \( t \), with period 1. By applying Theorem 2 to the function \( v \), writing back the bounds for \( u \) and using interpolation inequality (26), one ends up with (25).

We are now able to show the following theorem.
Theorem 4. Let \( L \) of the form (1), with coefficients satisfying (2). Let \( p \in [p_1(\alpha), p_2(\alpha)] \). Then, if \( f \in L^p(R) \), the problem

\[
(27) \quad Lu = f \quad \text{in } R
\]

\[
(28) \quad u_{|\partial R} = 0
\]

has a unique solution \( u \in W^{2,p}_{\text{loc}}(R) \cap C^0(\bar{R}) \) and:

\[
(29) \quad \|u\|_{L^\infty(R)} \leq C_{\alpha,p}\|Lu\|_{L^p(R)},
\]

\[
(30) \quad \|w^2D^2u\|_{L^p(R)} \leq C_{\alpha,p}\|Lu\|_{L^p(R)},
\]

\[
(31) \quad |u(x, t)| \leq C_{\alpha,p}\left\{\text{dist}((x, t), \partial R)\right\}^{2-3/p}\|Lu\|_{L^p(R)}.
\]

Proof. The uniqueness of the solution follows by Theorem 3. Indeed, if \( u_1 \in W^{2,p}_{\text{loc}}(R) \) and \( u_2 \in W^{2,p}_{\text{loc}}(R) \) are two solutions of (27) - (28), then \( u_1 - u_2 \in W^{2,p}_{\text{loc}}(R) \cap C^0(\bar{R}) \) vanishes on \( \partial R \) and \( L(u_1 - u_2) = 0 \) a.e. in \( R \). Thus, Theorem 3 applies and \( u_1 - u_2 \equiv 0 \).

Let us prove the existence. Let \( f_n \in C^\infty_0(R) \) and \( f_n \to f \) in \( L^p(R) \). Consider a sequence of operators:

\[
L_n = \alpha^{ij}_n D_{ij} + b^j D_j \partial / \partial t + \partial^2 / \partial t^2
\]

smooth and such that \( \alpha^{ij}_n \to \alpha^{ij}, b^j_n \to b^j \) a.e. in \( B \).

We may assume that the coefficients of \( L_n \) satisfy (2).

Let \( u_n \) be the solutions of

\[
(29) \quad L_n u_n = f_n \quad \text{in } R
\]

\[
(30) \quad u_{n|\partial R} = 0.
\]

Clearly \( u_n \in C^2(\bar{R}) \cap W^{2,p}(R) \) and, by Theorem 3 as well as Lemma 2:

\[
(32) \quad \|u_n\|_{L^\infty(R)} \leq C_{\alpha,p}\|L_n u_n\|_{L^p(R)}.
\]
(33) \[ \|u^2 D^2 u_n\|_{L^p(R)} \leq C_{\alpha,p} \|L_n u_n\|_{L^p(R)}. \]

Let us extend $f_n$ as a periodic (in $t$) function and solve the problem: $L_n v_n = f_n$, $v_n \in W^{2,p}_{\#}(R)$. Actually, the $v_n$ functions are smooth in $\bar{B} \times \mathbb{R}$.

From (19), recall that:

\[ \|v_n\|_{C^0,\sigma(\bar{R})} \leq C_{\alpha,p} \|f_n\|_{L^p(R)}, \quad \text{where} \quad \sigma = 2 - 3/p. \]

By a result of C. Pucci [10], since $L_n (v_n - u_n) = 0$ in $R$ and $u_{n|\partial R} = 0$, if $(x, t) \in R$ and $(\xi, \tau) \in \partial R$ such that $\text{dist}((x, t), (\xi, \tau)) = \text{dist}((x, t), \partial R)$, then:

\[ |v_n(x, t) - u_n(x, t) - v_n(\xi, \tau)| \leq C_{\alpha,p} \left\{ \text{dist}((x, t), \partial R) \right\}^\sigma \|v_n\|_{C^0,\sigma(\partial R)}, \]

and therefore:

\[ |u_n(x, t)| \leq |v_n(x, t) - u_n(x, t) - v_n(\xi, \tau)| + |v_n(x, t) - v_n(\xi, \tau)| \leq \]

\[ \leq C_{\alpha,p} \left\{ \text{dist}((x, t), \partial R) \right\}^\sigma \|v_n\|_{C^0,\sigma(\partial R)}. \]

Thus:

(34) \[ |u_n(x, t)| \leq C_{\alpha,p} \{\text{dist}((x, t), \partial R)\}^\sigma \|L_n u_n\|_{L^p(R)}. \]

Observe now that, as $L_n u_n = f_n \rightarrow f$ in $L^p(R)$, $\|L_n u_n\|_{L^p(R)}$ is bounded. As a consequence, by (32), there is a subsequence of solutions (let us call it again $u_n$) such that $u_n \rightarrow u \in W^{2,p}_{\#}(R \cup \partial_0 R)$, $u_{n|\partial R} \rightarrow u$ uniformly in every compact subset of $R \cup \partial_0 R$, so $u \in C^0(R)$. Then $u$ satisfies (29), (30), (31) and so $u \in C^0(\bar{R})$, $u = 0$ on $\partial R$.

On the other hand, if $\varphi$ is a smooth function with support in $R \cup \partial_0 R$, we have:

\[ \int_R (Lu - f) \varphi \, dx \, dt = \int_R L(u - u_n) \varphi \, dx \, dt + \int_R (L - L_n) u_n \varphi \, dx \, dt + \]

\[ + \int_R (f_n - f) \varphi \, dx \, dt. \]

Letting $n \rightarrow \infty$, clearly yields: $Lu = f$ a.e. in $R$.

The Theorem 4 is so completely proved.
Remark 3. Let $f \in L^3(R)$ and let $u$ to be a "good" solution to the problem $Lu = f$ in $R$, $u|_{\partial R} = 0$; namely, $u$ is the uniform limit on $\bar{R}$ of the sequence $\{u_n\}$ formed by the solutions to $L_n u_n = f$ in $R$, $u_n = 0$ on $\partial R$, where $L_n$ are regularizations of $L$ of the form (1), with coefficients satisfying (2) and converging pointwise to those of $L$ on $B$.

Then, as in Theorem 4, $u \in W^{2,p}_{\text{loc}}(R)$ (for some $p > 2$), $u \in C^0(\bar{R})$ and $Lu = f$ a.e. in $R$, $u|_{\partial R} = 0$.

Therefore Theorem 3 give us the uniqueness of "good" solutions to the Dirichlet problem for $L$, with no regularity assumptions on its coefficients $a^{ij}$, $b^i$.

REFERENCES


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